PROJECT DESCRIPTION

RESEARCH IN SET THEORY

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I propose to undertake research in the area of mathematical logic and foundations known as set theory, pursuing several projects that appear to be ripe for progress. The first project involves cross-boundary work between set theory and models of arithmetic, two sub-areas of mathematical logic. Despite the traditional distance between these areas, it is becoming increasingly clear that several of the fundamental open questions in models of arithmetic exhibit a deep set-theoretic nature, and an inter-speciality approach now seems called for. The most recent advances on Scott's problem, for example, involve a sophisticated blend of techniques from models of arithmetic and the Proper Forcing Axiom. The second project, at the heart of contemporary research in set theory, concerns forcing axioms and large cardinal indestructibility, a core area of my prior work. Recent advances have uncovered a surprisingly robust indestructibility phenomenon for comparatively weak large cardinals, and corresponding equiconsistency results for fragments of the Proper Forcing Axiom. In a third, more abstract project, I call attention to an emerging focus in set theory on set theoretic principles involving second and higher order features of the set theoretic universe, in a general context that includes the relation of the universe of sets to other more arbitrary models. The Inner Model Hypothesis of Sy Friedman [1], for example, exemplifies this emerging trend very well.

1. Proper Forcing in Models of Arithmetic

The field of models of arithmetic—models of PA—might usually be considered to stand somewhat apart from set theory, but it is becoming increasingly clear that a few of the central open questions exhibit a deep set-theoretic nature, and for these problems an inter-speciality approach now seems called for. This is particularly true for Scott's problem, the long-open question of whether every Scott set is the standard system of a model of PA. The most recent advances on this problem involve applications of the Proper Forcing Axiom and make connections with almost purely set-theoretic questions.

Date: September, 2007.

The *standard system* of a model $M \models \mathrm{PA}$, denoted $\mathrm{Ssy}(M)$, is the collection of subsets $A \subseteq \omega$ that are the standard parts of definable subsets of M with parameters. It is easy to see that this is a Boolean algebra, a subalgebra of the power set $P(\omega)$, closed under Turing reduction and with the property that if $T \subseteq {}^{<\omega}2$ is an infinite binary tree coded in the standard system, then there is an infinite branch through T coded there as well. Any such family is a *Scott set*, in honor of the fact that Scott [2] proved that every countable Scott set is the standard system of a model of PA. The concept extends to models of set theory in an appealing way. Specifically, if $M \models \mathrm{ZFC}$, then $\mathrm{Ssy}(M)$ is the set of all the standard parts of the reals of M; one can imagine riding a great lawnmower, cutting off the nonstandard parts of all the reals in M, leaving just their standard parts in a vast lawn of uniform height. This is the same, of course, as the standard system of the natural numbers of M, viewed as a model of PA. The proof of Scott's theorem shows, if ZFC is consistent, that every countable Scott set is the standard system of a model of ZFC.

The idea of Scott's proof is beautiful: he realized that the paths-through-trees property of Scott sets was enough to show that any consistent theory coded in a Scott set must have a consistent completion there as well; thus, he could carry out the Henkin proof of Gödel's completeness theorem *inside* a given Scott set, producing a model whose standard system would be contained within that Scott set as well. Iterating this with an elementary chain, he showed that every countable Scott set is the standard system of a model of PA (or ZFC). In the uncountable case, the question has remained open for over forty years:

Question 1. (Scott's Problem, 1962) *Is every Scott set the standard system of a model of* PA*?*

A breakthrough for Scott sets of size $\aleph_1$ came with the following.

Theorem 2. (Knight, Nadel [3]) *Every Scott set of size $\aleph_1$ is the standard system of a model of* PA*.*

Under the Continuum Hypothesis, therefore, the answer to Question 1 is yes. A key element of the proof is:

Theorem 3. (Ehrenfeucht's Lemma) *If X is a Scott set and M is a countable model of* PA *with $\mathrm{Ssy}(M) \subseteq X$, then for every $A \in X$ there is a countable elementary extension $M \prec N$ with $A \in \mathrm{Ssy}(N) \subseteq X$.*

Given a Scott set X of size $\aleph_1$, one may iterate Ehrenfeucht's lemma to build an elementary ω_1-chain, whose union witnesses Theorem 2.

Until the recent results of my student Victoria Gitman, which I mention below, there had been unfortunately little progress on Scott's problem beyond Theorem 2. In the mid-1990s, I began looking for solutions to Scott's problem via forcing. We had a variety of techniques for forcing the existence of the desired models, with many candidate partial orders, and a promising initial idea was to appeal to Martin's Axiom, but unfortunately none of the early arguments was ultimately suitable. Following up on these ideas, one of Enström's dissertation results [4], under Richard Kaye, also seemed promising: he proved under Martin's Axiom that every arithmetically closed Scott set X of size less than the continuum, such that X/Fin is c.c.c., is the standard system of a model of PA. But this hope was dashed when Gitman and I proved that X/Fin is c.c.c. if and only if X is countable, and so the Engström result does not actually move beyond Scott's original 1962 theorem. Recently, however, we have made a genuine advance. In her dissertation work under my supervision, Gitman injected additional set theory into the problem:

Theorem 4. (Gitman [5]) *Assuming the Proper Forcing Axiom, every arithmetically closed proper Scott set is the standard system of a model of* PA.

This theorem offers several advances, both in the statement itself and also in the methods of proof. The class of proper Scott sets, unlike the c.c.c. Scott sets, appears to be extensive (but not universal), and the new proof methods apply to other phenomena in models of arithmetic. For example, Gitman proved that under the Proper Forcing Axiom, if X is an arithmetically closed proper Scott set of size at most $\aleph_1$, then $\langle\omega, A\rangle_{A\in X}$ has a conservative elementary extension. Also, under the Proper Forcing Axiom, if M is a model of PA for which $\operatorname{Ssy}(M)$ is proper of size at most $\aleph_1$, then M has a minimal cofinal extension. Thus, in these instances, properness allows us to extend various other classical results beyond the countable case. The main question in the background here is the following almost purely set-theoretic question:

Question 5. *Which Scott sets are proper?*

Saharon Shelah has now become interested in the question. Since a Scott set X is simply a Boolean subalgebra of the power set $P(\omega)$, with a certain degree of closure, the question is very concrete. The question is, for which Scott sets is the quotient X/Fin proper as a notion of forcing? For example, if X is countable, then X/Fin is c.c.c. and hence proper; at the other extreme, if $X = P(\omega)$ is the entire power set, then X/Fin is countably closed and hence proper. What happens between these two extremes?

The question draws to mind similar questions about the properness of quotient algebras $P(\omega)/I$, where I is an ideal, which have been intensively studied in set theory. There seems, however, to be little direct connection between the properness of $P(\omega)/I$ and the properness of X/Fin. Providing a bound on Question 5, Ali Enayat recently proved in ZFC that there is an arithmetically closed Scott set X such that X/Fin collapses ω_1 and hence is not proper.

We are now pursuing several lines of inquiry, and several issues have been enormously clarified since the proposal I made last year. The main goal, of course, remains the original Scott's problem of Question 1. We are investigating the use of X/I, where X is a Scott set and I is an ideal on X, in place of X/Fin. We are exploring the extent to which PFA may outright imply the properness of certain Scott sets. We are investigating the use of PFA with other totally different partial orders. We are investigating the use of PFA and X/Fin in the context of other questions in models of arithmetic. Finally, we are exploring the possibility of a negative answer to Scott's problem; one idea, for example, is to build a Scott set from a Souslin tree, with the idea that any model having that standard system would provide a branch through the tree. This seems to be a rich and fruitful topic, and the subject is ripe for such cross-disciplinary work.

2. Forcing axioms and a new indestructibility phenomena

Large cardinal indestructibility concerns the interaction of two central topics of set-theoretic research, forcing and large cardinals. The basic question is: how are large cardinals affected by forcing? I view this as a fundamental concern in set theory, for surely we want to know, deeply, how our best tools interact. Laver's landmark result [6] showed that any supercompact cardinal κ can be made indestructible by κ-directed closed forcing, and the analysis continued with other large cardinals (e.g. [7]). We now have a deep understanding for many large cardinals of the kind of indestructibility that is possible, and some general tools, such as the lottery preparation [8], which I introduced as a uniform method for making large cardinals indestructible, even when there is no Laver function. The investigation of this indestructibility phenomenon has concentrated primarily on fairly large large cardinals—at least measurable and usually much larger—and much of our knowledge of the indestructibility of smaller large cardinals has been derivative of our knowledge about higher cardinals. The only known method of producing an indestructible weakly compact cardinal, for example, has been to begin with a supercompact cardinal and make its supercompactness indestructible. Of course, something like this is required in terms of consistency strength, because the existence of a weakly compact cardinal κ that is indestructible by κ-directed closed forcing implies the consistency of many Woodin cardinals. In addition, I proved in [9] that if the weak compactness of κ is made indestructible by κ-directed closed forcing by means of any progressively closed preparation, then κ was fully supercompact in the ground model. These results, however, rely fundamentally on the indestructibility of the weak compactness of κ by the forcing to collapse cardinals to κ, which may be somewhat unnatural in the weakly compact context. In particular, this previous analysis does not answer the following question:

Question 6. *What is the large cardinal strength of a weakly compact cardinal κ that is indestructible by all κ-directed closed κ^+-preserving forcing?*

The question is natural, because the weak compactness of κ is determined by objects of size κ, and so one naturally desires to preserve κ^+ in the weakly compact context. Previously, the best upper bound was a supercompact cardinal, until our result last year:

Theorem 7. (Hamkins, Johnstone [10]) *If κ is strongly unfoldable, then after suitable preparatory forcing, the strong unfoldability of κ becomes indestructible by all κ-closed κ^+-preserving forcing.*

This is a considerable reduction in the strength of the hypothesis from a supercompact cardinal, since the strongly unfoldable cardinals, a strengthening of the indescribable cardinals, are very low in the large cardinal hierarchy. For example, they are consistent with $V = L$ and relativize down to L. The result also provides a wider degree of indestructibility than requested, improving κ-directed closed to κ-closed forcing. Such an improvement is impossible with the larger large cardinals, because the forcing to add a slim κ-Kurepa tree is κ-closed, but necessarily destroys the measurability of κ and more; e.g. see [11].

Our proof introduces a new lifting method for large cardinal embeddings, not seen in previous indestructibility arguments. The method is to consider elementary substructures constructed in the relevant forcing extension $X \prec V[G][g]$ and then appeal to the approximation and cover property results of [9] to ensure that often enough these elementary substructures have $X \cap V \in V$, giving rise in V to suitable embeddings on the collapse, which can then be lifted to the extension. The construction uses ideas and methods from proper forcing, but with much larger elementary substructures. Indeed, the theory of κ-proper forcing was investigated for precisely this reason by my student Thomas Johnstone in his dissertation [12], and Theorem 7 grew out of that analysis.

Just as Baumgartner modified the Laver preparation to prove the relative consistency of the Proper Forcing Axiom from a supercompact cardinal, we are able to modify the lottery preparation to prove the relative consistency of fragments of the Proper Forcing Axiom from an unfoldable cardinal. The resulting construction, which we call the PFA lottery preparation, is a universal method for forcing fragments of PFA. The *Proper Forcing Axiom* PFA is the assertion that for any proper poset $\mathbb{Q}$ and every collection $\mathcal{D}$ of at most $\aleph_1$ many dense subsets of $\mathbb{Q}$, there is a filter $G \subseteq \mathbb{Q}$ meeting every dense set in $\mathcal{D}$. If Γ is a class of posets, then PFA(Γ) is the corresponding assertion restricted to proper $\mathbb{Q} \in \Gamma$. For example, the axiom PFA($\aleph_2$-preserving) asserts that for for every proper poset $\mathbb{Q}$ preserving $\aleph_2$ as a cardinal and every collection $\mathcal{D}$ of $\aleph_1$ many dense subsets of $\mathbb{Q}$, there is a filter $G \subseteq \mathbb{Q}$ meeting every element of $\mathcal{D}$. We define PFA($\aleph_3$-preserving) similarly.

Theorem 8. (Hamkins, Johnstone [13]) *If κ is strongly unfoldable and $0^\sharp$ does not exist, then the* PFA *lottery preparation of κ forces* PFA($\aleph_2$*-preserving*) *and* PFA($\aleph_3$*-preserving*)*.*

This was a very surprising advance, because previous applications to fragments of PFA and other forcing axioms had required the class of forcing to be closed under certain iterations, which is not true here. The $\neg 0^\sharp$ assumption, harmless in consistency strength because strong unfoldability relativizes to L, is used in a covering argument over L that replaces the covering property of [9] in the proof of Theorem 7; it can be weakened by using stronger core models. The actual phenomenon is that if κ is strongly unfoldable, then regardless of whether $0^\sharp$ exists or not, the PFA lottery preparation of κ forces PFA($\aleph_2$-covering) and PFA($\aleph_3$-covering), with $2^\omega = \kappa = \aleph_2$. The point is then that if there is sufficient covering for the background universe, then $\aleph_2$-covering forcing is the same as $\aleph_2$-preserving forcing.

Miyamoto [14] introduced the modified axiom $\mathrm{PFA}_{\mathfrak{c}}$, which asserts that for any proper poset $\mathbb{Q}$ and any family $\mathcal{A}$ consisting of at most $\aleph_1$ many maximal antichains, each of size at most continuum $\mathfrak{c}$, there is a filter in $\mathbb{Q}$ meeting every antichain in $\mathcal{A}$. Johnstone and I have observed that $\mathrm{PFA}_{\mathfrak{c}}$ holds in our model for Theorem 7. Consequently, building on Miyamoto's [14], we obtain:

Theorem 9. (Hamkins, Johnstone [13]) *The following theories are equiconsistent over* ZFC*:*

1. *There is an unfoldable cardinal* κ*.*
2. PFA($\aleph_2$-*preserving*) + PFA($\aleph_3$-*preserving*) + $\mathrm{PFA}_{\aleph_2} + 2^\omega = \aleph_2$
3. $\mathrm{PFA}_{\mathfrak{c}}$

This theorem fits neatly above the hierarchy of axioms recently considered by Neeman and Schimmerling, who have used a similar forcing argument in the case of certain indescribable cardinals to obtain the consistency of PFA($\mathfrak{c}^+$-c.c.) and others.

Question 10. *Do any of the principles* $\mathrm{PFA}_{\mathfrak{c}}$, PFA($\aleph_2$-*preserving*) *and* PFA($\aleph_3$-*preserving*) *imply any of the others? Are the latter principles equiconsistent with the former?*

All of the previous results about PFA extend easily to the case of semiproper forcing, by using a revised countable support in place of countable support, to form what we call the SPFA lottery preparation. This forcing shows that the existence of an unfoldable cardinal is equiconsistent with the theory SPFA($\aleph_2$-preserving) + SPFA($\aleph_3$-preserving) + $\mathrm{SPFA}_{\mathfrak{c}} + \mathfrak{c} = \aleph_2$.

In related joint work with Boban Velickovic, we have recently proved similar equiconsistency results concerning generalizations of the Bounded Proper Forcing Axiom. We are considering axioms of the form: (1) For every proper poset $\mathbb{Q}$ there is a proper poset $\mathbb{R}$ such that $H_{\omega_2} \prec H_{\omega_2}^{V^{\mathbb{Q}*\mathbb{R}}}$. This is a direct strengthening of the Bounded Proper Forcing Axiom. With parameters, we have the assertion: (1^+) For every proper poset $\mathbb{Q}$ and every $A \subseteq \omega_2$ there is a proper poset $\mathbb{R}$ and A^* such that $(H_{\omega_2}, A) \prec (H_{\omega_2}^{V^{\mathbb{Q}*\mathbb{R}}}, A^*)$. Dropping properness, we also consider the axiom: (2) For every poset $\mathbb{Q}$ there is a poset $\mathbb{R}$ such that $H_{\omega_1} \prec H_{\omega_1}^{V^{\mathbb{Q}*\mathbb{R}}}$, and its version ($2^+$) with parameters $A \subseteq \omega_1$. There are also embedding formulations: (1^{++}) For every proper $\mathbb{Q}$

and every transitive $M \models \mathrm{ZF}^-$ of size ω_2 there is a proper $\mathbb{R}$ and transitive N with an elementary embedding $j : M \to N$ in $V^{\mathbb{Q}*\mathbb{R}}$ with critical point ω_2 and $j(\omega_2) = \omega_2^{V^{\mathbb{Q}*\mathbb{R}}}$. The idea here is that these forcing axioms, because they express abstract elementary relations rather than combinatorial statements about dense sets, will help illuminate dense-set-free applications of forcing axioms. We have already proved large cardinal equiconsistencies throughout this family of forcing axioms. For example, Axioms 1 and 2 are equiconsistent with each other and with the existence of an inaccessible cardinal κ having arbitrarily large inaccessible γ with $V_\kappa \prec V_\gamma$. Axioms (1^+) and (2^+) are equiconsistent with the existence of a strongly unfoldable cardinal κ having inaccessible targets, witnessed by strong unfoldability embeddings $j : M \to N$ with $j(\kappa)$ inaccessible.

3. An emerging second order perspective in set theory

There appears to be emerging within set theory a new focus on set theoretic principles involving second and higher order features of the set theoretic universe, in a general context that includes the relation of the universe of sets to other more arbitrary models. The higher order nature of the phenomenon has sometimes introduced delicate meta-mathematical issues to be settled. To be sure, over the past four decades the subject of set theory has increasingly taken the *models* of set theory, rather than individual sets or set constructions, as the fundamental objects of study, and we can now build models of ZFC, using forcing and other methods, to exhibit precise, exacting features. As a result, set theory begins to exhibit a category-theoretic nature, for we have at bottom a collection of models, forming extensions of one another and connected by large cardinal embeddings and other maps. The new part is to investigate set theoretic principles of a model that explicitly consider its relation to other members of this vast collection of models.

An excellent example of this new perspective is the Inner Model Hypothesis of Sy Friedman [1]. The Inner Model Hypothesis asserts of a model of set theory V, that any statement φ true in any inner model of an outer model $W \supseteq V$, is already true in an inner model of V. Recent work of Friedman, Welch and Woodin has provided improved upper and lower bounds for the consistency strength. Although the IMH is not a first order property of V, it can be formalized as a statement about a fixed countable transitive model V, and this is the kind of meta-mathematical issue to which I alluded earlier.

The Inner Model Hypothesis has a clear affinity with what I have called the Maximality Principle, the scheme asserting of any statement that can be forced in a such a way that it holds in all further forcing extensions, that it is already true (see the earlier account of Stavi and Väänänen in [15] and my work in [16], [17]). The Maximality Principle is surely a forcing axiom, with forcing axiom consequences, but because it is expressed by asserting a certain relation of the universe to its forcing extensions, it perfectly illustrates the trend to which I am calling attention. The Maximality Principle is equiconsistent with ZFC, but if one allows real parameters, then it begins to have large cardinal strength, and the Necessary Maximality Principle, asserting that $\mathrm{MP}(\mathbb{R})$ holds in every forcing extension using the reals of that extension, implies $\mathrm{AD}^{L(\mathbb{R})}$. Woodin provided an upper bound of "$\mathrm{AD}_{\mathbb{R}} + \Theta$ regular."

The fundamental concepts arising in the Maximality Principles of "true in all forcing extensions" and "true in some forcing extension" have a clear modal aspect that naturally suggest an investigation of the modal properties of forcing. Benedikt Löwe and I carried out just such an investigation in [18]. The basic definitions from [16] are that a set-theoretic statement φ is *forceable* or *possible*, written $\Diamond\,\varphi$, if it holds in a forcing extension and *necessary*, written $\Box\,\varphi$, if it holds in all forcing extensions. Although these operators are eliminable, using the first-order definability of forcing in set theory, the question is what are the fundamental modal features of forcing. For example, a statement φ is forceably necessary, written $\Diamond\,\Box\,\varphi$, if it can be forced in such a way that it remains true in all further forcing extensions. Thus, the Maximality Principle can be expressed as the scheme asserting that every forceably necessary statement is true, written $\Diamond\,\Box\,\varphi \implies \varphi$, which is the principal axiom of the modal theory S5. The main question, another clear instance of the emerging trend that I have identified, was to find out exactly what are the valid modal assertions under this forcing interpretation. The answer is provided by:

Theorem 11. (Hamkins, Löwe [18]) *If* ZFC *is consistent, then the* ZFC-*provably valid principles of forcing are exactly those in the modal theory S4.2.*

For any fixed model $W \models$ ZFC, we proved that the valid principles of forcing over W constitute a modal theory between S4.2 and S5, and both of these endpoints are realized. It is not known whether it is possible to have a model exhibiting a strictly intermediate modal theory between S4.2 and S5. This possibility is connected with the question whether every valid principle of forcing over a fixed model W remains valid for forcing over all forcing extensions of W.

Question 12. *Is there a model of* ZFC *whose valid principles of forcing form a theory other than S4.2 or S5?*

It is natural also to restrict attention to a specific class of forcing.

Question 13. *What are the valid principles of c.c.c. forcing? Of proper forcing? Of class forcing? Of arbitrary extensions?*

Class forcing and arbitrary extensions involve the meta-mathematical complication that the corresponding possibility and necessitation operators are no longer first-order expressible. The work on Question 13 has been surprisingly difficult, even for the easiest cases, and has led to some interesting, subtle questions in forcing combinatorics. For example, in the modal logic of collapse forcing the question arose whether there could be a model W of ZFC such that further forcing over W to collapse cardinals to ω would not change the theory of the model. In other words, could the elementary equivalence $W \equiv W[G]$ hold for all such extensions? Such a model of set theory would have valid principles of collapse forcing that are beyond S5, a hard upper bound for the other natural classes of forcing. After some work, Mitchell provided such a model by performing Radin forcing over a model of $o(\kappa) = \kappa^+$, and he and Philip Welch both gave lower bounds with large values of $o(\kappa)$.

Another attractive example is Laver's [19] recent theorem (proved independently by Woodin [20]) that every model of set theory is a definable class in all its forcing extensions, using parameters from the ground model. This result illustrates the theme of considering the universe of set theory in a general context including all its forcing extensions. Laver's theorem, which builds on work of mine [9], led my student Reitz and I to introduce the Ground Axiom, the principle asserting that the universe was not obtained by set forcing over any inner model. Although this appears to be a second order assertion, Reitz [21], [22] proved in his dissertation (under my supervision) that it is first order expressible. The natural models of the Ground Axiom, such as L, $L[0^\sharp]$, $L[\mu]$ and many instances of K, all exhibit many structural regularities, but Reitz proved that none of these are consequences of the Ground Axiom, and furthermore every model of ZFC has a class forcing extension, preserving any desired initial segment V_α, which is a model of the Ground Axiom. We define that W is a *ground* for V if $V = W[G]$ is a forcing extension of W by set forcing $G \subseteq \mathbb{P} \in W$. The model W is a *bedrock* for V, if it is a ground for V and there is no deeper ground inside W. In other words, W is a bedrock for V if it is a ground for V and satisfies the Ground Axiom.

Question 14. *Can a model of set theory have two distinct bedrock models?*

In joint work by myself, Reitz and Fuchs, we are now deepening the analysis. Continuing the geological metaphor, we define the *mantle* of V to be the intersection of all ground models of V. The *limit mantle* is the intersection of all ground models of all forcing extensions of V. This is the same as the intersection of the multiverse of V, as defined by Woodin in [20]. For any model of ZFC, the limit mantle is always a model at least of ZF. Our work in this area begins with:

Theorem 15. (Fuchs, Hamkins, Reitz) *Every model of* ZFC *is the mantle of another model of* ZFC*.*

It follows that the mantle of V is not necessarily a ground model of V, even when it is a model of ZFC. One can therefore iteratively take the mantle of the mantle and so on, and we have proved that this process can strictly continue. We can achieve the result of Theorem 16 while also controlling both the HOD and the limit HOD, the intersection of the HODs of all forcing extensions. We do not currently know, however, whether the mantle and limit mantle are actually different.

Question 16. *Can the mantle and the limit mantle differ?*

One might hope to prove even that every model V of ZFC is the limit mantle of a model W of the Ground Axiom, so that the mantle of W is W. This would provide a very attractive answer to Question 16. We also have as yet no example where the mantle or limit mantle does not satisfy ZFC, or where the mantle does not satisfy ZF. This work is at an early stage, and we are excited to get to work.

In another project in this theme, I am currently attempting to extend Simpson's theorem from models of PA to models of Gödel-Bernays GBC set theory. Specifically, Simpson proved in [23] that for every countable model M of PA one can amend a class $A \subseteq M$, preserving PA in the expanded language, such that every object in M is definable without parameters in (M, A). Ali Enayat [24] extended this result to ZFC by proving that every countable model of ZFC has a class forcing extension in which every set is definable without parameters. In joint work with myself, David Linetsky and Jonas Reitz, we plan to extend this to Gödel-Bernays:

Proposition 17. *Every countable model M of* GBC *has an extension N in which every set and class is definable without parameters.*

The definitions are not uniform, of course, and the property of having every object definable is not itself first-order. The classes of the extension N include all the classes of M and more. If the classes of M are all definable from parameters, then this theorem reduces to the case of Enayat. We have proved the proposition when the classes of M are all definable from one fixed class. We plan to reduce the general case to the single-class case by performing meta-class forcing to collapse the powerclass of the ordinals to the ordinals, but the difficulties of this meta-class forcing remain to be worked out.

Finally, in this theme if one has in mind the category of the models of set theory and the large cardinal embeddings between them, it seems very natural to inquire under what circumstances will an extension $V \subseteq W$ have the property that the large cardinal embeddings of W necessarily lift embeddings in V? Investigating this in [25], [26] and [9], I can answer the question for a large class of extensions.

Theorem 18. (Hamkins [9]) *If $V \subseteq W$ exhibits the δ-approximation and cover properties, then every ultrapower embedding in W with critical point above δ lifts an embedding definable in V.*

If one performs small forcing followed by closed forcing (closed to the size of the small forcing), then the resulting extension will have the approximation and cover properties, and so the class of extensions appearing in the hypothesis of this theorem is quite extensive, including all of the common progressively closed Easton iterations, such as the Laver preparation, the canonical forcing of the GCH, the Silver iteration and many others.

Question 19. *To what extent does the lifting property hold? Are there other more general classes of extensions $V \subseteq W$ for which every large cardinal ultrapower embedding of W lifts an embedding of V?*

4. Other Logic Activities

Through a variety of other activities, I have attempted to promote mathematical logic and support the logic community. While these activities do not constitute research as such, I believe that they help the success of the logic community, and I intend to continue with them.

First, I have been an energetic organizer of conferences and seminars, and I am proud to be a part of the increasingly vibrant New York logic scene. I have organized or co-organized five conferences at CUNY since 1997, including the three biennial NYC Logic Conferences, with multi-day programs and parallel sessions. I have served as an advisor for the New York Graduate Student Logic Conferences, which attracted large international groups of upcoming logicians. With Roman Kossak and now Hans Schoutens, I have co-organized the CUNY Logic Workshop, our weekly research seminar, in which we have been fortunate to host a distinguished collection of speakers, since 1996; and I now also run the weekly New York Set Theory Seminar. I have served on the MAMLS advisory board, which sponsors three or four conferences yearly, since 1997. I oversee the MAMLS web site (mamls.org) and the New York Logic web site (nylogic.org), which hosts web sites for the CUNY Logic Workshop, the NY Logic Colloquium and the NYC Logic Conferences, among others. I have encouraged the founders of the New York Women in Mathematics Network (NYWMN), who have now launched their activities.

Second, I am active in graduate education. At the CUNY Graduate Center, I have recently had four excellent PhD students complete their dissertations, successfully graduating and now all holding tenure-track faculty positions. A further group of students is in progress, on track to graduate in coming years. We have managed to establish a lively atmosphere for logic at The CUNY Graduate Center, with advanced topic courses and seminars of all kinds, attracting graduate students and faculty from surrounding universities. Frankly, our students are thriving.

5. Broader Impact of the Proposed Research

The greatest impact of the proposed research will be the progress it makes towards solving the open questions that were posed, some of which have been open for decades. These questions lie within the main stream of contemporary research in logic. Recent advances of mine and others indicate the promise of future progress. In addition, the proposed support activities will benefit the larger mathematical community. In particular, I have been active in graduate education, with four highly-talented PhD students now graduated and holding tenure-track positions (but with whom I am still working closely), and additional talented students in progress. I have served on nine Ph.D. dissertation committees, supervising five of them, at The City University of New York, Columbia University and the University of Amsterdam. In addition, I have served on four Masters thesis committees, supervising one of them. The conferences I have organized previously have attracted a diverse spectrum of logicians, mathematicians, computer scientists, philosophers and other researchers. I aspire to serve through these activities as a kind of ambassador for mathematics and logic, promoting interest in logic and attracting students to mathematics.

6. Prior Research and NSF Support

My prior research was supported in part by National Science Foundation grant #9970993, $74,400, 1999-2002, entitled "Indestructibility Phenomena of Large Cardinals." Since the time of my Ph.D. (UC Berkeley, 1994), my research program has pursued the following major lines of inquiry, which I shall subsequently explain:

- Indestructibility of large cardinals
  [27], [28], [29], [30], [31], [32], [8], [33], [34], [35], [10]
- New large cardinal axioms [36], [33], [37]
- Large cardinal combinatorics, infinitary combinatorics
  [38], [39], [40], [41], [42], [43], [44]
- Amenability of generic embeddings to the ground model
  [25], [45], [26], [34], [9]
- Forcing axioms and generic absoluteness [16], [17], [46], [47], [13]
- The modal logic of forcing [16], [18]
- The automorphism tower problem, group theory/set theory
  [48], [49], [50], [43], [44]
- Infinitary computability
  [51], [52], [53], [54], [55], [56], [57], [58],[59], [60],[61], [62]
- Computability theory [63]
- (mathematical/philosophical work) [64], [65], [66]
- (book reviews) [67], [68], [69]

The main line of my research has been to investigate the interaction of forcing and large cardinals in set theory. I have been motivated by the belief that it is critically important to understand in the most fundamental way how the principal tools of one's subject interact. A major theme in this line, therefore, has been the indestructibility phenomenon of large cardinals, a theme represented in this research proposal and explained in detail above. Large cardinal combinatorics, of course, permeates all this work, and some of my most difficult, technically detailed work has been explicitly concerned with infinitary combinatorics. Another major component of this theme has been my investigation of the relations of the large cardinal embeddings of a forcing extension to those in the ground model. My work in [25], [26] and [9] shows that for a surprisingly broad class of forcing extensions, every suitably closed embedding $j : V[G] \to M[j(G)]$ in the extension lifts an embedding $j \upharpoonright V : V \to M$ definable in the ground model. Such extensions, consequently, have no new large cardinals. That work has led to a general consideration of how the universe of sets relates to its forcing extensions, an idea I discuss in the more abstract third project of the proposal above. My previous work on this idea would include work on the Maximality Principals, the modal logic of forcing and the Ground Axiom, among others. Some of this work can also be characterized as a part of the theory of forcing axioms and generic absoluteness.

My work on the automorphism tower problem is interdisciplinary between set theory and group theory. The automorphism tower of a group is the result of iteratively computing the automorphism group, mapping each group into the next by inner automorphisms: $G \to \mathrm{Aut}(G) \to \mathrm{Aut}(\mathrm{Aut}(G)) \to \cdots$ and so on, taking direct limits at limit stages. The question is whether there is a fixed point, a group that is isomorphic to its automorphism group by the natural map. The classical result [70] is that any finite centerless group has a terminating automorphism tower. This was gradually strengthened to centerless Černikov groups [71] and centerless polycyclic groups [72], until Simon Thomas [73] [74] proved that every centerless group has a terminating automorphism tower. In [48], I generalized this result to all groups, proving that every group has a terminating transfinite automorphism tower. Subsequently, Thomas and I revealed in [49] the set-theoretic nature of the automorphism tower of a group, showing that a single group can have wildly different automorphism towers in different models of set theory. The latest work [43] [44], using strong notions of rigidity for Souslin trees, shows that such groups exist under $\diamondsuit$.

I have made an investigation of infinitary computation, introducing the model of infinite time Turing machines with Lewis in [51] in order to do so. The basic idea is to extend the operation of ordinary Turing machines into transfinite ordinal time, giving rise to an infinitary computability theory on the reals. This theory lies on the ample boundaries between computability theory and descriptive set theory, with increasing links to the much earlier work on higher recursion theory. The growing community of researchers in this area includes several prominent set theorists, as well as an enormous group of enthusiastic young researchers and graduate students. There have now been a surprising number of Masters theses explicitly concerned with infinite time Turing machines. A sophisticated body of research is developing, and in January 2007 we held the first Bonn International Workshop on Ordinal Computation, focussed on this and related models of infinitary computability. Some of the most recent work is building additional connections between infinitary computability and descriptive set theory, including the theory of Borel equivalence relations.

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