Superstrong and other large cardinals are never Laver indestructible

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Mathematics, Philosophy, Computer Science

ASL Annual Meeting
Special session in memory of Richard Laver

Richard Laver, 1942-2012

Figure : Richard Laver, 1974, photo by George Bergman

The main result on which I shall speak is deeply connected with two topics where Richard Laver made fundamental contributions.

- Large cardinal indestructibility phenomenon
- Ground model definability theorem

Main Theorem

Theorem (Bagaria, Hamkins, Tsaprounis, Usuba)

Superstrong and many other kinds of large cardinals are never Laver indestructible. Indeed, they are all superdestructible.

For example, after adding a Cohen subset to κ, it cannot be superstrong, weakly superstrong, and so on.

Joint work with Joan Bagaria, Konstantino Tsaprounis and Toshimichi Usuba.

"Superstrong and other large cardinals are never Laver indestructible," to appear in Archive for Mathematical Logic (special issue in honor of Richard Laver).

http://jdh.hamkins.org/superstrong-never-indestructible.

Laver indestructibility phenomenon

Begins with Laver's seminal result:

Theorem (Laver, 1978)

If κ is a supercompact cardinal, then there is a forcing extension $V[G]$, over which the supercompactness of κ is indestructible by any subsequent $<\kappa$-directed closed forcing.

- Laver preparation unified earlier special case instances
- Introduced the Laver diamond principle, now generalized to many large cardinals
- Large cardinal indestructibility is now pervasive

Universal indestructibility

Theorem (Hamkins, Apter 1999)

Given a high-jump cardinal, there is a transitive model with a supercompact cardinal exhibiting *universal indestructibility*:

> *Every supercompact cardinal, every θ-supercompact cardinal, every measurable cardinal, every Ramsey cardinal, every indescribable cardinal, every weakly compact cardinal and so on, is Laver indestructible.*

The proof uses trial-by-fire forcing. At stage γ, destroy as much of γ as possible. Whatever survives is therefore indestructible.

Universal indestructibility is inconsistent with two or more supercompact cardinals.

Small forcing ruins indestructibility

Theorem (Hamkins, Shelah 1998)

No supercompact cardinal remains indestructible after nontrivial small forcing.

A new slick proof of the main case:

Apter noticed that if κ is an indestructible supercompact cardinal, then $V_\kappa \subseteq \mathrm{HOD}$ via the continuum coding axiom CCA, namely, every set in V_κ is coded (unboundedly often) into the GCH pattern below κ. Code above κ and apply reflection.

Small forcing adds a set that is not coded unboundedly often. So κ is no longer indestructible. $\square$

The original argument works more generally, with measurable, partially supercompact, partially strong...

Ground model definability theorem

Theorem (Laver 2007, Woodin)

For any forcing extension

$$V \subseteq V[G]$$

where $G \subseteq \mathbb{P} \in V$ is V-generic, the ground model V is a definable class in the extension $V[G]$.

This theorem answers a question that could have been asked over forty years earlier.

I view this theorem as absolutely fundamental to a deeper understanding of the nature of forcing.

Stronger results and further developments

Laver adopted my proof of ground-model definability, using

Definition (Hamkins)

1. $V \subseteq W$ has δ *cover property* if every $A \subseteq V$ with $A \in W$, $|A|^W < \delta$ is covered $A \subseteq B$ by some $B \in V$ with $|B|^V < \delta$.
2. $V \subseteq W$ has δ *approximation property* if every $A \subseteq V$ with $A \in W$ and all small approximations $A \cap a \in V$, whenever $|a|^V < \delta$, is already in the ground model $A \in V$.

Key Lemma

If $\mathbb{P}$ is absolutely δ-c.c. and nontrivial and $\Vdash \dot{\mathbb{Q}}$ is $<\!\delta$-closed, then $\mathbb{P} * \dot{\mathbb{Q}}$ has the δ-approximation and cover properties over ground model.

Generalized ground-model definability

Theorem (Hamkins)

If $V \subseteq W$ has the δ-approximation and δ-cover properties and correct δ^+, then V is definable in W.

Essentially, for sufficiently closed θ, the rank initial segment V_θ is the unique subset of W_θ with δ-approximation and cover properties and the correct ${}^{<\delta}2$.

So we can define V in W using parameter $r = ({}^{<\delta}2)^V$.

This theorem covers all set forcing, but also many common instances of class forcing and other non-forcing extensions.

Upon learning of Laver's theorem, Jonas Reitz and I formulated

Definition (Hamkins,Reitz)

The *Ground Axiom* is the assertion that the universe V has no nontrivial grounds.

That is, $V \models \mathrm{GA}$ if there is no transitive inner model $W \models \mathrm{ZFC}$ such that $V = W[G]$ for some nontrivial W-generic filter $G \subseteq \mathbb{P} \in W$.

- GA is first-order expressible.
- Natural models of GA are highly-structured: L, $L[0^\sharp]$, $L[\vec{E}]$,...
- Meanwhile, GA follows from CCA, which is forceable by class forcing, while preserving any V_θ.
- (Hamkins,Reitz,Woodin) GA is consistent with $V \neq \mathrm{HOD}$.

The grounds form a parameterized family

Theorem

There is a parameterized family $\{ W_r \mid r \in V \}$ such that

1. *Every W_r is a ground of V and $r \in W_r$.*
2. *Every ground of V is W_r for some r.*
3. *The relation "$x \in W_r$" is first order.*

This reduces second-order statements about grounds to first-order statements about parameters.

For example, the ground axiom asserts $\forall r\ W_r = V$.

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Set-theoretic Geology

The ground model definability theorem is the first theorem of *set-theoretic geology*, the study of the structure of all the ground models of the universe and its forcing extensions. (Fuchs, Hamkins, Reitz)

- Bedrock is a minimal ground; solid bedrock is least ground.
- Bottomless models.
- Downward directed grounds hypothesis: the grounds are downward directed. (Open!)
- Mantle = intersection of all grounds.
- generic mantle = $\bigcap$ grounds of all forcing extensions.
- gM is the largest forcing-invariant class.
- Ancient paradise. Should mantle be highly-structured?
- Every model of ZFC is mantle, generic mantle of another model.

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Superstrong and many other kinds of large cardinals are never Laver indestructible. Indeed, they are all superdestructible.

For example, after adding a Cohen subset to κ, it cannot be superstrong, weakly superstrong, and so on.

The proof makes detailed use of the ground model definability analysis between V and $V[G]$.

For example, the assertion $V = W_r[G]$, obtained by forcing over W_r with the W_r-generic filter $G \subseteq \mathbb{Q} \in W_r$, has complexity $\Pi_2(\mathbb{Q}, G, r)$,

Main Theorem

1. Superstrong cardinals are never Laver indestructible.
2. Almost huge, huge, superhuge and rank-into-rank cardinals...
3. Extendible, 1-extendible and even 0-extendible cardinals...
4. Uplifting, pseudo-uplifting, weakly superstrong, superstrongly unfoldable, strongly uplifting cardinals...
5. Σ_n-reflecting and Σ_n-correct cardinals ($n \geq 3$)...
6. Indeed, the Σ_3-extendible cardinals are never indestructible.

Each is superdestructible: if κ exhibits any of these large cardinal properties, with target θ, then after any nontrivial strategically $<\kappa$-closed forcing $\mathbb{Q} \in V_\theta$, the cardinal κ will exhibit none of them (with target θ or larger).

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Proof is motivated by the question

Question (Hamkins)

After forcing with $\mathrm{Add}(\kappa, 1)$, does the cardinal κ become definable?

More specifically, If $G \subseteq \kappa$ is V-generic, then in $V[G]$ can we rule out the existence of an inner model $W \subseteq V[G]$ with $W[H] = V[G]$, where $H \subseteq \gamma$ is W-generic for $\mathrm{Add}(\gamma, 1)$ for some other cardinal γ?

Theorem (Bagaria, Hamkins, Tsaprounis, Usuba)

The answer to the question is yes.

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Theorem (Bagaria, Hamkins, Tsaprounis, Usuba)

The answer to the question is yes.

Theorem

If $\delta < \kappa$ are inaccessible cardinals, then V is not simultaneously a forcing extension over one ground model by adding a subset to κ and over another ground model adding a Cohen subset to δ.

Proof. Suppose $V = M[G] = N[H]$, where $G \subseteq \kappa$ and $H \subseteq \delta$ are the Cohen sets. Let $G_1 \subseteq \kappa$ be another Cohen subset of κ, and form $V[G_1] = M[G * G_1] = N[H][G_1]$. The latter extension has the δ^+-approximation and cover properties, and so $N = W_r^{V[G_1]}$, where $r = ({}^{<\delta}2)^N$, and $V[G_1] = N[H][G_1]$ satisfies the assertion,

> *"The universe is a forcing extension of the ground W_r defined by parameter r, by $\mathrm{Add}(\delta, 1)$ followed by $\mathrm{Add}(\kappa, 1)$."*

Parameters are in M, so $M[G]$ also satisfies assertion, giving $M[G] = W_r^V[H_0][G_0]$. Thus, $V[G_1] = W_r^V[H_0][G_0][G_1] = N[H][G_1]$. The definability theorem now gives $W_r^V = N$. So $N[H] \subseteq N[H_0]$ and thus $M[G] \subseteq N[H_0]$, so $G_0 \in N[H_0]$, contradiction. QED

Main theorem proof sketch

The main theorem now follows as a consequence of the definability lemma.

Suppose that κ is superstrong in $V[G]$, where $G \subseteq \kappa$ is V-generic for $\mathrm{Add}(\kappa, 1)$. Consider superstrongness embedding $j : V[G] \to M[j(G)]$. Note that $M[j(G)]_{j(\kappa)} = V[G]_{j(\kappa)} = V_{j(\kappa)}[G]$. This is a model of ZFC in which we have just added a Cohen subset to κ. So κ is definable in $M[j(G)]_{j(\kappa)}$, but $\kappa \notin \mathrm{ran}(j)$, a contradiction.

For the other large cardinals, a more refined argument can be pushed through.

Strongest version

Theorem (Bagaria, Hamkins, Tsaprounis, Usuba)

Suppose that $V_\kappa \prec_{\Sigma_2} V_\lambda$ for some $\lambda \geq \eta$ and that $G \subseteq \mathbb{Q}$ is V-generic for nontrivial strategically $<\kappa$-closed forcing $\mathbb{Q} \in V_\eta$. Then for all $\theta \geq \eta$,

$$V_\kappa = V[G]_\kappa \not\prec_{\Sigma_3} V[G]_\theta.$$

Alternative slick proof

Observation (Apter)

If κ is a Laver indestructible supercompact cardinal, then $V_\kappa \subseteq \mathrm{HOD}$.

In fact, if the Σ_2-correctness of κ is indestructible, then $V_\kappa \models$ CCA, every set of ordinals is coded into the GCH pattern.

That is a Π_3 assertion that is true in V_κ.

But if we add a Cohen subset $G \subseteq \kappa$, then in $V[G]$, the set G is not coded into the GCH pattern. So there can be no θ with $V_\kappa = V[G]_\kappa \prec_{\Sigma_3} V[G]_\theta$. So κ is not Σ_3-extendible in $V[G]$.

Thus, κ is not superstrong, weakly superstrong, superstrongly unfoldable, and so on.

Thank you.

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