The mate-in-n problem of infinite chess is decidable

Joel David Hamkins

Visiting Fellow, Isaac Newton Institute
Cambridge, Spring 2012

The City University of New York
College of Staten Island of CUNY and
The CUNY Graduate Center

CiE 2012 Cambridge

This talk includes joint work with:

- Dan Brumleve, Joel David Hamkins, Philipp Schlicht, "The mate-in-n problem of infinite chess is decidable," to appear in LNCS proceedings volume for CiE 2012.
- C. D. A. Evans, J. D. Hamkins, W. H. Woodin, "Transfinite game values in infinite chess," in preparation.

A preprint of the mate-in-n paper is available on my web page:

http://jdh.hamkins.org

Infinite chess

Infinite chess is chess played on an infinite edgeless chess board, arranged like the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

The familiar chess pieces—kings, queens, bishops, knights, rooks and pawns—move about according to their usual chess rules, with bishops on diagonals, rooks on ranks and files and so on, with each player striving to place the opposing king into checkmate.

There is no standard starting configuration in infinite chess, but rather a game proceeds by setting up a particular position on the board and then playing from that position.

Clarifying the rules

Let me clarify a few of the rules as they relate to infinite chess.

- At most one king of each color
- There is no boundary, hence no pawn promotion
- There is no castling and no *en passant*
- Abandon the 50 move rule as limiting
- Infinite play is a draw
- We may abandon the three-fold repetition rule

The mate-in-n problem

There is a genre of chess problems, the mate-in-n problems

- White to mate in 2
- Black to mate in 3

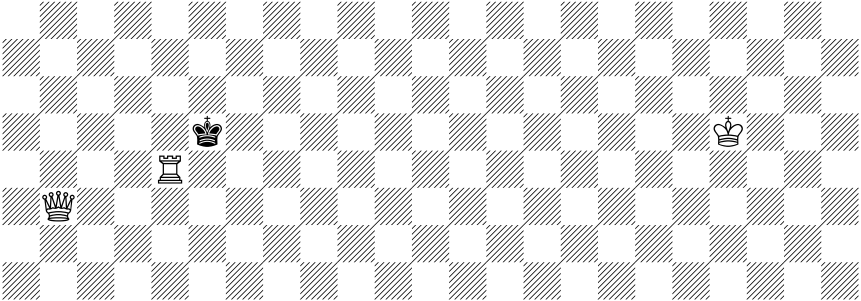
Consider the infinite chess analogue.

The mate-in-n problem

Given a finite position in infinite chess, can a designated player force a win in at most n moves?

A mate-in-12 problem

White to move on an infinite, edgeless board.



Can white force mate in 12 moves?

The mate-in-*n* problem: decidable?

Question

Is the mate-in-*n* problem computably decidable?

A naive formulation of white to mate-in-3, with white to move:

> *There is a white move, such that for every black reply,*
> *there is a white move, such that for every black reply,*
> *there is a white move, which delivers checkmate.*

$$\exists w_1 \forall b_1 \exists w_2 \forall b_2 \exists w_3 \cdots$$

Very high arithmetical complexity, $2n$ alternating quantifiers.

Infinite game tree

In finite $n \times n$ chess, one can search the entire game tree.

In infinite chess, the game tree is not only infinite, but infinitely branching.

One cannot expect to search it, even to finite depth. The naive, brute-force-search manner of deciding the mate-in-n problem is inadequate.

Nevertheless, C. D. A. Evans and I conjectured it was decidable.

The infinitary mate-in-n problem is decidable

Theorem (Brumleve, Hamkins, Schlicht)

The mate-in-n problem of infinite chess is decidable.

Furthermore, there is a computable strategy for optimal play from such a position.

The first-order structure of chess $\mathfrak{Ch}$

To prove the theorem, we introduce the first-order structure of chess $\mathfrak{Ch}$.

- Domain consists of all finite chess positions: description of pieces, locations, whether captured, whose turn.
- Relations for the fundamental chess concepts: $\mathrm{OneMove}(p, q)$, $\mathrm{WhiteInCheck}(p)$, $\mathrm{WhiteMated}(p)$, and so on.

For a given finite list A of piece types, consider substructure $\mathfrak{Ch}_A$ of positions with just those pieces.

The mate-in-n problem is expressible in the theory of the structure of chess $\mathfrak{Ch}$, using $2n$ many alternating quantifiers.

The structure of chess

Key Lemma

For any finite list A of chess-piece types, the structure of chess $\mathfrak{Ch}_A$ is automatic.

An *automatic* structure is one whose domain and relations constitute a regular language, one that can be recognized by a finite automata.

Fact (Khoussainov, Nerode)

The theory of any automatic structure is decidable.

Use pumping lemma idea to reduce infinitary to finitary search—just like the equivalence relation on chess moves.

Conclusion: the mate-in-n problem is decidable. □

Alternative proof via Presburger arithmetic

For a fixed list A of piece types, the structure of chess $\mathfrak{Ch}_A$ is definable inside Presburger arithmetic $\langle \mathbb{N}, +, < \rangle$.

The key points are that bishops, queens, rooks move on straight lines, which are definable in Presburger arithmetic, and new pieces are not introduced during play.

Since Presburger arithmetic admits elimination of quantifiers, the theory is decidable. Hence, the theory of the structure of chess is decidable. So the mate-in-n problem is decidable.

The winning-position problem still open

Our theorem does not settle the decidability of winning-position problem.

Question (Richard Stanley)

Is the winning-position problem for infinite chess decidable?

This question remains open, and is MathOverflow-hard.

The point is that a player may have a winning strategy from a position, but it is not mate-in-n for any n. So the winning positions is not even c.e., indeed, it is not even clear whether it is arithmetic, or even hyperarithmetic.

These positions are precisely the positions with transfinite game value.

Requiring computable play matters

Consider whether it matters to require the players to play according to a computable procedure.

Theorem

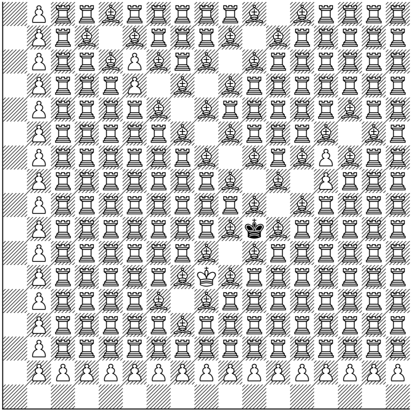
There is a computable position in infinite chess such that:

1. *The position is drawn.*
2. *But white has a computable strategy defeating all computable strategies for black.*

So yes, it matters.

A position where computable play matters

Fix a computable tree $T \subseteq 2^{<\omega}$ with no computable branch.



White can force black through the tree via zugzwang.

Alternative, with filled-in channels

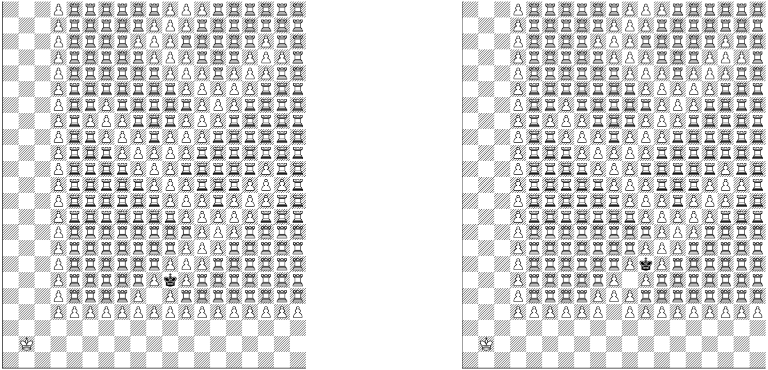
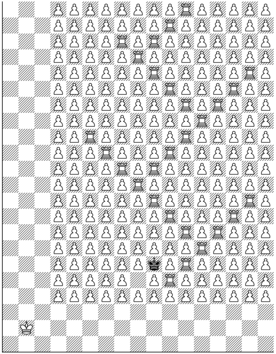


Figure: Starting position at left; position after pawn check and response at right.

Simplified version



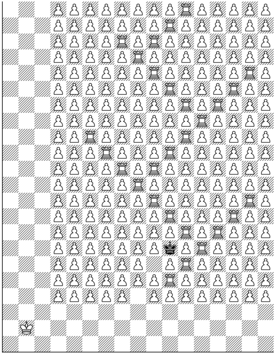
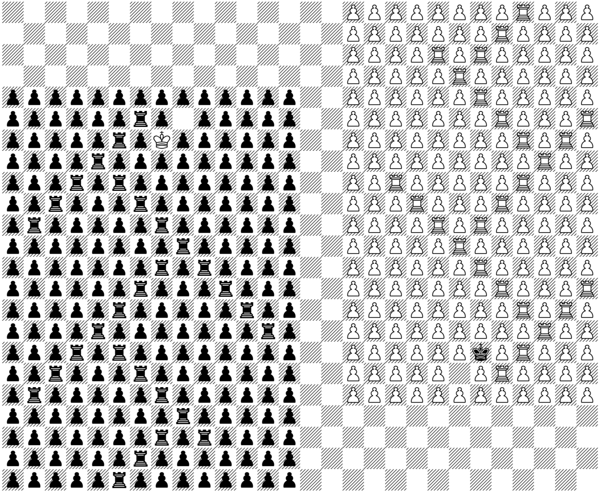


Figure: White forces black to climb the tree via successive pawn checks; response shown at right

Drawn position, but won for first player, if play is computable.



Game value

An *open* game is a game that, if won, is won after finitely many moves.

Such games support a theory of transfinite *game values*, generalizing the mate-in-n idea.

The ordinal value of a position measures the distance for white to win.

The omega one of chess

We define $\omega_1^{\mathfrak{Ch}}$ to be the supremum of the game values of the finite positions of infinite chess.

Question

What is the value of $\omega_1^{\mathfrak{Ch}}$?

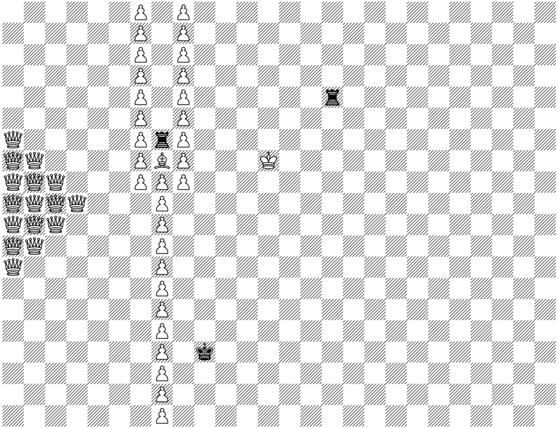
Current known lower bounds: $\omega \cdot n + k$.

Cory Evans and I have candidates for value ω^2.

Theorem (Andreas Blass, independently, Philip Welch)

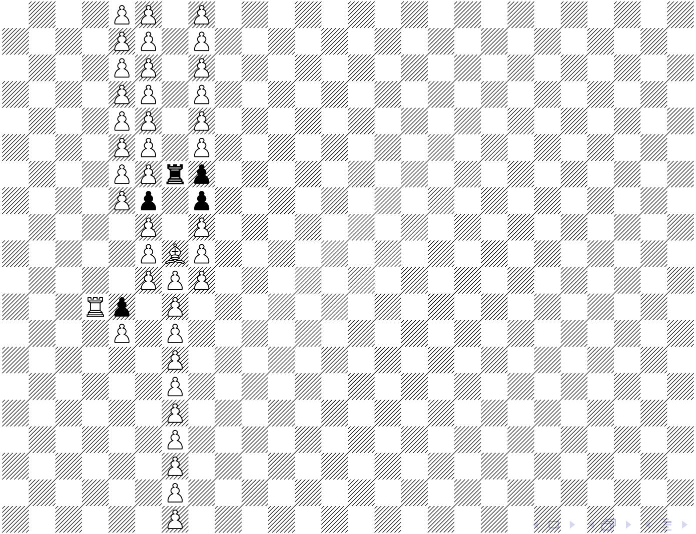
The omega one of chess $\omega_1^{\mathfrak{Ch}}$ is at most ω_1^{ck}. Indeed, if a designated player has a winning strategy from a finite position p, then there is such a strategy with hyperarithmetic complexity.

Releasing the hordes, value ω^2



White aims to open the portcullis.

Basic lock and key



Infinite 3D chess

Three-dimensional chess goes back a century. Dr. Ferdinand Maack invented Raumschach $8 \times 8 \times 8$ chess in 1907, eventually settling on $5 \times 5 \times 5$ as providing a better game, and founding a Hamburg club in 1919.

Purists play with no board.

Spock and Kirk played tri-chess on a physical 3D board.

Consider 3-D infinite chess.

The omega one of 3D infinite chess

Theorem (Evans, Hamkins, Woodin)

The ω_1 of three-dimensional infinite chess with infinite positions is true ω_1.

That is, every countable ordinal arises as the game value of a position in three-dimensional infinite chess.

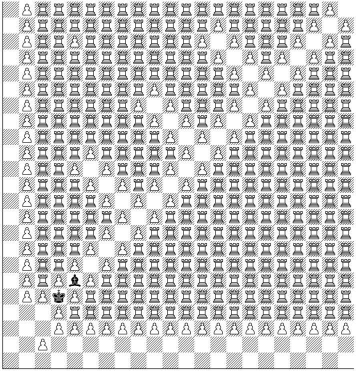
The ω_1 of 3D infinite chess

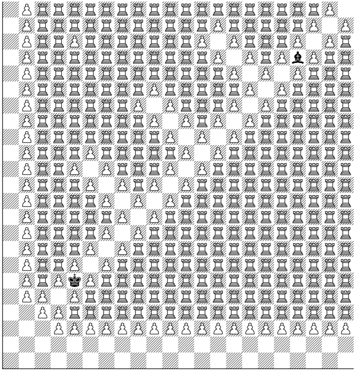
The basic idea is to code an arbitrary well-founded tree $T \subseteq \omega^\omega$ as a position in infinite chess.

The difficulty is that the tree must be infinite-branching to achieve large ranks. How to force black to make a choice?

Branching node layer in 3D infinite chess

Black has mate-in-2 elsewhere. Play is tightly forced.





Thank you.

Joel David Hamkins
The City University of New York

http://jdh.hamkins.org

http://cantorsattic.info