The multiverse perspective in set theory

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The central discovery in set theory

To my way of thinking, the central set-theoretic discovery of the last fifty years is the discovery that there is an enormous diversity of set-theoretic worlds.

A principal set-theoretic activity, nowadays, is the construction of yet another set-theoretic world, using the methods of forcing, ultrapowers, relative definability and all our other methods. We often build these worlds to exhibit specific desirable or unusual features or to have certain specific relations to other worlds.

Set-theoretic experience

Set theorists have a truly rich experience in these worlds, exhaustively exploring them, investigating how they relate to one another and studying the methods used to construct them.

These diverse worlds instantiate various concepts of set, which we have come to know as robust and legitimate, each suitable as a foundation for the mathematics that takes place inside of the corresponding world.

To my way of thinking, the traditional Universe view in set theory has largely failed to explain this fundamental phenomenon, the phenomenon of diverse set-theoretic possibility.

The set-theoretic multiverse

The multiverse perspective, in constrast, embraces this mathematical experience, and takes the phenomenon as a fundamental aspect of mathematical ontology.

On the multiverse view, we do not have a single absolute concept of set, but rather many distinct and often closely related concepts of set.

On the multiverse view, there are many set-theoretic worlds.

Each may be regarded as just as “real” as the universe is on the universe view.

Thus, the multiverse perspective separates the issue of realism from the issue of uniqueness of the universe.

The analogy with geometry

The analogy with geometry is strong.

In geometry, the classical notions of point, line, plane, space, which were once thought to describe unique absolute geometrical concepts, have splintered into an array of thousands of distinct yet related geometrical concepts, each giving rise to a corresponding Euclidean or non-Euclidean geometrical world.

In set theory, what was thought to be a unique absolute background concept of set, has splintered into a diverse array of distinct yet closely related set theoretic concepts, each giving rise to a corresponding set-theoretic world.

The multiverse view on CH

On the multiverse view, the Continuum Hypothesis is a settled question. It is incorrect to describe CH as an open question.

CH is solved not by the independence result, but rather solved by the extensive knowledge we have gained about how CH behaves in the multiverse, about how we may force CH or $\neg\mathrm{CH}$ while preserving this or that other desirable feature, and about how we may or may not find CH in an inner model of a certain kind.

Of course, not every question about CH is solved, but the main and most important facts about CH are deeply understood.

Those facts are what constitute the answer to the question of CH.

Dream solution is now impossible

The Dream Solution

1. Find a new natural set-theoretic truth Ψ, a missing axiom.
2. Prove that Ψ settles CH.

I have argued:

(i) our set-theoretic experience in diverse set-theoretic worlds shows that the corresponding concepts of set in those worlds are perfectly robust and satisfactory as concepts of set;

(ii) thus, whatever philosophical support might be made for Ψ, if we know Ψ violates CH or $\neg\mathrm{CH}$, then our experience in the contrary set-theoretic contexts will prevent us from accepting Ψ as a natural set-theoretic truth;

(iii) and so the proposed dream solution will fail.

The multiverse view on $V = L$

The axiom of constructibility $V = L$ is often rejected on maximality grounds: L seems restrictive.

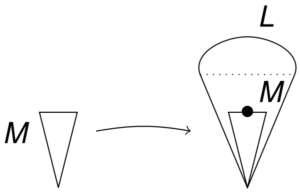
Much of this perspective relies on the concept of ordinals as a “finished” totality, so that larger universe concepts can grow only outward from L, never upward.

On the multiverse view, no universe knows all the ordinals, and every universe may become countable inside a taller, better universe.

On this view, every set-theoretic world can be extended to a world in which $V = L$ holds.

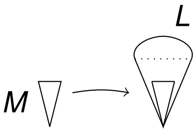
L is compatible with strength

- (Schoenfield) L and V have transitive models of exactly the same strong theories.
- Every countable transitive set is a countable transitive set in the well-founded part of an ω-model of $V = L$.

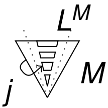


Extending models to L

3 (Barwise) Every countable model of ZF has an end-extension to a model of $\mathrm{ZFC} + V{=}L$.



4 (Hamkins) Every countable model of set theory M is isomorphic to a submodel of its own constructible universe $j : M \to L^M$.



$$x \in y \quad \longleftrightarrow \quad j(x) \in j(y)$$

L as rewarder of the patient

Thus, on the multiverse view, L becomes a rewarder of the patient.

The ordinals are an unfinished business, and as new ordinals come in, sets we thought were not constructible are now seen to be constructible.

With such an upwardly extensible concept of ordinal, L does not seem so restrictive after all.

The multiverse view on definite arithmetic truth

Many mathematicians regard the natural numbers $0, 1, 2, \ldots$ as having a privileged mathematical existence, a Platonic realm in which assertions have definite, absolute truth values, independently of our ability to prove or discover them.

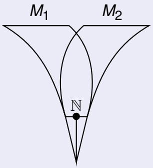
Further, the definite nature of arithmetic truth is often seen as a consequence of the definiteness of the structure of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ itself.

I have argued against this inference—the theory of truth for a structure is a higher-order ontological claim, and one cannot deduce a definite nature for truth in a structure just from the definite nature of the structure itself.

Definiteness of objects $\not\to$ Definiteness of truth

Hamkins & Yang

There are models of set theory that agree on the structure of the natural numbers, but disagree on arithmetic truth.



$M_1, M_2 \models \mathrm{ZFC}$

$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$

M_1 believes $\mathbb{N} \models \sigma$

M_2 believes $\mathbb{N} \models \neg\sigma$

Similar phenomenon with $\mathbb{R}$ and projective truth, H_{ω_2} and so on.

Conclusion: definiteness of truth does not seem to follow from definiteness of structure.

Thank you.

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