Must there be numbers we cannot describe or define?

Pointwise definability and the Math Tea argument

Joel David Hamkins

Visiting Fellow, Isaac Newton Institute
Cambridge, March-April 2012

The City University of New York
College of Staten Island of CUNY and
The CUNY Graduate Center

Bristol, April 2012

This talk includes joint work with

Jonas Reitz, New York City College of Technology, and

David Linetsky, CUNY Graduate Center.

The math tea argument

Heard at a good math tea anywhere:

> *"There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions."*

Does this argument withstand scrutiny?

> *"I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it."*
>
> *–Horatio, age 8*

The math tea argument

Heard at a good math tea anywhere:

> *"There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions."*

Does this argument withstand scrutiny?

> *"I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it."*
>
> *–Horatio, age 8*

Definability

An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying some assertion $\mathcal{M} \models \varphi[r]$.

- Nothing is definable in $\langle \mathbb{R}, < \rangle$.
- Algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.
- More in $\langle \mathbb{R}, +, \cdot, 0, 1, \mathbb{Z}, \sin(x), e^x, \ldots \rangle$.
- Even more in $\langle H_{\omega_2}, \in \rangle$ or in $\langle V_{\omega+5}, \in \rangle$.
- $\langle V_{\omega+\omega}, \in \rangle \ldots$

In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

Definability

An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying some assertion $\mathcal{M} \models \varphi[r]$.

- Nothing is definable in $\langle \mathbb{R}, < \rangle$.
- Algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.
- More in $\langle \mathbb{R}, +, \cdot, 0, 1, \mathbb{Z}, \sin(x), e^x, \ldots \rangle$.
- Even more in $\langle H_{\omega_2}, \in \rangle$ or in $\langle V_{\omega+5}, \in \rangle$.
- $\langle V_{\omega+\omega}, \in \rangle \ldots$

In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

Definability

An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying some assertion $\mathcal{M} \models \varphi[r]$.

- Nothing is definable in $\langle \mathbb{R}, < \rangle$.
- Algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.
- More in $\langle \mathbb{R}, +, \cdot, 0, 1, \mathbb{Z}, \sin(x), e^x, \ldots \rangle$.
- Even more in $\langle H_{\omega_2}, \in \rangle$ or in $\langle V_{\omega+5}, \in \rangle$.
- $\langle V_{\omega+\omega}, \in \rangle \ldots$

In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

Definability

An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying some assertion $\mathcal{M} \models \varphi[r]$.

- Nothing is definable in $\langle \mathbb{R}, < \rangle$.
- Algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.
- More in $\langle \mathbb{R}, +, \cdot, 0, 1, \mathbb{Z}, \sin(x), e^x, \ldots \rangle$.
- Even more in $\langle H_{\omega_2}, \in \rangle$ or in $\langle V_{\omega+5}, \in \rangle$.
- $\langle V_{\omega+\omega}, \in \rangle \ldots$

In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

Definability

An object is *definable* in a structure $\mathcal{M}$ if it is the unique object r satisfying some assertion $\mathcal{M} \models \varphi[r]$.

- Nothing is definable in $\langle \mathbb{R}, < \rangle$.
- Algebraic reals are definable in $\langle \mathbb{R}, +, \cdot, 0, 1, < \rangle$.
- More in $\langle \mathbb{R}, +, \cdot, 0, 1, \mathbb{Z}, \sin(x), e^x, \ldots \rangle$.
- Even more in $\langle H_{\omega_2}, \in \rangle$ or in $\langle V_{\omega+5}, \in \rangle$.
- $\langle V_{\omega+\omega}, \in \rangle \ldots$

In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

Cheating

It would be a kind of cheating to define an object r in a structure or language that was itself not definable:

- such as a constant with value r,
- a unary relation holding only at r,
- or to define objects in $\langle V_\alpha, \in\rangle$ when α is not itself definable. (This amounts to using α as a parameter.)

We are thereby pushed:

- to allow only countable languages, and
- to consider only structures that are themselves definable with respect to the set-theoretic background $\langle V, \in\rangle$.

In a fixed structure

In a fixed structure $\mathcal{M}$ in a countable language, the math tea argument is fine: there are only countably many definitions, but uncountably many reals.

We simply associate each definable object r with a formula ψ_r that defines it. With access to such a definability map

$$r \mapsto \psi_r,$$

we may diagonalize against it to produce a real that is not definable.

Meta-mathematical obtacle

When defining reals *r* over $\langle V, \in\rangle$, however, a subtle meta-mathematical obstacle arises:

The property of *being definable in* $\langle V, \in\rangle$ is not first order expressible in set theory.

As in Tarski's theorem on the non-definability of truth, in general we may have no way to express "*x* is defined by formula ψ".

But the obstacle is less severe than in Tarski's theorem, since some models of ZFC do have such a definition.

Pointwise definability

The Theme

To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?

Definition

A structure $\mathcal{M}$ is *pointwise definable* if every element of $\mathcal{M}$ is definable without parameters in $\mathcal{M}$.

In such models, all objects are discernible; every object satisfies a distinct principal complete 1-type. Being pointwise definable is not first-order expressible, since it is not preserved by elementary extensions. Pointwise definable models (in countable language) are countable.

Pointwise definability

The Theme

To what extent is it possible that every real or indeed, every object in the set-theoretic universe, is definable without parameters?

Definition

A structure $\mathcal{M}$ is *pointwise definable* if every element of $\mathcal{M}$ is definable without parameters in $\mathcal{M}$.

In such models, all objects are discernible; every object satisfies a distinct principal complete 1-type. Being pointwise definable is not first-order expressible, since it is not preserved by elementary extensions. Pointwise definable models (in countable language) are countable.

The main theorem will be that every countable model M of ZFC, and indeed, of GBC, has an extension $M[G]$ that is pointwise definable. Every set and class is definable in $M[G]$ without parameters.

Easy folklore observations

Theorem

If ZFC is consistent, then there are continuum many non-isomorphic pointwise definable models of ZFC.

Proof.

Consider any $M \models \mathrm{ZFC} + V = \mathrm{HOD}$. There are definable Skolem functions. The collection of definable elements of M is closed under the these Skolem functions, hence elementary, hence pointwise definable. So every completion of $\mathrm{ZFC} + V = \mathrm{HOD}$ has a pointwise definable model. By Gödel-Rosser, there are continuum many completions. □

Note that pointwise definable models with same theory are isomorphic. So these are exactly all the pointwise definable models of ZFC.

Characterization of pointwise definability

That idea is fully general.

Observation

The following are equivalent:

1. M is a pointwise definable model of ZFC.
2. M consists of the definable elements of a model of $\mathrm{ZFC} + V = \mathrm{HOD}$.
3. M is a prime model of $\mathrm{ZFC} + V = \mathrm{HOD}$.

Pointwise definability is a strong form of $V = \mathrm{HOD}$.

We might introduce the notation $V = D$ or $V = HD$, but we don't want to suggest that pointwise definability is first-order expressible.

Transitive pointwise definable models

Theorem

If there is a transitive model of ZFC*, then there are continuum many transitive pointwise-definable models of* ZFC*.*

Proof.

Fix transitive $N \models \mathrm{ZFC} + V = \mathrm{HOD}$. The definable elements of N form an elementary substructure, whose Mostowski collapse is pointwise definable.

For continuum many such models, force to add a Cohen real $N[c]$, and then force $V = \mathrm{HOD}$ in $N[c][G]$ by coding into the GCH pattern, and make c definable. The definable elements of $N[c][G]$ include c and have pointwise definable Mostowski collapse. There is a perfect set of such c. □

Minimal Transitive Model

Theorem

The minimal transitive model of ZFC *is pointwise definable.*

Proof.

If L_α is the minimal transitive model of ZFC, then by condensation the definable hull of $\emptyset$ in L_α collapses to L_α, and so every element of L_α is definable in L_α. □

The argument generalizes to show that if L_β is pointwise definable, then the next $\hat{\beta} > \beta$ with $L_{\hat{\beta}} \models \mathrm{ZFC}$, if it exists, is also pointwise definable. And more...

Pointwise definable L_α's

Theorem

If there is an uncountable transitive model of ZF*, then there are arbitrarily large* $\alpha < \omega_1^L$ *for which* L_α *is a pointwise definable model of* ZFC*.*

Proof.

Actual hypothesis: there are unboundedly many $\alpha < \omega_1^L$ for which $L_\alpha \models \mathrm{ZFC}$. Every real of L is definable in some countable L_ξ, since otherwise consider the least counterexample. Every $\xi < \omega_1^L$ is definable in $L_\alpha \models \mathrm{ZFC}$, where α is the $(\xi + 1)^{\text{th}}$ ordinal for which $L_\alpha \models \mathrm{ZFC}$, because this L_α sees exactly ξ smaller α. So every real is in the definable hull of some $L_\alpha \models \mathrm{ZFC}$, whose collapse is pointwise definable. □

Every real is in a pointwise definable model

Theorem

If there is an uncountable transitive model of ZF*, then every real is an element of a pointwise definable* ω*-standard model of* ZFC*.*

Proof.

The previous theorem shows that the conclusion is true in L. The statement of the conclusion has complexity Π^1_2. So it is true in V. □

The models can be well-founded as high in the countable ordinals as desired. If a real z codes α, then any ω-model containing z will be well-founded at least to α.

Extending arbitrary transitive models

Theorem

Every countable transitive model M of set theory has an end-extension to a (possibly nonstandard) model $M^+ \models \mathrm{ZFC} + V = L$, such that M^+ is pointwise definable.

Proof.

Code M by a real. By the previous argument, this real is in a pointwise definable model M^+ of $V = L$. Note that this real decodes to M inside M^+, and M^+ is well-founded at least to the height M. □

Curious instances

These results admit curious instances when applied to nonconstructible reals.

Force to collapse ω_1^V in $V[g]$. By the previous, there is an ω-standard model $M \models \mathrm{ZFC} + V = L$ in which g exists and is definable. So M is well-founded beyond ω_1^V, but thinks g is constructible at some stage $\tilde{\alpha}$, even though we know $g \notin L$. The model M thinks $V = L$ and ω_1^V is countable.

In particular, if $0^\sharp$ exists, then there is a pointwise definable model M of $\mathrm{ZFC} + V = L$, well-founded high in the countable ordinals (e.g. past many indiscernibles), such that $0^\sharp \in M$.

Thus, true $0^\sharp$ can exist unrecognized but definable in a model of $\mathrm{ZFC} + V = L$ that is well-founded far beyond the true ω_1^L! (Can even arrange that $M \equiv L^V$.)

Curious instances

These results admit curious instances when applied to nonconstructible reals.

Force to collapse ω_1^V in $V[g]$. By the previous, there is an ω-standard model $M \models \mathrm{ZFC} + V = L$ in which g exists and is definable. So M is well-founded beyond ω_1^V, but thinks g is constructible at some stage $\tilde{\alpha}$, even though we know $g \notin L$. The model M thinks $V = L$ and ω_1^V is countable.

In particular, if $0^\sharp$ exists, then there is a pointwise definable model M of $\mathrm{ZFC} + V = L$, well-founded high in the countable ordinals (e.g. past many indiscernibles), such that $0^\sharp \in M$.

Thus, true $0^\sharp$ can exist unrecognized but definable in a model of $\mathrm{ZFC} + V = L$ that is well-founded far beyond the true ω_1^L! (Can even arrange that $M \equiv L^V$.)

The range of possibility

(i) There is no uniform definition of class of definable elements.

Specifically, there is no formula $\mathrm{df}(x)$ in the language of set theory that is satisfied in any model $M \models \mathrm{ZFC}$ exactly by the definable elements. To see this, consider $\forall x\, df(x)$ in a pointwise definable model and elementary extensions.

(ii) In some models, the class of definable elements is nevertheless definable.

For example, in a pointwise definable model.

(iii) In others, the definable elements do not form a class.

Consider any nontrivial ultrapower of a pointwise definable model.

More possibilities

(iv) The definable elements may be a class, but not $r \mapsto \psi_r$.

This is true in a pointwise definable model.

(v) The definable elements can be a set, along with $r \mapsto \psi_r$.

True in V if there is γ with $V_\gamma \prec V$.

(vi) No model has a *definable* definability map $r \mapsto \psi_r$.

Diagonalize against $r \mapsto \psi_r$.

The surviving content of the math-tea argument: in any model with $r \mapsto \psi_r$, the definable reals do not exhaust all the reals.

Pointwise definable ZFC extensions

Consider now the positive results, where we (Hamkins, Linetsky, Reitz) obtain pointwise definability not by discarding non-definable elements, but by preserving them to a pointwise definable extension.

Theorem

Every countable model of ZFC *has a pointwise definable class forcing extension.*

An earlier independent version of this theorem was mentioned by Ali Enayat.

Pointwise definable ZFC extensions

Consider now the positive results, where we (Hamkins, Linetsky, Reitz) obtain pointwise definability not by discarding non-definable elements, but by preserving them to a pointwise definable extension.

Theorem

Every countable model of ZFC *has a pointwise definable class forcing extension.*

An earlier independent version of this theorem was mentioned by Ali Enayat.

Paris models

Enayat was focused on the *Paris* models, models of ZF in which every ordinal is definable without parameters.

Theorem (Enayat)

If L has an uncountable transitive model of ZF, then there are Paris models of arbitrarily large cardinality.

These are very large models, with countable height, where every ordinal is definable without parameters. The proof uses model-theoretic methods and $\mathcal{L}_{\omega_1,\omega}$ logic, such as Morley's two-cardinal theorem, and a result of Harvey Friedman showing that every model $M \models \mathrm{ZF}$ has extensions with same ordinals of size $\beth_\alpha$, where $\alpha = \mathrm{ORD}^M$.

Simpson's Theorem

Our proof uses a PA result of Simpson, applied to ZFC.

Theorem (Simpson 1974)

Let $\langle M, \in\rangle$ be a countable model of ZFC. Then, there is an M-generic class $U \subseteq M$ such that $\langle M, \in, U\rangle \models \mathrm{ZFC}(U)$ and every element of M is definable in $\langle M, \in, U\rangle$.

Proof.

Use $\mathbb{Q} = \mathrm{Add}(\mathrm{ORD}, 1)$. Enumerate sets of ordinals of M as $\langle a_n \mid n < \omega\rangle$. Enumerate dense classes $\langle D_n \mid n < \omega\rangle$, where D_n is defined by $\varphi_n(x, a_i)_{i<n}$. Define descending p_n so that p_{n+1} is the shortest extension of p_n in D_n, followed by a block listing a_n and end-marker. Resulting filter $U \subset \mathrm{ORD}$ is M-generic, but every a_n is definable in $\langle M, \in, U\rangle$. □

Pointwise Definable ZFC extensions

Theorem

Every countable model of ZFC *has a pointwise definable class forcing extension.*

Proof.

Suppose $M \models \mathrm{ZFC}$ is countable. First get U as in Simpson's theorem, so $M[U] \models \mathrm{ZFC}(U)$ and $\langle M, \in, U\rangle$ is pointwise definable. Now code U into the GCH pattern (or whatever coding), and also force $V = \mathrm{HOD}$. In final model $M[G]$, a class forcing extension, M and U are definable without parameters, so every ordinal is definable, so $M[G]$ is pointwise definable. □

Preserving large cardinals

The proof accommodates basically any large cardinal notion. In moving from M to $M[U]$ via Simpson's theorem, we performed class forcing to add a Cohen generic class of ordinals, which adds no new sets. And then we performed coding forcing and $V = \mathrm{HOD}$ forcing to $M[G]$, which can be done so as to preserve large cardinals.

Gödel-Bernays set theory

Gödel-Bernays set theory, or von Neumann-Gödel-Bernays set theory, is a second-order set theory that is conservative over ZFC.

Models have form $\langle M, S, \in\rangle$, where $\langle M, \in\rangle \models \mathrm{ZFC}$ and $S \subseteq P(M)$ is a family of *classes*, such that instances of Replacement and Separation are allowed to use finitely many class parameters (but not to quantify over classes). Plus, we have a global choice class.

GBC is conservative over ZFC since every ZFC model $\langle M, \in\rangle$ can be extended to a GBC model $\langle M, S, \in\rangle$ by adding a generic global well-ordering and letting S consist of the definable (with set parameters) classes of M relative to it.

Forcing works fine over GBC models.

Main GBC theorem

Theorem

Every countable model of Gödel-Bernays set theory has a pointwise definable extension, where every set and class is first-order definable without parameters.

Thus, even when we augment our ZFC model with a large family of non-definable classes, we may nevertheless make those classes (and all sets) definable in an extension of the model.

In the end, we have a pure ZFC model, while retaining all original classes, and making them all definable without parameters.

Special case: principal models

A GBC model $\langle M, S, \in\rangle$ is *principal* if there is $X \in S$ such that every class in S is definable in $\langle M, \in, X\rangle$.

Natural examples include the ZFC definability extensions; and the principal GBC models are closed under class forcing.

A non-example is obtained by successive forcing extensions $M, M[G_0], M[G_0, G_1], \cdots$, whose union is non-principal.

No model $\langle M, S, \in\rangle$ of Kelly-Morse set theory is principal as a GBC model, since KM proves the existence of a truth predicate relative to any one class.

For example, if κ is inaccessible, then $\langle V_\kappa, V_{\kappa+1}, \in\rangle$ is non-principal.

Pointwise definability for principle GBC models

Theorem

Every countable principal GBC model has a class forcing extension that is pointwise definable, in which every set and class is first-order definable without parameters.

Proof.

Suppose that $\langle M, S, \in\rangle$ is a principal GBC model with principal class $X \in S$. By Simpson's argument, add a generic class U such that every set is definable from U. Now force to code U and X into GCH pattern, as well as forcing $V = \text{HOD}$. Get forcing extension $M[G]$ in which X and U and hence M are definable. Thus, $M[G]$ is pointwise definable, and every class of S is definable in $M[G]$. □

Extending to a Principal GBC Model

In order to achieve the full GBC theorem, it suffices to show that every GBC model $\langle M, S, \in\rangle$ can be extended to a principal model.

The initial idea was to use meta-class forcing to code up all the classes in one class.

For example, if κ is inaccessible, then $\langle V_\kappa, V_{\kappa+1}, \in\rangle$ is extended to a principal GBC model $V_\kappa[G]$ by forcing with $\mathrm{Coll}(\kappa, 2^\kappa)$.

More generally, one can do something similar over KM models, adding a generic collapse class.

But this forcing idea does not seem to go through for GBC models...

Extending to a Principal GBC Model

Nevertheless...

Theorem (Kossak, Schmerl, indep. Friedman)

Every countable GBC *model* $\mathcal{M} = \langle M, S, \in\rangle$ *has an extension to a principal* GBC *model* $\mathcal{M}[Y] = \langle M[Y], S[Y], \in\rangle$.

Proof.

Kossak, Schmerl proved for PA, but argument extends to GBC. $\mathcal{M}[Y]$ is not a forcing extension. Enumerate the classes of ordinals as $\langle A_n \mid n < \omega\rangle$. Build descending sequence $\mathbb{Q}_0 \supset \mathbb{Q}_1 \supset \mathbb{Q}_2 \supset \cdots$ with each $\mathbb{Q}_n \subseteq 2^{<\mathrm{ORD}}$ a perfect tree. Each $\mathbb{Q}_n \cong \mathrm{Add}(\mathrm{ORD}, 1)$. Build $\mathbb{Q}_{n+1}$ so that any branch through $\mathbb{Q}_n$ is Σ_n-generic for $\mathbb{Q}_n$ and also codes A_n. Lengthen stem so that there is common branch $Y \subseteq \bigcap_n \mathbb{Q}_n$. Thus, $\mathcal{M}[Y] \models \mathrm{ZFC}(Y)$, and every A_n is definable. □

Pointwise definable GBC extensions

Putting it together...

Theorem

Every countable model of Gödel-Bernays set theory has a pointwise definable extension, where every set and class is first-order definable without parameters.

Proof.

If $\mathcal{M}$ is a countable GBC model, we may first extend it to $\mathcal{M}[Y]$ that is principal, and then apply the special case of principal GBC models to find a forcing extension $\mathcal{M}[Y][G]$ in which every set and class is definable without parameters. □

Summary Conclusion

Returning to the math-tea argument...

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Summary Conclusion

Returning to the math-tea argument...

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Summary Conclusion

Returning to the math-tea argument...

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Summary Conclusion

Returning to the math-tea argument. . .

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Summary Conclusion

Returning to the math-tea argument. . .

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Summary Conclusion

Returning to the math-tea argument. . .

- In seeking to define more and more reals, we are pushed to enlarge our context by considering larger structures or higher-order descriptions.
- In any fixed such context, there will be only countably many definable objects.
- The full context of definability-in-V is not actually expressible,
- and for all we know, every object in the universe *is* uniquely describable.
- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Thank you.

Joel David Hamkins
The City University of New York

http://jdh.hamkins.org