Infinite chess

The mate-in-*n* problem is decidable and the omega one of chess

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This talk includes joint work with:

- Dan Brumleve, Joel David Hamkins, Philipp Schlicht, “The mate-in-*n* problem of infinite chess is decidable,” to appear in LNCS proceedings volume for CiE 2012.
- C. D. A. Evans, J. D. Hamkins, W. H. Woodin, “Transfinite game values in infinite chess,” in preparation.

A preprint of the mate-in-*n* paper is available on my web page:

http://jdh.hamkins.org

Infinite chess

Infinite chess is chess played on an infinite edgeless chess board, arranged like the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

The familiar chess pieces—kings, queens, bishops, knights, rooks and pawns—move about according to their usual chess rules, with bishops on diagonals, rooks on ranks and files and so on, with each player striving to place the opposing king into checkmate.

There is no standard starting configuration in infinite chess, but rather a game proceeds by setting up a particular position on the board and then playing from that position.

Clarifying the rules

Let me clarify a few of the rules as they relate to infinite chess.

- At most one king of each color
- There is no boundary, hence no pawn promotion
- There is no castling and no *en passant*
- Abandon the 50 move rule as limiting
- Infinite play is a draw
- We may abandon the three-fold repetition rule
- For arbitrary starting positions, one must clarify some weird boundary cases, e.g. White to move, black in check, white in checkmate

The mate-in-*n* problem

There is a genre of chess problems, the mate-in-*n* problems

- White to mate in 2
- Black to mate in 3

Consider the corresponding class of problems in infinite chess.

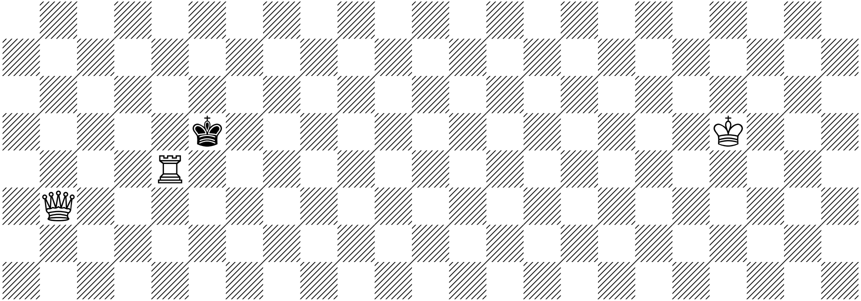
The mate-in-*n* problem

Given a finite position in infinite chess, can a designated player force a win in at most *n* moves?

A mate-in-12 problem

White to move on an infinite, edgeless board.

Can white force mate in 12 moves?



The mate-in-n problem

The mate-in-n problem

Given a finite position in infinite chess, can a designated player force a win in at most n moves?

Question

Is the mate-in-n problem computably decidable?

A naive formulation of white to mate-in-3, with white to move:

> *There is a white move, such that for every black reply, there is a white move, such that for every black reply, there is a white move, which delivers checkmate.*

$$\exists w_1 \forall b_1 \exists w_2 \forall b_2 \exists w_3 \cdots$$

Very high arithmetical complexity, $2n$ alternating quantifiers.

Infinite game tree

In finite $n \times n$ chess, one can search the entire game tree, which is finite.

In infinite chess, the game tree is not only infinite, but infinitely branching. The unbounded pieces (queens, rooks, bishops) may have infinitely many possible moves.

Thus, one cannot expect computably to search the entire game tree, even to finite depth.

The naive, brute-force-search manner of deciding the mate-in-n problem is inadequate.

Various decidability problems in infinite chess

- Winning-position problem: can a designated player force a win?
- Mate-in-n: can a player force a win in at most n moves?
- Drawn-position problem: can a player force a draw?
- Stalemate-position: can a player force a stalemate?
- Draw-in-n
- Stalemate-in-n
- Draw-in-n by k-repetition: can a designated player force the position into a set of k positions inside of which play can be forced to remain?

Formulated directly, each of these problems leads to statements of very high arithmetical complexity (or worse), which would give little reason to suggest they are decidable.

Equivalence relation on unbounded moves

Even though a position may allow infinitely many legal moves, there seems to be an (eventually periodic) equivalence for sufficiently distant moves in their effect on a position.

For example, if there is a distant checkmate move by a bishop, then any sufficiently distant move in that direction will deliver checkmate.

One can push this kind of reasoning further, two and three moves out. (Parity issues, eventual periodicity)

After observing the situation for some (very) small values of n, C. D. A. Evans and I conjectured that the mate-in-n problem for infinite chess is decidable.

The infinitary mate-in-n problem is decidable

Theorem (Brumleve, Hamkins, Schlicht)

The mate-in-n problem of infinite chess is decidable.

1. *There is a computable strategy for optimal play from a mate-in-n position, achieving the checkmate in the fewest number of moves.*
2. *There is a computable strategy for optimal opposing play from a mate-in-n position, to delay checkmate as long as possible.*
3. *There is a computable strategy to enable any player to avoid checkmate for k moves from a given position, if this is possible.*
4. *The stalemate-in-n and draw-in-n-by-k-repetition problems are also decidable, with computable strategies giving optimal play.*

The first-order structure of chess $\mathfrak{Ch}$

To prove the theorem, we introduce the first-order structure of chess $\mathfrak{Ch}$.

- Domain consists of all finite chess positions: description of pieces, locations, whether captured, whose turn.
- Basic relations: $\mathrm{OneMove}(p, q)$, $\mathrm{WhiteInCheck}(p)$, $\mathrm{WhiteMated}(p)$, and so on.

For a given finite list A of piece types, consider reduct $\mathfrak{Ch}_A$ to positions with those pieces.

The mate-in-n problem is expressible in the theory of the structure of chess $\mathfrak{Ch}$, using $2n$ many alternating quantifiers.

The structure of chess

Key Lemma

For any finite list A of chess-piece types, the structure of chess $\mathfrak{Ch}_A$ is automatic.

An *automatic* structure is one whose domain and relations constitute a regular language, one that can be recognized by a finite automata.

Fact (Khoussainov, Nerode)

The theory of any automatic structure is decidable.

Use pumping lemma idea to reduce infinitary to finitary search—just like the equivalence relation on chess moves.

Conclusion: the mate-in-n problem is decidable. □

Alternative proof via Presburger arithmetic

An alternative proof arises from Presburger arithmetic $\langle \mathbb{N}, +, < \rangle$.

For a fixed list A of piece types, the structure of chess $\mathfrak{Ch}_A$ is definable inside $\langle \mathbb{N}, +, < \rangle$.

The point is that bishops, queens, rooks move on straight lines, which are definable in Presburger arithmetic.

Since Presburger arithmetic admits elimination of quantifiers, the theory is decidable. Hence, the theory of the structure of chess is decidable. So the mate-in-n problem is decidable.

The winning-position problem still open

Our theorem does not show that the winning-position problem is decidable.

Question (Richard Stanley)

Is the winning-position problem for infinite chess decidable?

This question remains open, and is mathoverflow-hard.

The point is that a player may have a winning strategy from a position, but it is not mate-in-n for any n. So the winning positions is not even c.e., indeed, it is not even clear whether it is arithmetic, or even hyperarithmetic.

These positions are precisely the positions with transfinite game value.

Playing according to a deterministic computable procedure

A position is computable if there is a computable function giving the locations and number of the pieces. (Complications...)

A strategy is a rule for playing, given an initial position and the sequence of moves so far.

A computable strategy is such a rule that is a computable function.

Question

Can requiring the players to play according to a computable procedure affect our judgement of a position?

Requiring computable play matters

Yes. Requiring players to play according to a deterministic computable procedure can affect our judgement of whether a given computable position is drawn or a win.

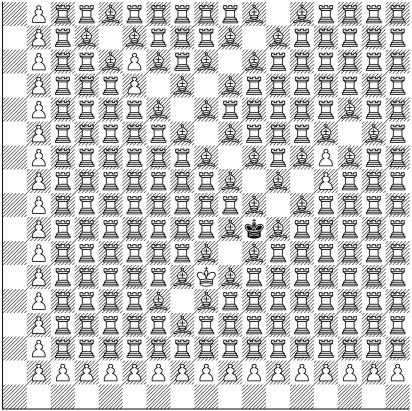
Theorem

There is a position in infinite chess such that:

1. *The position is computable.*
2. *The position is drawn. Neither player has a winning strategy, and either player can force a draw.*
3. *But among computable strategies, the position is a win for white. White has a computable strategy defeating every computable strategy for black.*

A position where computable play matters

Fix a computable tree $T \subseteq 2^{<\omega}$ with no computable branch.



White can force black through the tree via zugzwang.

Alternative, with filled-in channels

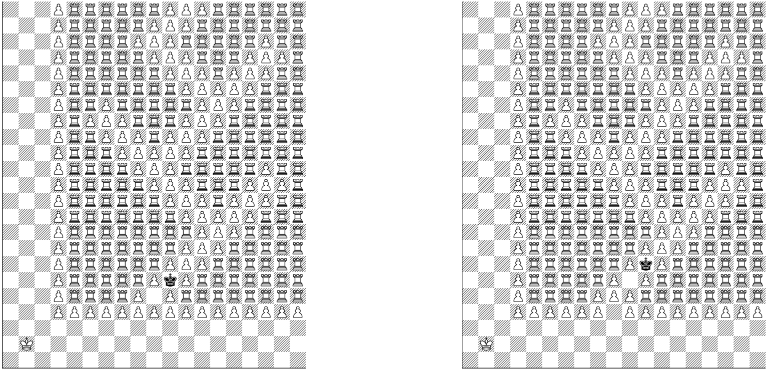


Figure: Starting position at left; position after pawn check and response at right.

Simplified version

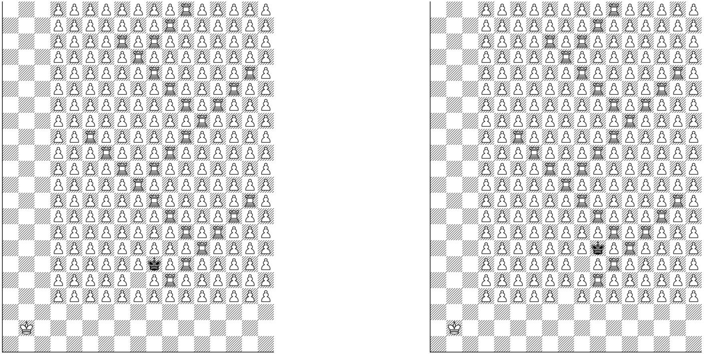
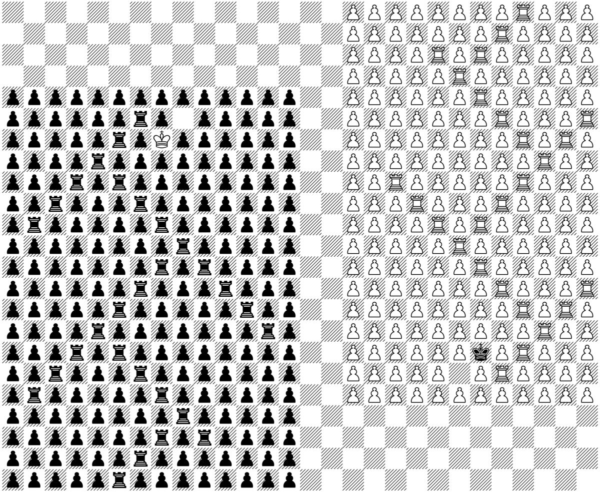


Figure: White forces black to climb the tree via successive pawn checks; response shown at right

Drawn position, but won for first player, if play is computable.



Game value

An *open* game is a game that, if won, is won after finitely many moves. Such games support a theory of transfinite *game values*, generalizing the mate-in-*n* idea.

- A position that is already won has value 0.
- A position one move away from being one has value 1.
- A mate-in-*n* position has value *n*.

The values continue transfinitely:

- value is $\alpha + 1$, if white may play to a position with value α.
- if black to play, value is supremum of the values of the positions to which black may play.

Concrete meaning of the game values

If a position has a value, then white can win by the value-reducing strategy. If a position has no value, then black can win or draw by the value-maintaining strategy.

Value ω means black to move, but every move creates mate-in-n, for arbitrarily large n. Thus, black can announce at the outset a lower bound on the length of the game.

Value $\omega + 10$ means white can force to a value ω position in at most 10 moves.

Value ω^2 means black to play, but every move creates value $\omega \cdot n + k$. Black can announce the number of future announcements he will make, for each of which play will take that long before the next announcement.

The omega one of chess

We define ω_1^{chess} to be the supremum of the game values of the finite positions of infinite chess.

Question

What is the value of ω_1^{chess}?

Current known lower bounds: $\omega \cdot n + k$.

Cory Evans and I have candidates for value ω^2.

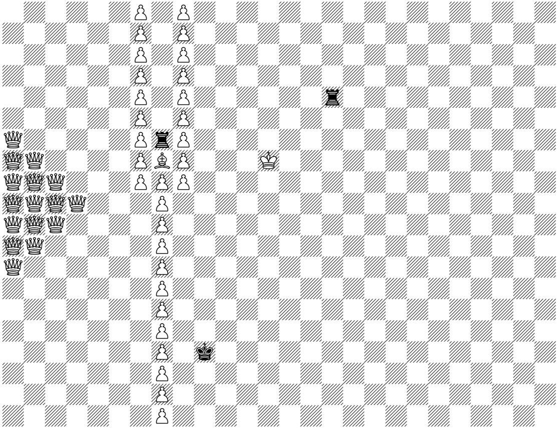
Theorem (Andreas Blass, independently, Philip Welch)

The omega one of chess ω_1^{chess} is at most ω_1^{ck}. Indeed, if a designated player has a winning strategy from a finite position p, then there is such a strategy with hyperarithmetic complexity.

Values of infinite positions

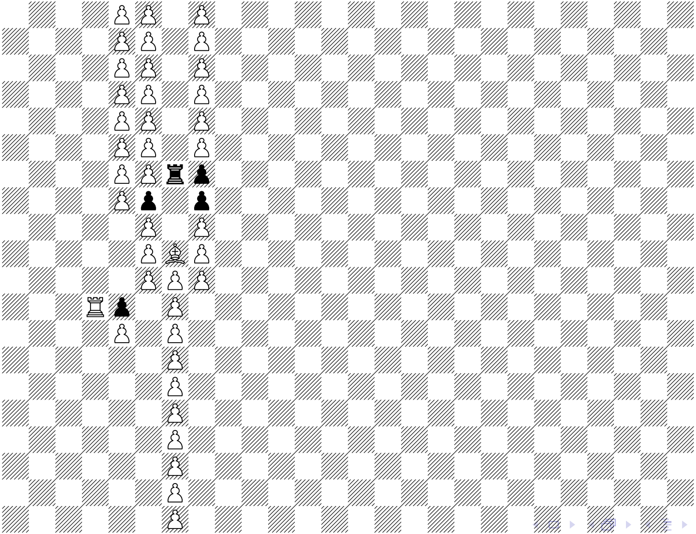
Let me now present a few interesting infinite positions with high transfinite game values.

Releasing the hordes, value ω^2



White aims to open the portcullis.

Basic lock and key



Value ω^3

Iterated lock and key arrangements have value $\omega^2 \cdot n + k$.

Cory Evans and I have assembled infinite positions with value exceeding ω^3.

The idea is to have one diagonal infinite branch, going into "rooms", with value $\omega^2 \cdot n$.

Conjecture

The omega one of infinite positions in infinite chess is true ω_1.

We haven't been able to prove this in two-dimensional infinite chess, but in three dimensions...

Infinite 3D chess

Three-dimensional chess goes back a century. Dr. Ferdinand Maack invented Raumschach $8 \times 8 \times 8$ chess in 1907, eventually settling on $5 \times 5 \times 5$ as providing a better game, and founding a Hamburg club in 1919.

Purists play with no board. But in several Star Trek episodes, Spock and Kirk play tri-chess on a physical 3D board.

We may consider the infinitary version.

One must specify the piece movement; there is room for reasonable disagreement.

The omega one of 3D infinite chess

Theorem (Evans, Hamkins,Woodin)

The ω_1 of three-dimensional infinite chess with infinite positions is true ω_1.

That is, every countable ordinal arises as the game value of a position in three-dimensional infinite chess.

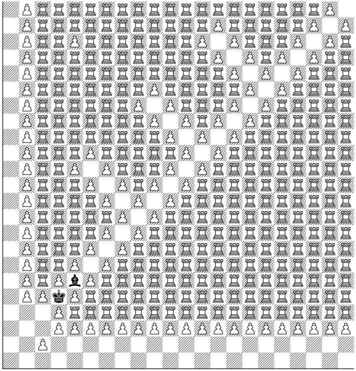
The ω_1 of 3D infinite chess

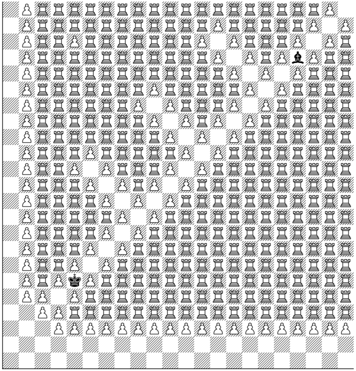
The basic idea is to code an arbitrary well-founded tree $T \subseteq \omega^\omega$ as a position in infinite chess.

The difficulty is that the tree must be infinite-branching to achieve large ranks. How to force black to make a choice?

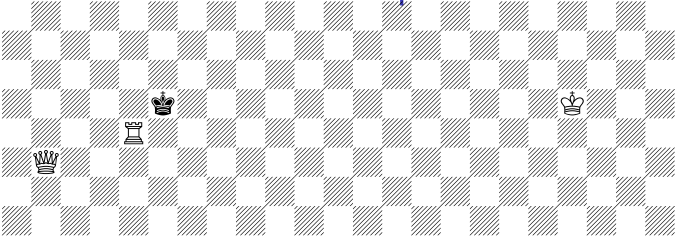
Branching node layer in 3D infinite chess

Black has mate-in-2 elsewhere. Play is tightly forced.





Solution to the mate-in-12 problem



White can mate in 13, but not 12. Bring queen up diagonally with check, forcing black king back, then check with rook and queen successively to deliver mate on 13th move. (Alternatively, push left with white king.) There is no mate in 12, since the kings must be brought together.

White can force stalemate in 12 moves, using king left idea. White can force draw by repetition in 3 moves, by trapping king in a 4x4 box.

Thank you.

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