# The hierarchy of equivalence relations on $\mathbb{N}$ under computable reducibility

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This talk includes joint work with Sam Coskey and Russell Miller. Our paper is

> *S. Coskey, J. D. Hamkins, R. Miller, "The hierarchy of equivalence relations on the natural numbers under computable reducibility," forthcoming.*

A preprint is available on my web page: http://jdh.hamkins.org

# Other researchers, related work

Some of the ideas have arisen independently several times.

- Claudio Bernadi and Andrea Sorbi, “Classifying positive equivalence relations,” JSL 48(3):529–538, 1983.
- Su Gao and Peter Gerdes, “Computably enumerable equivalence relations,” Studia Logica, 67(1):27–59, 2001.
- Julia F. Knight, Sara Miller, M. Vanden Boom, “Turing computable embeddings,” JSL 72(3):901–918, 2007.
- Sam Buss, Yijià Chen, Jörg Flum, Sy-David Friedman and Moritz Müller, “Strong isomorphism reductions in complexity theory,” JSL, 2011.
- Akaterina B. Fokina, Sy-David Friedman, “On Sigma-1-1 equivalence relations over the natural numbers,” MLQ, to appear.

# Borel equivalence relations

Many naturally arising equivalence relations, such as isomorphisms of various countable structures (groups, graphs, fields, linear orders, isometries of separable Banach spaces), can be viewed as equivalence relations, often Borel relations, on a standard Borel space.

The classification problems for those structures is the problem of assigning invariants to the equivalence classes.

The concept of *Borel reducibility*, due to H. Friedman and L. Stanley, allows us to compare the relative difficulty of classification problems, to allow us to say that a classification problem is wild or tame.

The subject has been enormously successful, a very fruitful interaction of logic with the rest of the mathematics.

# Borel reducibility

One equivalence relation is *Borel reducible* to another, $E \leq_B F$, if there is a Borel function $f$ such that

$$a \; E \; b \qquad \Longleftrightarrow \qquad f(a) \; F \; f(b).$$

Thus, $f$ maps $E$-classes to $F$-classes and thereby provides a classification of $E$ using $F$-classes.

In this case, the $E$-classification problem is no harder than the $F$-classification problem.

Goal of the Borel theory: to study the hierarchy of such classification problems under Borel reducibility (and mathematicians care).

# Computable reducibility

We aim to adapt the Borel theory to the computability context, with equivalence relations on $\mathbb{N}$.

Namely, $E$ is *computably reducible* to $F$, written $E \leq_c F$, when there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$n \; E \; m \iff f(n) \; F \; f(m).$$

Versions of this notion have arisen several times independently: Bernadi, Sorbi; Gao, Gerdes; Fokina, Friedman; Coskey, Hamkins, Miller.

Our new focus: natural relations on c.e. structures. The resulting theory is an attractive blend of methods and ideas from computability theory, descriptive set theory and other parts of mathematics.

# Reducibility is not relative computability

Although $E \leq_c F$ implies $E \leq_T F$ via

$$n \; E \; m \iff f(n) \; F \; f(m),$$

nevertheless the Turing degree of $E$ as a set of pairs does not determine it's place in the hierarchy of reducibility.

The Borel theory has many examples of low-complexity equivalence relations with wild classification problems; and the same for computable reducibility.

- Much of computable model theory is about the comparative complexity of individual structures.
- Computable reducibility is about the comparative difficulty of classification problems.

# Computable relations are at the bottom

**Theorem**

*The computable equivalence relations on $\mathbb{N}$ are exactly those computably bireducible with $=_{\mathbb{N}}$ or with $=_n$.*

$=_n$ is equality on $\{0, 1, \ldots, n-1\}$, with rest of $\mathbb{N}$ in $[n-1]$.

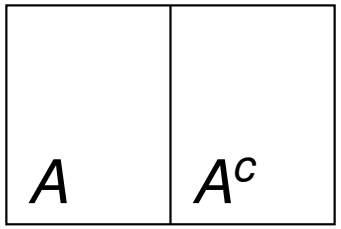
**Proof.**

If $E$ has $\geq n$ classes, then $=_n$ reduces to $E$. And if $E$ is computable (or even just c.e. or co-c.e.) with $n$ classes, then conversely. If $E$ is $\Pi^0_1$ with infinitely many classes, then $=_{\mathbb{N}}$ reduces to $E$. And computable $E$ reduces to $=_{\mathbb{N}}$. $\square$

Thus, the computable equivalence relations are classified by their number of equivalence classes.

# A copy of the many-one degrees

For $A \subseteq \mathbb{N}$, consider the relation $E_{A,A^c}$ with two classes:




If $A$ is c.e., this has complexity $\Delta^0_2$.

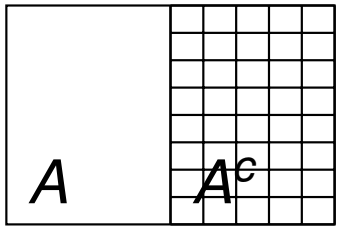
**Theorem**

*The relation $E_{A,A^c}$ reduces to $E_{B,B^c}$ if and only if $A$ is many-one reducible to $B$ or to $B^c$.* □

Thus, the hierarchy contains essentially a copy of the many-one degrees, even among the relations with only two classes.

# A copy of the 1-degrees

For $A \subseteq \mathbb{N}$, consider the relation $E_A$ with classes:




Thus, $n \; E_A \; m$ iff $n, m \in A$ or $n = m$. If $A$ is c.e., then so is $E_A$.

**Theorem**

*If $A, B \subseteq \mathbb{N}$ are c.e. non-computable, then $E_A \leq_c E_B$ iff $A \leq_1 B$.*

**Proof.**

A 1-reduction of $A$ to $B$ also reduces $E_A$ to $E_B$. Conversely, a reduction of $E_A$ to $E_B$ gives rise to a 1-reduction $A \leq_1 B$. □

# c.e. relations incomparable with $=_{\mathbb{N}}$

## Corollary

*There exist c.e. relations incomparable with $=_{\mathbb{N}}$.*

## Proof.

Consider the relation $E_A$ where $A$ is a simple set (c.e. co-infinite set whose complement contains no infinite c.e. sets). Since $A$ is not computable, neither is $E_A$, and it follows that $E_A$ is not reducible to $=_{\mathbb{N}}$. On the other hand, if $f$ is a computable reduction from $=_{\mathbb{N}}$ to $E_A$ then there exists $n \in \mathbb{N}$ such that $f(\mathbb{N} \setminus \{n\}) \subseteq A^c$. But this is a contradiction, since $f(\mathbb{N} \setminus \{n\})$ is c.e. and infinite. $\square$

# Universal c.e. relation

There is a universal c.e. equivalence relation.

**Theorem**

*There is a c.e. relation $U_{ce}$ to which every c.e. equivalence relation is reducible.*

**Proof.**

Let $E_e$ be the $e^{\text{th}}$ c.e. equivalence relation (take symmetric, transitive reflexive closure of $W_e$). Define $(e, a)\ U_{ce}\ (e', a')$ iff $e = e'$ and $a\ E_e\ a'$. This is c.e., and $E_e$ reduces via $a \mapsto (e, a)$. $\square$

# Orbit equivalence relations

The *orbit equivalence relation* of the action of a group $\Gamma$ on $\mathbb{N}$ is

$$x \; E_\Gamma \; y \qquad \text{iff} \qquad \exists \gamma \in \Gamma \;\; y = \gamma x.$$

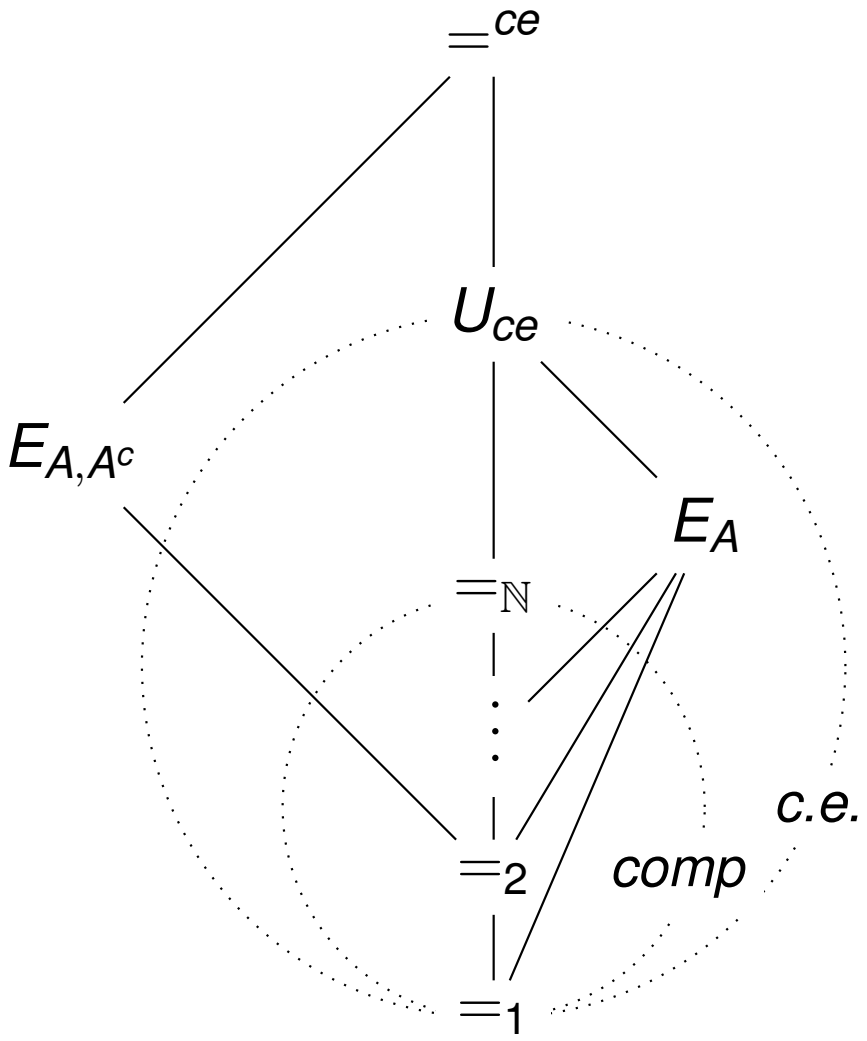
## Theorem

*The c.e. equivalence relations are precisely the orbit equivalence relations induced by computable actions of computable groups.*

## Proof.

Every such orbit relation is c.e. Conversely, if $E$ is c.e., let $\Gamma$ be a computable copy of the free group $F_\omega$ with generators $x_i$. If $(n, m) \in E$ at stage $s$, let $x_{\langle s,n,m\rangle}$ act by swapping $n$ and $m$ only. This is a computable action with orbit $E$. $\square$

# Summary of the hierarchy so far



# Equivalence relations on c.e. sets

A deeper theory arises in the context of equivalence relations on c.e. sets, such as isomorphism relations on c.e. graphs and other c.e. structures. We begin by adapting the key relations of the Borel theory by considering them as relations on c.e. sets.

An equivalence relation $E$ on the c.e. sets gives rise to

$$e \; E^{ce} \; e' \qquad \Longleftrightarrow \qquad W_e \; E \; W_{e'},$$

For example, the equality relation on c.e. sets,

$$e =^{ce} e' \qquad \Longleftrightarrow \qquad W_e = W_{e'}.$$

This is a $\Pi^0_2$-complete set of pairs. Indeed, it has a $\Pi^0_2$-complete class $[e]$, where $W_e = \mathbb{N}$.

# Equality on c.e. sets, $=^{ce}$

**Theorem**

*Every c.e. equivalence relation lies properly below $=^{ce}$.*

**Proof.**

Let $E$ be an arbitrary c.e. relation. Define a reduction from $E$ to $=^{ce}$ by mapping each $n$ to a program $f(n)$ which enumerates $[n]_E$.

Conversely, there can be no reduction from $=^{ce}$ to $E$ since $=^{ce}$ is $\Pi^0_2$-complete and $E$ is c.e. $\square$

# Almost equality, $E_0^{ce}$

The almost-equality relation $E_0$ similarly plays an essential role in the Borel theory.

$$A \; E_0 \; B \quad \iff \quad \text{symmetric difference } A \triangle B \text{ is finite}$$

**Theorem**

*Equality $=^{ce}$ lies strictly below almost-equality $E_0^{ce}$.*

**Proof.**

To reduce $=^{ce}$ to $E_0^{ce}$, amplify differences: given $e$, let $W_{f(e)}$ have $(n,0),(n,1),(n,2),\ldots$ whenever $n \in W_e$. Thus, $W_e \neq W_{e'} \implies W_{f(e)} \triangle W_{f(e')}$ is infinite. Conversely, there is no computable reduction from $E_0^{ce}$ to $=^{ce}$, since $E_0^{ce}$ is $\Sigma^0_3$-complete, having a class equivalent to COF, whilst $=^{ce}$ is just $\Pi^0_2$. $\square$

# Analogue of Silver, Glimm-Effros dichotomy?

To what extent will the computable reducibility hierarchy mirror the structure of Borel reducibility?

For example, one might hope for an analogues of Silver's theorem, that $=^{ce}$ is minimal in some large class. Unfortunately, this fails, as we shall see.

Are there any relations lying properly between $=^{ce}$ and $E_0^{ce}$? More generally, is there a form of the Glimm-Effros dichotomy in this context? In other words, is there a large collection of equivalence relations $E$ such that if $=^{ce}$ lies strictly below $E$, then $E_0^{ce}$ lies below $E$?

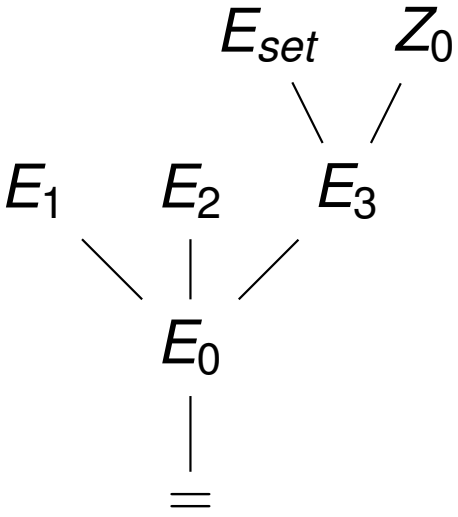
# More c.e. analogues of Borel relations

We may code subsets of $\mathbb{N} \times \mathbb{N}$ with subsets of $\mathbb{N}$.

- $(A_n)\ E_1\ (B_n)$ iff for almost all $n$, $A_n = B_n$.
- $A\ E_2\ B$ iff $\sum_{n \in A \triangle B} 1/n < \infty$.
- $(A_n)\ E_3\ (B_n)$ iff for all $n$, $A_n\ E_0\ B_n$.
- $(A_n)\ E_{\text{set}}\ (B_n)$ iff $\{\, A_n \mid n \in \mathbb{N} \,\} = \{\, B_n \mid n \in \mathbb{N} \,\}$.
- Density: $A\ Z_0\ B$ iff $\lim |(A \triangle B) \cap n|/n = 0$.

We consider the analogues on c.e. sets.

The complete diagram of Borel reducibilities on left.

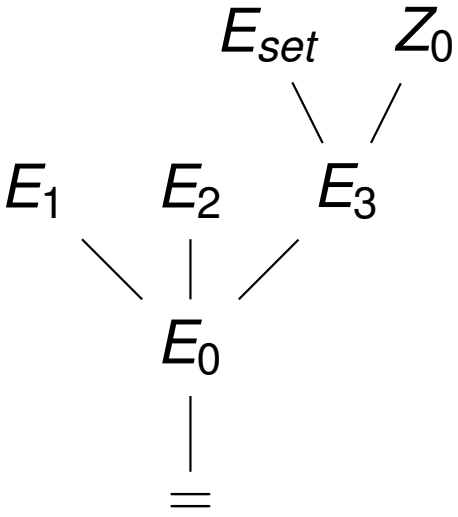


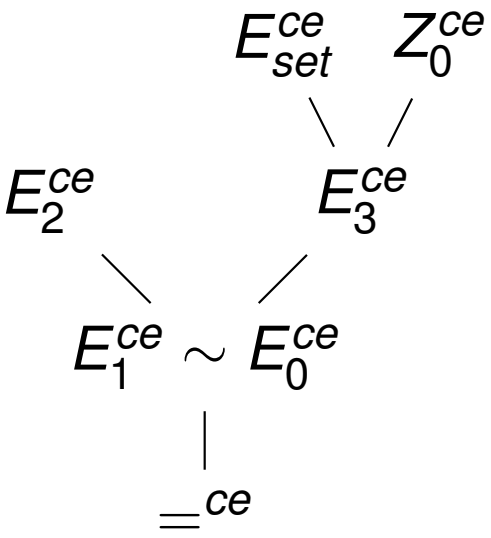




The c.e. analogues on right. (Not sure if this is complete.)

The complete diagram of Borel reducibilities on left.




The c.e. analogues on right. (Not sure if this is complete.)

## Theorem

*The positive reductions of the previous diagram hold.*

## Proof.

Use classical Borel reductions, which are computable here.

- $E_0$ reduces to $E_3$ by the map $A \mapsto \mathbb{N} \times A$.
- $E_0$ reduces to $E_1$ via $A \mapsto \{\, \langle x, y \rangle \mid y \geq x \;\&\; y \in A \,\}$, so that the column $x$ equals $A - \{0, \ldots, x-1\}$.
- $E_0 \leq E_2$ via $A \mapsto \bigsqcup_{n \in A} I_n$, where $\sum_{i \in I_n} 1/i \geq 1$.
- $E_0 \leq_c Z_0$ via $f(A) = \bigcup_{n \in A} [2^n, 2^{n+1})$.
- $E_3 \leq Z_0$ via $(A_n) \mapsto \bigcup \pi_n(f(A_n))$, where $I_n$ has density $1/2^n$ and $\pi_n : I_n \cong \mathbb{N}$.
- To reduce $E_3 \leq_c E_{\text{set}}$, for each $s \in 2^{<\mathbb{N}}$, place $1^n \frown 0 \frown s \frown (A \smallsetminus |s|)$ as a column of $f((A_n))$.

# $E_3$ does not reduce

**Theorem**

*$E_3^{ce}$ is not computably reducible to any of $E_0^{ce}$, $E_1^{ce}$ or $E_2^{ce}$.*

**Proof.**

A direct arithmetic complexity argument.

First, $E_0^{ce}$, $E_1^{ce}$, and $E_2^{ce}$ are all easily $\Sigma_3^0$.

Meanwhile, $E_3^{ce}$ is not $\Sigma_3^0$, and indeed, has a $\Pi_3^0$-complete class, the set of indices $e$ for c.e. subsets of $\mathbb{N} \times \mathbb{N}$ with every column finite. $\square$

# $E_1^{ce}$ bireducible with $E_0^{ce}$

We were very surprised by the following:

**Theorem**

*$E_1^{ce} \leq E_0^{ce}$. Hence, $E_1^{ce}$ and $E_0^{ce}$ are computably bireducible.*

(See our paper for the proof.)

This is a deviation from the situation in the Borel theory, where $E_0 <_B E_1$.

Are there other surprises?

# Complexity below equality $=^{ce}$

In contrast to the Borel theory, in the computable theory there is a rich collection of equivalence relations properly below equality $=^{ce}$.

This is a departure from the Borel theory, where Silver's theorem implies that $=$ is continuously reducible to every Borel equivalence relation (with uncountably many classes).

We shall also show that there are relations on c.e. sets which are computably incomparable with $=^{ce}$.

# Same minimum, Same maximum relations

Same minimum:

$$e \; E^{ce}_{\min} \; e' \quad \Longleftrightarrow \quad \min(W_e) = \min(W_{e'})$$

Same maximum:

$$e \; E^{ce}_{\max} \; e' \quad \Longleftrightarrow \quad \max(W_e) = \max(W_{e'})$$

Both are reducible to $=^{ce}$, by saturating $W_e$ upwards for $E_{\min}$ and downwards for $E_{\max}$.

These saturation reductions are computable *selectors*, choosing a representative of each class.

# Max not reducible to Min

**Theorem**

*$E^{ce}_{\max}$ is not computably reducible to $E^{ce}_{\min}$. Consequently, $E^{ce}_{\min}$ lies properly below $=^{ce}$.*

**Proof.**

This holds because $E^{ce}_{\min}$ is $\Delta^0_2$, while $E^{ce}_{\max}$ is $\Pi^0_2$ complete. Even the $E_{\max}$ class $\mathrm{INF} = \{\, e \mid |W_e| = \aleph_0 \,\}$ is $\Pi^0_2$ complete. $\square$

In fact, we show next that $E^{ce}_{\max}$ and $E^{ce}_{\min}$ are *incomparable*, and both lie properly below $=^{ce}$.

# Monotonicity lemma

- $f$ well-defined: $W_e = W_{e'} \implies W_{f(e)} = W_{f(e')}$.
- $f$ monotone: $W_e \subseteq W_{e'} \implies W_{f(e)} \subseteq W_{f(e')}$.

## Monotonicity Lemma

Every computable well-defined $f : \mathbb{N} \to \mathbb{N}$ is monotone.

## Proof.

Suppose $W_e \subseteq W_{e'}$ and $x \in W_{f(e)}$. Using the recursion theorem, let $W_p$ look like $W_e$ until $x$ appears in $W_{f(p)}$, and then $W_p$ looks like $W_{e'}$. Note that $x$ must appear in $W_{f(p)}$, since otherwise $W_p = W_e$ and so $W_{f(p)} = W_{f(e)}$. But now $W_p = W_{e'}$ and so $x \in W_{f(p)} = W_{f(e')}$, as desired. $\square$

# Min not reducible to Max

**Theorem**

*$E^{ce}_{\min}$ does not reduce to $E^{ce}_{\max}$. Consequently, $E^{ce}_{\max} <_c =^{ce}$.*

**Proof.**

Suppose $f : E^{ce}_{\min} \leq_c E^{ce}_{\max}$. By saturating downwards, assume without loss $f$ is well-defined. For each $n$, let $W_{e_n} = [n, \infty)$, a monotone descreasing sequence. By monotonicity lemma, $W_{f(e_n)}$ is therefore also monotone decreasing. Moreover, since $\min(W_{e_n})$ are distinct, it follows that $\max(W_{f(e_n)})$ are distinct, impossible. $\square$

# Same median relation, $E^{ce}_{\text{med}}$

$$e \; E^{ce}_{\text{med}} \; e' \iff \text{median}(W_e) = \text{median}(W_{e'})$$

**Theorem**

*$E^{ce}_{\text{med}}$ lies between $E^{ce}_{\text{max}}$ and $E^{ce}_0$ in the reducibility hierarchy.*

**Proof.**

$E^{ce}_{\text{max}} \leq_c E^{ce}_{\text{med}}$ by saturating downwards.

$E^{ce}_{\text{med}} \leq_c E^{ce}_0$ via $f$, where as long as $r_s = \text{median}(W_{e,s})$ doesn't change, $W_{f(e)}$ enumerates multiples of $r_s$ into $W_{f(e)}$. When $r_s$ does change, $W_{f(e)}$ saturates below current max and now enumerates multiples of new $r_s$. This reduces, since if $W_e$ has median $r$, then this stabilizes, so $W_{f(e)}$ is eventually $r\mathbb{N}$. $\square$

# Median incomparable with Min and equality

**Theorem**

*$E^{ce}_{\text{med}}$ is incomparable with both $E^{ce}_{\text{min}}$ and $=^{ce}$.*

**Proof.**

Suppose $f : E^{ce}_{\text{med}} \leq_c =^{ce}$. So $f$ well-defined, hence monotone. Consider $W_{e_1} = \{1\}$, $W_{e_2} = \{1, 2\}$, $W_{e_3} = \{0, 1, 2\}$. By monotonicity, $W_{f(e_1)} \subsetneq W_{f(e_2)} \subsetneq W_{f(e_3)}$, but $\mathrm{med}(W_{e_1}) = \mathrm{med}(W_{e_3})$, contradiction.

Suppose $f : E^{ce}_{\text{min}} \leq_c E^{ce}_{\text{med}}$. Fix $W_{e_0}$ nonempty with $W_{f(e_0)}$ finite median $r_0$. Find $\min(W_{e_0}) < \min(W_{e_1}) < \min(W_{e_2}) < \cdots < \min(W_{e_N})$, $N = 2r_0 + 2$, with $r_i = \mathrm{med}(W_{f(e_i)})$ finite. Consider program $e$, which enumerates $\min(W_{e_N})$ into $W_e$ until $\mathrm{med}(W_{f(e),s_N}) = r_N$. Now enumerate $\min(W_{e_{N-1}})$ into $W_e$, and so on. After $N$ iterations, $\mathrm{med}(W_{f(e)}) = \mathrm{med}(W_{f(e_0)}) = r_0$, but $W_{f(e)}$ has $\geq N = 2r_0 + 2$ elements, impossible. $\square$

# Reducibility diagram

$$\begin{tikzcd}
 & E_0^{ce} \arrow[dl, dash] \arrow[dr, dash] & \\
=^{ce} \arrow[dr, dash] & & E_{\text{med}}^{ce} \arrow[dl, dash] \\
E_{\text{min}}^{ce} \arrow[ur, dash] & & E_{\text{max}}^{ce}
\end{tikzcd}$$

This diagram is complete for these relations.

# Generalization to other orders

For computable linear ordering $L$ and $W \subseteq L$, consider Dedekind cut

$$\mathrm{cut}_L(W) = \{\, l \in L \mid \exists w \in W (l <_L w) \,\}.$$

Same cut relation

$$e \; E_L \; e' \quad \Longleftrightarrow \quad \mathrm{cut}_L(W_e) = \mathrm{cut}_L(W_{e'})$$

Same hull relation

$$e \; H_L \; e' \quad \Longleftrightarrow \quad W_e, W_{e'} \text{ have same convex hull in } L$$

- Both $E_L$ and $H_L$ reduce to $=^{ce}$ by saturating $W_e$.
- Both $E_L$ and $E_{L^*}$ computably reduce to $H_L$.
- Note $E_{\max} = E_\omega$ and $E_{\min} = E_{\omega^*}$.
- $=^{ce}$ is bireducible with $E_{\mathbb{Q}}$, via $e \mapsto$ Dedekind cut in $\mathbb{Q}$ of the real number $\sum_{n \in W_e} 1/3^{n+1}$

# Complete embeddings

Let $\overline{L}$ be the c.e. cuts of $L$. Define *computably embeddable* $\overline{L_1} \hookrightarrow_c \overline{L_2}$, if there is computable $\alpha : \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{cut}_{L_1}(W_e) < \mathrm{cut}_{L_1}(W_{e'}) \iff \mathrm{cut}_{L_2}(W_{\alpha(e)}) < \mathrm{cut}_{L_2}(W_{\alpha(e')}).$$

**Theorem**

*If $L_1, L_2$ computable linear orders, then $E_{L_1} \leq E_{L_2}$ iff $\overline{L_1} \hookrightarrow_c \overline{L_2}$.*

**Proof.**

If $f : E_{L_1} \leq_c E_{L_2}$, wlog $f$ is well-defined. By monotonicy, $f$ preserves cut order, so $f$ witnesses $\overline{L_1} \hookrightarrow_c \overline{L_2}$.

If $\alpha : \overline{L_1} \hookrightarrow_c \overline{L_2}$, let $W_{f(e)} = \mathrm{cut}_{L_2}(W_{\alpha(e)})$, which is a reduction of $E_{L_1}$ to $E_{L_2}$. $\square$

The result generalizes to partial orders.

# Cuts and Hulls in computable ordinals

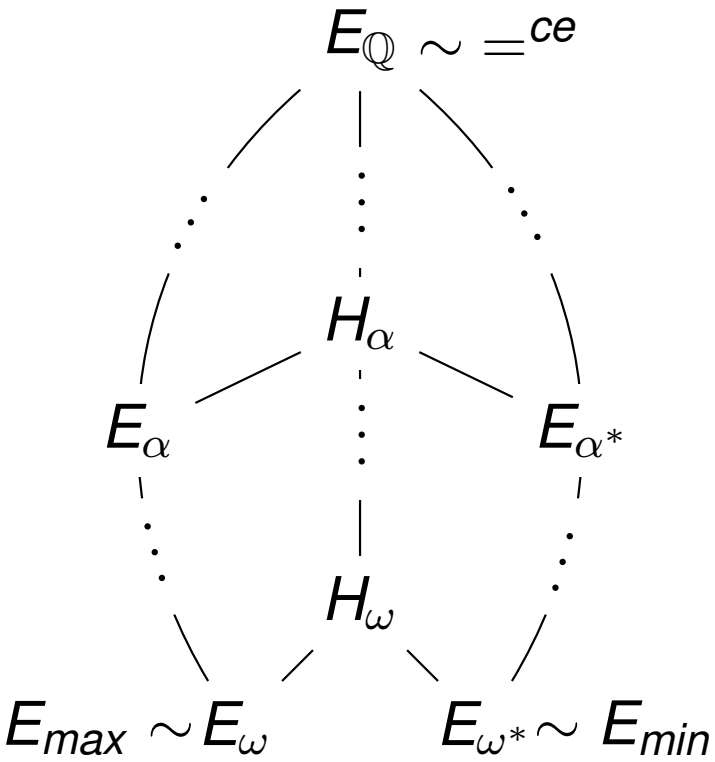



Figure: Diagram showing the cut and hull relations for computable ordinals $\alpha$ and their reverse orderings $\alpha^*$. In 2017 this will be the first diagram to land on Gliese 581g.

# Enumerable relations

A major focus for Borel theory: countable Borel equivalence relations.

The Lusin/Novikov theorem: every countable Borel equivalence relation has a uniform Borel enumeration of each class.

The Feldman/Moore argument uses this to show every countable Borel equivalence relation is the orbit relation of a Borel action of a countable group.

We develop the computable analogue of this theory.

# Computable analogue of Lusin/Novikov

An equivalence relation $E$ on c.e. sets is *enumerable in the indices* if there is computable $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with

$$W_e \; E \; W_{e'} \quad \iff \quad \exists n \; W_{\alpha(n,e)} = W_{e'}.$$

This is the computable analogue of the Lusin/Novikov property for countable Borel equivalence relations.

For example, $E_0^{ce}$ has this property, since let $W_{\alpha(n,e)}$ be $W_e$ modified to have $n^{\text{th}}$ finite set at bottom. As $n$ varies, $W_{\alpha(e,n)}$ enumerates all finite modifications of $W_e$, and so $\alpha$ witnesses that $E_0^{ce}$ is enumerable in the indices.

# Enumerable relations reduce to $E_{\text{set}}$

**Theorem**

*If $E^{ce}$ is enumerable in the indices, then $E^{ce} \leq E^{ce}_{\text{set}}$.*

**Proof.**

Simply map program $e$ to a program for a subset of $\mathbb{N} \times \mathbb{N}$ which puts $W_{\alpha(n,e)}$ on the $n^{\text{th}}$ column. $\square$

Of course $E^{ce}_{\text{set}}$ is not itself enumerable, since enumerable relations are easily seen to be $\Sigma^0_3$, whereas $E^{ce}_{\text{set}}$ is $\Pi^0_3$ complete.

# Computable group actions

The action of a computable group $\Gamma$ acting on the c.e. sets is *computable in the indices* if there is computable $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with $W_{\alpha(\gamma,e)} = \gamma W_e$.

For example, left translation action of $\Gamma$ on c.e. subsets of $\Gamma$.

Induced orbit equivalence relation

$$e \; E_\Gamma^{ce} \; e' \quad \iff \quad \exists \gamma \in \Gamma \; W_{e'} = \gamma W_e.$$

One would prefer an analogue of Feldman/Moore, saying that every enumerable relation is the orbit relation of an action computable in the indices.

Unfortunately, this is not the case.

# Counterexamples to orbit equivalence relations

## Theorem

*If $E$ is an equivalence relation on c.e. sets and $|[W_e]_E| \geq 2$ for some $e$ with $W_e \subseteq W_{e'}$ for all $e' \; E^{ce} \; e$, then $E$ is not an orbit equivalence of any action computable in the indices.*

## Proof.

Suppose $E$ is orbit equivalence of computable action of $\Gamma$, witnessed by $\alpha$. For each $\gamma$, the map $e \mapsto \alpha(\gamma, e)$ is well-defined on c.e. sets, hence monotone. Choose $e' \; E^{ce} \; e$ with $W_e \subsetneq W_{e'}$ and $\gamma W_{e'} = W_e$. So $\gamma W_e \subseteq \gamma W_{e'} = W_e$, so by hypothesis, $\gamma W_e = W_e$ also, a contradiction. $\square$

# $E_0^{ce}$ is not an orbit equivalence

**Corollary**

*$E_0^{ce}$ is not induced by any action which is computable in the indices.*

**Proof.**

The empty set $W_e = \emptyset$ is minimal in its $E_0$ equivalence class, which consists of all the finite sets. □

Nevertheless, we don't know:

**Question**

Is $E_0^{ce}$ computably *bireducible* with an orbit relation induced by an action which is computable in the indices?

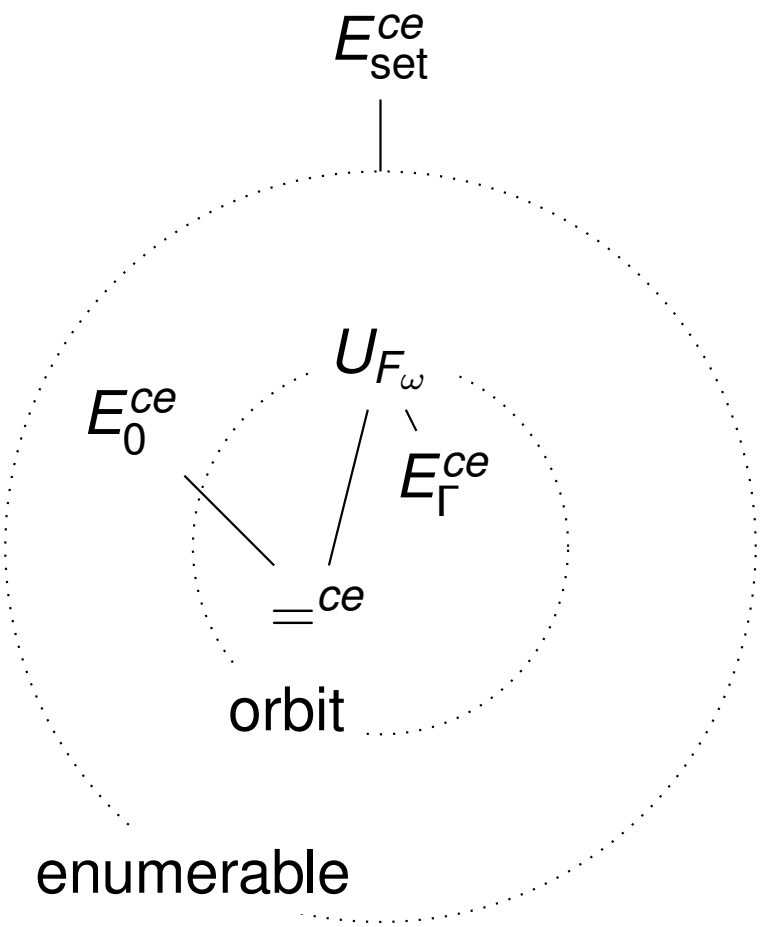
# Universal orbit equivalence

Like their Borel counterparts, the class of orbit relations induced by actions computable in the indices exhibits a universal element, which gives hope that the structure of the orbit equivalence relations on c.e. sets will mirror the countable Borel equivalence relations.

**Theorem**

*There is $E_\infty^{ce}$ induced by an action computable in the indices, and $E_\Gamma \leq E_\infty^{ce}$ whenever $E_\Gamma$ arises from an action computable in the indices.*

The proof uses the free group on $\omega$ generators to simulate all other actions.




The enumerable relations, with orbit equivalence relations.

# Isomorphism of c.e. structures

## Definition

Let $\cong^{ce}_{\text{bin}}$ denote the isomorphism relation on the codes for c.e. binary relations. That is, let $e \cong^{ce}_{\text{bin}} e'$ if and only if $W_e$ and $W_{e'}$, thought of as binary relations on $\mathbb{N}$, are isomorphic.

We remark that in order to analyze the isomorphism on arbitrary $\mathcal{L}$-structures, it is enough to consider just the binary relations, since if $\mathcal{L}$ is a computable language then the isomorphism relation $\cong^{ce}_{\mathcal{L}}$ on the c.e. $\mathcal{L}$-structures is computably reducible to $\cong^{ce}_{\text{bin}}$.

# What is isomorphism on c.e. graphs?

Two different ways to treat isomorphism of c.e. structures.

- Could restrict $\cong^{ce}_{\text{bin}}$ to indices for $W_e$ that are the right kind of structure. (Problem: these relations are not total)
- Or, identify structures as c.e. substructures of fixed universal structure.

For example, fix a computable copy $\Gamma$ of the countable random graph, and we define $\cong^{ce}_{\text{graph}}$ to be the isomorphism relation on the c.e. subsets of $\Gamma$.

Similarly, we consider c.e. linear orders as subsets of a copy of $\mathbb{Q}$, and c.e. trees as the downward closure of c.e. subsets of $\mathbb{N}^{<\mathbb{N}}$.

In some cases, the two approaches are equivalent.

# Isomorphisms of c.e. graphs, linear orders, trees

**Theorem**

*Isomorphism of binary relations $\cong^{ce}_{\text{bin}}$ is computably bireducible with each of isomorphisms of graphs, linear orders and trees.*

$$\cong^{ce}_{\text{graph}} \qquad \cong^{ce}_{\text{lo}} \qquad \cong^{ce}_{\text{tree}}$$

The classical reductions go through easily.

Part of the reason it works is that these classical reductions use only the *positive* information about the structures; they need to know when two elements are related, but not instances of non-relation.

# Same set is strictly below $\cong_{\text{bin}}$

The isomorphism relation of binary relations $\cong^{ce}_{\text{bin}}$ is very high in our hierarchy. For example,

**Theorem**

*$E^{ce}_{\text{set}}$ lies properly below $\cong^{ce}_{\text{bin}}$.*

**Proof.**

To reduce $E^{ce}_{\text{set}}$ to $\cong^{ce}_{\text{bin}}$, let $W_{f(e)}$ be a code for $W_e$ as a hereditarily countable set. In other words, $W_{f(e)}$ is a well-founded tree coding the transitive closure of $\{W_e\}$.

The absence of any reverse reduction follows from complexity, since $E^{ce}_{\text{set}}$ is $\Pi^0_4$. $\square$

# Isomorphism of c.e. groups

Isomorphism of c.e. groups admits several coding methods.

We could consider indices $e$ such that $W_e$, as a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, is the graph of a group operation, and then define $\cong^{ce}_{\text{group}}$ to be the restriction of the isomorphism relation $\cong^{ce}_{\text{tern}}$ on the c.e. ternary relations to this set.

Alternatively we can code a group by a presentation, a set of words in $F_\omega$, thinking of the group as the corresponding quotient. Thus, $e \cong^{ce}_{\text{pres}} e'$ iff $W_e$ and $W_{e'}$, as sets of relations in $F_\omega$, determine isomorphic quotient groups.

Thus, the classification problem for groups splits into two separate problems: for computable group operations, and for computably presented groups.

# Two versions of group isomorphism

**Theorem**

$\cong^{ce}_{\text{group}} \leq \cong^{ce}_{\text{bin}} \leq \cong^{ce}_{\text{pres}}$.

We suspect that neither reduction is reversible.

# Computable isomorphisms

Consider now the case of *computable* isomorphisms, rather than arbitrary isomorphisms.

$e \simeq^{ce}_{\text{bin}} e'$ iff $W_e, W_{e'}$, as binary relations, are isomorphic by a computable bijection.

It happens that most of the previous reductions, such as from binary relations to graphs, go through for computable isomorphisms.

Thus, the computable isomorphism relation on the class of c.e. binary relations, graphs, linear orders and trees are all computably bireducible to each other.

# Computable isomorphisms

But they are lower in the hierarchy:

**Theorem**

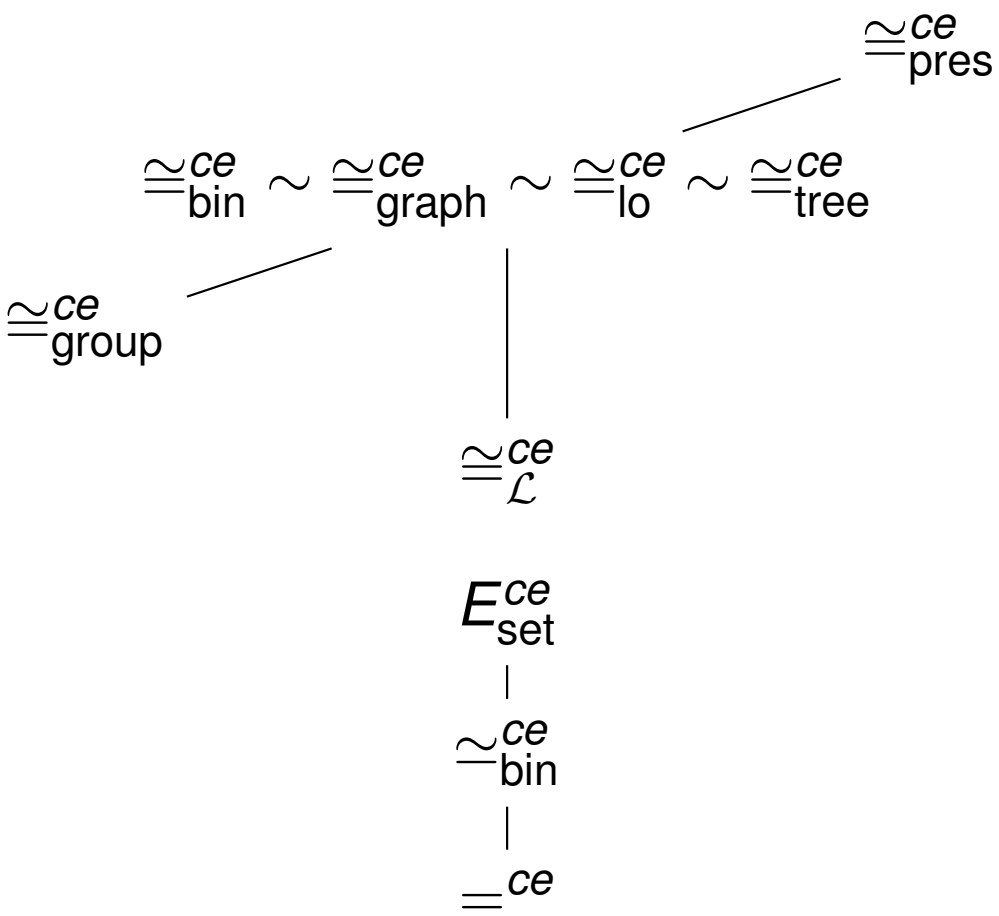
$\simeq^{ce}_{\text{bin}}$ *lies properly below* $E^{ce}_{\text{set}}$.

**Proof.**

For the reduction, the basic idea is to reduce to $E_{\text{set}}$ by making a subset of $\mathbb{N} \times \mathbb{N}$ whose slices enumerate all computable copies of the original structure (plus all finite sets).

Conversely, $E^{ce}_{\text{set}}$ is not reducible to $\simeq^{ce}_{\text{bin}}$ because $\simeq^{ce}_{\text{bin}}$ is $\Sigma^0_3$, but $E^{ce}_{\text{set}}$ has a $\Pi^0_3$-complete equivalence class. $\square$

# Isomorphisms and computable isomorphisms



# Relations from computability theory

Consider now the principal equivalence relations of computability theory, viewed as equivalence relations on the c.e. sets.

- Turing equivalence, $=_T$
- Many-one equivalence, $=_m$
- one-equivalence, $=_1$

## Theorem

*$=^{ce}$ lies properly below each of $=^{ce}_T$, $=^{ce}_1$, and $=^{ce}_m$ in the computable reducibility hierarchy.*

## Proof.

Reduce to all at once. Fix strong antichain: c.e. sets $A_i$ with $A_i \not\leq_T \bigoplus_{j \neq i} A_j$. Let $W_{f(e)} \subseteq \mathbb{N} \times \mathbb{N}$ have $k^{\text{th}}$ column $A_{n_k}$, when $n_k$ is $k^{\text{th}}$ element enumerated into $W_e$.

If $W_e = W_{e'}$, then $W_{f(e)} \equiv_1 W_{f(e')}$, by suitable permutation of columns. Conversely, if $W_e \neq W_{e'}$, then $W_{f(e)} \not\equiv_T W_{f(e')}$, for if $i \in W_e \setminus W_{e'}$, then $A_i \leq_T W_{f(e)}$ but not $W_{f(e')}$.

None of the relations reduces to $=^{ce}$, because they are $\Sigma^0_3$-complete, whilst $=^{ce}$ is $\Pi^0_2$. $\square$

**Theorem**

*$=^{ce}_m$ is computably reducible to $=^{ce}_1$.*

**Proof.**

Simply let $W_{f(e)} = W_e \times \mathbb{N}$. If $\phi : \mathbb{N} \to \mathbb{N}$ many-one reduces $W_e$ to $W_{e'}$, then $(m, n) \mapsto (\phi(m), \langle m, n \rangle)$ one-one reduces $W_{f(e)}$ to $W_{f(e')}$. Converse is similar. □

**Theorem**

*$=^{ce}_1$ is computably reducible to $\simeq^{ce}_{\text{bin}}$.*

**Proof.**

This is because $=^{ce}_1$ is the computable isomorphism relation on the set of c.e. *unary* relations. So $=^{ce}_1 \leq \simeq^{ce}_{\text{bin}}$ via $f$ where $W_{f(e)}$ codes the graph on $\mathbb{N} \cup \{\star\}$ where $\star \to n$ iff $n \in W_e$. □

# Degree relations

$$
\begin{tikzcd}
 & \simeq^{ce}_{\text{bin}} \arrow[d, dash] \\
 & =^{ce}_{1} \arrow[d, dash] \\
=_T \arrow[dr, dash] & =^{ce}_{m} \arrow[d, dash] \\
 & =^{ce}
\end{tikzcd}
$$

# Difference hierarchy

We have extended the analysis to the difference hierarchy.

For $e_1, \ldots, e_n$, the corresponding $n$-c.e. set is

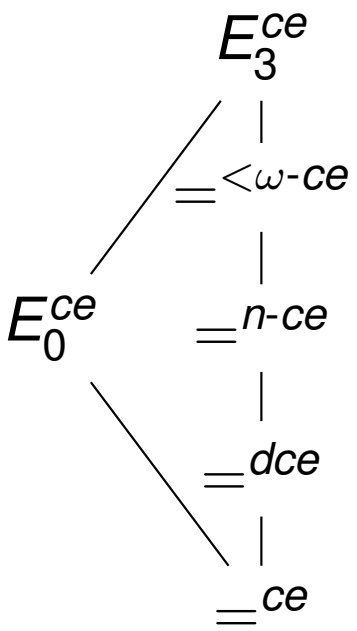
$$W_{(e_i)} = (W_{e_1} \smallsetminus W_{e_2}) \cup (W_{e_3} \smallsetminus W_{e_4}) \cup \cdots \smallsetminus \cup W_{e_n}.$$

Every equivalence relation $E$ on $n$-c.e. sets has corresponding relation

$$(e_i)\ E^{n\text{-}ce}\ (f_i) \quad \Longleftrightarrow \quad W_{(e_i)}\ E\ W_{(f_i)}$$

It is easy to see that $E^{n\text{-}ce} \leq E^{n+1\text{-}ce}$. Just fix $e$ with $W_e = \emptyset$, and use $(e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n, e)$.

# Relations in the difference hierarchy



Thank you.

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