# The hierarchy of equivalence relations on $\mathbb{N}$ under computable reducibility

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This talk includes joint work with Sam Coskey and Russell Miller. Our paper is

*S. Coskey, J. D. Hamkins, R. Miller, "The hierarchy of equivalence relations on the natural numbers under computable reducibility," forthcoming.*

A preprint is available on my web page: http://jdh.hamkins.org

# Other researchers, related work

Some of the ideas have arisen independently several times.

- Claudio Bernadi and Andrea Sorbi, “Classifying positive equivalence relations,” JSL 48(3):529–538, 1983.
- Su Gao and Peter Gerdes, “Computably enumerable equivalence relations,” Studia Logica, 67(1):27–59, 2001.
- Julia F. Knight, Sara Miller, M. Vanden Boom, “Turing computable embeddings,” JSL 72(3):901–918, 2007.
- Sam Buss, Yijià Chen, Jörg Flum, Sy-David Friedman and Moritz Müller, “Strong isomorphism reductions in complexity theory,” JSL, 2011.
- Akaterina B. Fokina, Sy-David Friedman, “On Sigma-1-1 equivalence relations over the natural numbers,” MLQ, to appear.

# Borel equivalence relations

Many natural equivalence relations in mathematics can be viewed as equivalence relations on a standard Borel space.

- isomorphism of countable structures
- isomorphism of countable groups, graphs, linear orders
- isometries of separable Banach spaces, etc.

The concept of *Borel reducibility*, due to Friedman and Stanley, allows us to compare the difficulty of classification problems. We can say that a classification problem is wild or tame.

The subject has been an enormous success, a very fruitful interaction of logic with the rest of mathematics.

# Borel reducibility

One equivalence relation is *Borel reducible* to another, $E \leq_B F$, if there is a Borel function $f$ such that

$$a \, E \, b \qquad \Longleftrightarrow \qquad f(a) \, F \, f(b).$$

Thus, $f$ maps $E$-classes to $F$-classes and thereby provides a classification of $E$ using $F$-classes.

In this case, the $E$-classification problem is no harder than the $F$-classification problem.

Goal of the Borel theory: to study the hierarchy of such classification problems under Borel reducibility (and mathematicians care).

# A computable analogue

We aim to adapt the Borel theory to the computability context, with equivalence relations on $\mathbb{N}$.

Namely, $E$ is *computably reducible* to $F$, written $E \leq_c F$, when there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

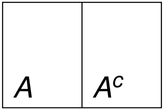
$$n \mathrel{E} m \iff f(n) \mathrel{F} f(m).$$

Versions of this notion have arisen several times independently: Bernadi, Sorbi; Gao, Gerdes; Fokina, Friedman; Coskey, Hamkins, Miller.

Our new focus: natural relations on c.e. structures. The resulting theory is an attractive blend of methods and ideas from computability theory, descriptive set theory and other parts of mathematics.

# A copy of the many-one degrees

For $A \subseteq \mathbb{N}$, consider the relation $E_{A,A^c}$ with two classes:




If $A$ is c.e., this has complexity $\Delta^0_2$.

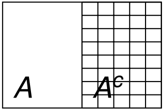
**Theorem**

*The relation $E_{A,A^c}$ reduces to $E_{B,B^c}$ if and only if $A$ is many-one reducible to $B$ or to $B^c$.* □

Thus, the hierarchy contains essentially a copy of the many-one degrees, even among the relations with only two classes.

# A copy of the 1-degrees

For $A \subseteq \mathbb{N}$, consider the relation $E_A$ with classes:




Thus, $n \; E_A \; m$ iff $n, m \in A$ or $n = m$. If $A$ is c.e., then so is $E_A$.

**Theorem**

*If $A, B \subseteq \mathbb{N}$ are c.e. non-computable, then $E_A \leq_c E_B$ iff $A \leq_1 B$.*

**Proof.**

A 1-reduction of $A$ to $B$ also reduces $E_A$ to $E_B$. Conversely, a reduction of $E_A$ to $E_B$ gives rise to a 1-reduction $A \leq_1 B$. □

# Orbit equivalence relations

The *orbit equivalence relation* of the action of a group $\Gamma$ on $\mathbb{N}$ is

$$x \, E_\Gamma \, y \qquad \text{iff} \qquad \exists \gamma \in \Gamma \;\; y = \gamma x.$$

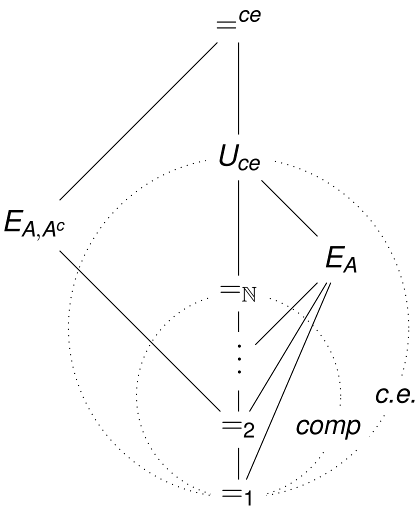
## Theorem

*The c.e. equivalence relations are precisely the orbit equivalence relations induced by computable actions of computable groups.*

## Proof.

Every such orbit relation is c.e. Conversely, if $E$ is c.e., let $\Gamma$ be a computable copy of the free group $F_\omega$ with generators $x_i$. If $(n, m) \in E$ at stage $s$, let $x_{\langle s,n,m \rangle}$ act by swapping $n$ and $m$ only. This is a computable action with orbit $E$. □

# Summary of the hierarchy so far



# Equivalence relations on c.e. sets

A deeper theory arises in the context of equivalence relations on c.e. sets, such as isomorphism relations on c.e. graphs and other c.e. structures. We begin by adapting the key relations of the Borel theory by considering them as relations on c.e. sets.

An equivalence relation $E$ on the c.e. sets gives rise to

$$e \; E^{ce} \; e' \qquad \iff \qquad W_e \; E \; W_{e'},$$

For example, the equality relation on c.e. sets,

$$e =^{ce} e' \qquad \iff \qquad W_e = W_{e'}.$$

This is a $\Pi^0_2$-complete set of pairs. Indeed, it has a $\Pi^0_2$-complete class $[e]$, where $W_e = \mathbb{N}$.

# Equality on c.e. sets, $=^{ce}$

**Theorem**

*Every c.e. equivalence relation lies properly below $=^{ce}$.*

**Proof.**

Let $E$ be an arbitrary c.e. relation. Define a reduction from $E$ to $=^{ce}$ by mapping each $n$ to a program $f(n)$ which enumerates $[n]_E$.

Conversely, there can be no reduction from $=^{ce}$ to $E$ since $=^{ce}$ is $\Pi^0_2$-complete and $E$ is c.e. □

# Almost equality, $E_0^{ce}$

The almost-equality relation $E_0$ similarly plays an essential role in the Borel theory.

$$A \; E_0 \; B \quad \iff \quad \text{symmetric difference } A \triangle B \text{ is finite}$$

**Theorem**

*Equality $=^{ce}$ lies strictly below almost-equality $E_0^{ce}$.*

**Proof.**

To reduce $=^{ce}$ to $E_0^{ce}$, amplify differences: given $e$, let $W_{f(e)}$ have $(n,0),(n,1),(n,2),\ldots$ whenever $n \in W_e$. Thus, $W_e \neq W_{e'} \implies W_{f(e)} \triangle W_{f(e')}$ is infinite. Conversely, there is no computable reduction from $E_0^{ce}$ to $=^{ce}$, since $E_0^{ce}$ is $\Sigma_3^0$-complete, having a class equivalent to COF, whilst $=^{ce}$ is just $\Pi_2^0$. □

# Analogue of Silver, Glimm-Effros dichotomy?

To what extent will the computable reducibility hierarchy mirror the structure of Borel reducibility?

For example, one might hope for an analogues of Silver's theorem, that $=^{ce}$ is minimal in some large class. Unfortunately, this fails, as we shall see.

Are there any relations lying properly between $=^{ce}$ and $E_0^{ce}$? More generally, is there a form of the Glimm-Effros dichotomy in this context? In other words, is there a large collection of equivalence relations $E$ such that if $=^{ce}$ lies strictly below $E$, then $E_0^{ce}$ lies below $E$?

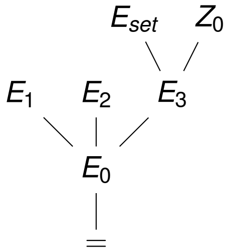
# More c.e. analogues of Borel relations

We may code subsets of $\mathbb{N} \times \mathbb{N}$ with subsets of $\mathbb{N}$.

- $(A_n)\ E_1\ (B_n)$ iff for almost all $n$, $A_n = B_n$.
- $A\ E_2\ B$ iff $\sum_{n \in A \triangle B} 1/n < \infty$.
- $(A_n)\ E_3\ (B_n)$ iff for all $n$, $A_n\ E_0\ B_n$.
- $(A_n)\ E_{\text{set}}\ (B_n)$ iff $\{\, A_n \mid n \in \mathbb{N}\,\} = \{\, B_n \mid n \in \mathbb{N}\,\}$.
- Density: $A\ Z_0\ B$ iff $\lim |(A \triangle B) \cap n|/n = 0$.

We consider the analogues on c.e. sets.

The complete diagram of Borel reducibilities on left.

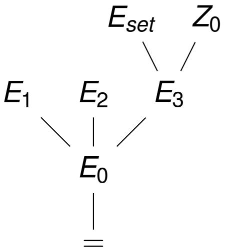


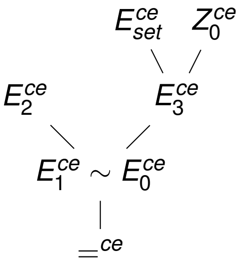




The c.e. analogues on right. (Not sure if this is complete.)

The complete diagram of Borel reducibilities on left.




The c.e. analogues on right. (Not sure if this is complete.)

# Complexity below equality $=^{ce}$

In contrast to the Borel theory, in the computable theory there is a rich collection of equivalence relations properly below equality $=^{ce}$.

This is a departure from the Borel theory, where Silver's theorem implies that $=$ is continuously reducible to every Borel equivalence relation (with uncountably many classes).

# Same minimum, same maximum relations

Same minimum:

$$e \; E^{ce}_{\text{min}} \; e' \quad \iff \quad \min(W_e) = \min(W_{e'})$$

Same maximum:

$$e \; E^{ce}_{\text{max}} \; e' \quad \iff \quad \max(W_e) = \max(W_{e'})$$

Both are reducible to $=^{ce}$, by saturating $W_e$ upwards for $E_{\text{min}}$ and downwards for $E_{\text{max}}$.

These saturation reductions are computable *selectors*, choosing a representative of each class.

# Max not reducible to Min

**Theorem**

*$E^{ce}_{\max}$ is not computably reducible to $E^{ce}_{\min}$. Consequently, $E^{ce}_{\min}$ lies properly below $=^{ce}$.*

**Proof.**

This holds because $E^{ce}_{\min}$ is $\Delta^0_2$, while $E^{ce}_{\max}$ is $\Pi^0_2$ complete. Even the $E_{\max}$ class $\mathrm{INF} = \{\, e \mid |W_e| = \aleph_0 \,\}$ is $\Pi^0_2$ complete. □

In fact, we show next that $E^{ce}_{\max}$ and $E^{ce}_{\min}$ are *incomparable*, and both lie properly below $=^{ce}$.

# Monotonicity lemma

- $f$ well-defined: $W_e = W_{e'} \implies W_{f(e)} = W_{f(e')}$.
- $f$ monotone: $W_e \subseteq W_{e'} \implies W_{f(e)} \subseteq W_{f(e')}$.

## Monotonicity Lemma

Every computable well-defined $f : \mathbb{N} \to \mathbb{N}$ is monotone.

## Proof.

Suppose $W_e \subseteq W_{e'}$ and $x \in W_{f(e)}$. Using the recursion theorem, let $W_p$ look like $W_e$ until $x$ appears in $W_{f(p)}$, and then $W_p$ looks like $W_{e'}$. Note that $x$ must appear in $W_{f(p)}$, since otherwise $W_p = W_e$ and so $W_{f(p)} = W_{f(e)}$. But now $W_p = W_{e'}$ and so $x \in W_{f(p)} = W_{f(e')}$, as desired. □

# Min not reducible to Max

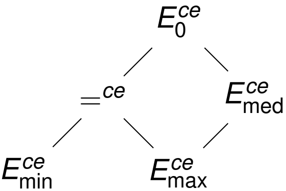
## Theorem

*$E^{ce}_{\min}$ does not reduce to $E^{ce}_{\max}$. Consequently, $E^{ce}_{\max} <_c =^{ce}$.*

## Proof.

Suppose $f : E^{ce}_{\min} \leq_c E^{ce}_{\max}$. By saturating downwards, assume without loss $f$ is well-defined. For each $n$, let $W_{e_n} = [n, \infty)$, a monotone descreasing sequence. By monotonicity lemma, $W_{f(e_n)}$ is therefore also monotone decreasing. Moreover, since $\min(W_{e_n})$ are distinct, it follows that $\max(W_{f(e_n)})$ are distinct, impossible. □

# Reducibility diagram




This diagram is complete for these relations.

# Generalization to other orders

For computable linear ordering $L$ and $W \subseteq L$, consider Dedekind cut

$$\mathrm{cut}_L(W) = \{\, l \in L \mid \exists w \in W (l <_L w) \,\}.$$

Same cut relation

$$e\ E_L\ e' \iff \mathrm{cut}_L(W_e) = \mathrm{cut}_L(W_{e'})$$

Same hull relation

$$e\ H_L\ e' \iff W_e, W_{e'} \text{ have same convex hull in } L$$

- Both $E_L$ and $H_L$ reduce to $=^{ce}$ by saturating $W_e$.
- Both $E_L$ and $E_{L^*}$ computably reduce to $H_L$.
- Note $E_{\max} = E_\omega$ and $E_{\min} = E_{\omega^*}$.

# Complete embeddings

Consider the computable Dedekind completion: $\overline{L}$ is the set of c.e. cuts of $L$.

Define *computably embeddable* $\overline{L_1} \hookrightarrow_c \overline{L_2}$, if there is computable $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{cut}_{L_1}(W_e) < \mathrm{cut}_{L_1}(W_{e'}) \iff \mathrm{cut}_{L_2}(W_{\alpha(e)}) < \mathrm{cut}_{L_2}(W_{\alpha(e')}).$$

## Theorem

*If $L_1, L_2$ computable linear orders, then $E_{L_1} \leq E_{L_2}$ iff $\overline{L_1} \hookrightarrow_c \overline{L_2}$.*

The result generalizes to partial orders.

# Cuts and Hulls in computable ordinals

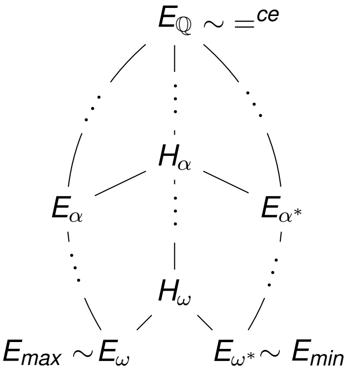



Figure: Diagram showing the cut and hull relations for computable ordinals $\alpha$ and their reverse orderings $\alpha^*$. In 2017 this will be the first diagram to land on Gliese 581g.

# Enumerable relations

A major focus for Borel theory: countable Borel equivalence relations.

The Lusin/Novikov theorem: every countable Borel equivalence relation has a uniform Borel enumeration of each class.

The Feldman/Moore argument uses this to show every countable Borel equivalence relation is the orbit relation of a Borel action of a countable group.

We develop the computable analogue of this theory.

# Computable analogue of Lusin/Novikov

An equivalence relation $E$ on c.e. sets is *enumerable in the indices* if there is computable $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with

$$W_e \; E \; W_{e'} \quad \iff \quad \exists n \; W_{\alpha(n,e)} = W_{e'}.$$

This is the computable analogue of the Lusin/Novikov property for countable Borel equivalence relations.

For example, $E_0^{ce}$ has this property, since let $W_{\alpha(n,e)}$ be $W_e$ modified to have $n^{\text{th}}$ finite set at bottom. As $n$ varies, $W_{\alpha(e,n)}$ enumerates all finite modifications of $W_e$, and so $\alpha$ witnesses that $E_0^{ce}$ is enumerable in the indices.

# Enumerable relations reduce to $E_{\text{set}}$

**Theorem**

*If $E^{ce}$ is enumerable in the indices, then $E^{ce} \leq E^{ce}_{\text{set}}$.*

**Proof.**

Simply map program $e$ to a program for a subset of $\mathbb{N} \times \mathbb{N}$ which puts $W_{\alpha(n,e)}$ on the $n^{\text{th}}$ column. □

Of course $E^{ce}_{\text{set}}$ is not itself enumerable, since enumerable relations are easily seen to be $\Sigma^0_3$, whereas $E^{ce}_{\text{set}}$ is $\Pi^0_3$ complete.

# Computable group actions

The action of a computable group Γ acting on the c.e. sets is *computable in the indices* if there is computable $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $W_{\alpha(\gamma,e)} = \gamma W_e$.

For example, left translation action of Γ on c.e. subsets of Γ.

Induced orbit equivalence relation

$$e \; E_\Gamma^{ce} \; e' \quad \iff \quad \exists \gamma \in \Gamma \; W_{e'} = \gamma W_e.$$

One would prefer an analogue of Feldman/Moore, saying that every enumerable relation is the orbit relation of an action computable in the indices.

Unfortunately, this is not the case.

# Counterexamples to orbit equivalence relations

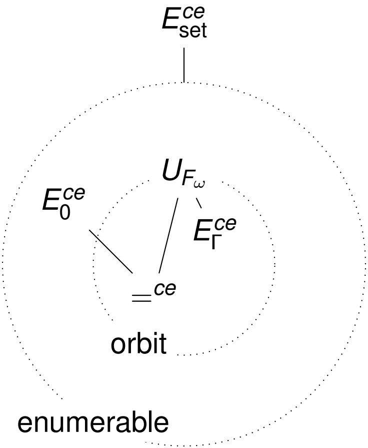
**Theorem**

*If $E$ is an equivalence relation on c.e. sets and $|[W_e]_E| \geq 2$ for some $e$ with $W_e \subseteq W_{e'}$ for all $e'$ $E^{ce}$ $e$, then $E$ is not an orbit equivalence of any action computable in the indices.*

The proof uses monotonicity: the action can't map the larger sets back to the minimal element $W_e$.

**Corollary**

*$E_0^{ce}$ is not induced by any action which is computable in the indices.*




The enumerable relations, with orbit equivalence relations.

# Isomorphism of c.e. structures

## Definition

Let $\cong^{ce}_{\text{bin}}$ denote the isomorphism relation on the codes for c.e. binary relations. That is, let $e \cong^{ce}_{\text{bin}} e'$ if and only if $W_e$ and $W_{e'}$, thought of as binary relations on $\mathbb{N}$, are isomorphic.

We remark that in order to analyze the isomorphism on arbitrary $\mathcal{L}$-structures, it is enough to consider just the binary relations, since if $\mathcal{L}$ is a computable language then the isomorphism relation $\cong^{ce}_{\mathcal{L}}$ on the c.e. $\mathcal{L}$-structures is computably reducible to $\cong^{ce}_{\text{bin}}$.

# What is isomorphism on c.e. graphs?

Two different ways to treat isomorphism of c.e. structures.

- Could restrict $\cong^{ce}_{\text{bin}}$ to indices for $W_e$ that are the right kind of structure. (Problem: these relations are not total)
- Or, identify structures as c.e. substructures of fixed universal structure.

For example, fix a computable copy $\Gamma$ of the countable random graph, and we define $\cong^{ce}_{\text{graph}}$ to be the isomorphism relation on the c.e. subsets of $\Gamma$.

Similarly, we consider c.e. linear orders as subsets of a copy of $\mathbb{Q}$, and c.e. trees as the downward closure of c.e. subsets of $\mathbb{N}^{<\mathbb{N}}$.

In some cases, the two approaches are equivalent.

# Isomorphisms of c.e. graphs, linear orders, trees

## Theorem

*Isomorphism of binary relations $\cong^{ce}_{\mathrm{bin}}$ is computably bireducible with each of isomorphisms of graphs, linear orders and trees.*

$$\cong^{ce}_{\mathrm{graph}} \qquad \cong^{ce}_{\mathrm{lo}} \qquad \cong^{ce}_{\mathrm{tree}}$$

The classical reductions go through easily.

Part of the reason it works is that these classical reductions use only the *positive* information about the structures; they need to know when two elements are related, but not instances of non-relation.

# Same set is strictly below $\cong_{\text{bin}}$

The isomorphism relation of binary relations $\cong^{ce}_{\text{bin}}$ is very high in our hierarchy. For example,

**Theorem**

*$E^{ce}_{\text{set}}$ lies properly below $\cong^{ce}_{\text{bin}}$.*

**Proof.**

To reduce $E^{ce}_{\text{set}}$ to $\cong^{ce}_{\text{bin}}$, let $W_{f(e)}$ be a code for $W_e$ as a hereditarily countable set. In other words, $W_{f(e)}$ is a well-founded tree coding the transitive closure of $\{W_e\}$.

The absence of any reverse reduction follows from complexity, since $E^{ce}_{\text{set}}$ is $\Pi^0_4$. □

# Isomorphism of c.e. groups

Isomorphism of c.e. groups admits several coding methods.

We could consider indices $e$ such that $W_e$, as a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, is the graph of a group operation, with corresponding isomorphism relation $\cong^{ce}_{\text{group}}$ on such indices.

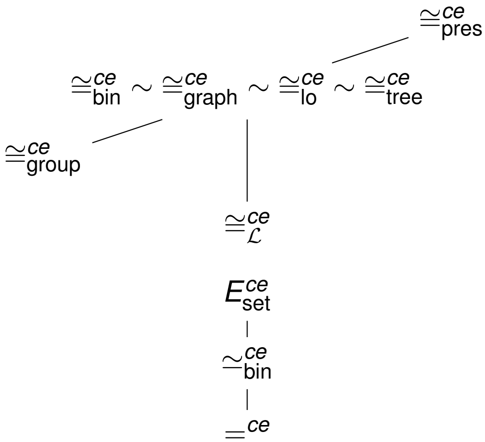
Alternatively we can code a group by a presentation, a set of words in $F_\omega$, thinking of the group as the corresponding quotient. Thus, $e \cong^{ce}_{\text{pres}} e'$ iff $W_e$ and $W_{e'}$, as sets of relations in $F_\omega$, have isomorphic quotients.

Thus, we have two classification problems: for computable group operations, and for computably presented groups.

**Theorem**

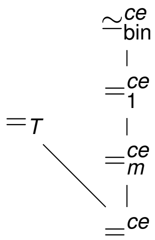
$\cong^{ce}_{\text{group}} \;\leq\; \cong^{ce}_{\text{bin}} \;\leq\; \cong^{ce}_{\text{pres}}.$

# Isomorphisms and computable isomorphisms



# Relations from computability theory

Consider the principal equivalences of computability, as relations on (indices for) the c.e. sets.




- Turing equivalence, $=_T$
- Many-one equivalence, $=_m$
- one-equivalence, $=_1$

# Many open problems remain

This research area is wide open, and there are easily dozens of open questions that would be interesting to work on. Pick your favorite equivalence relations on c.e. sets or structures, and find out how they fit into the hierarchy.

There are many dissertation problems hiding here.

Furthermore, it is a mathematically fulfilling research area

- Interesting questions, easy to get started
- New, exciting, deep, ripe for progress
- Connected deeply with logic and computability
- Connected deeply with the rest of mathematics

Thank you.

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