

Large cardinals need not be large in HOD

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Some of this material is joint work with Cheng Yong and Sy-David Friedman. See our article at:

jdh.hamkins.org/large-cardinals-need-not-be-large-in-hod

Some is also joint with Moti Gitik.



Motivating Questions

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- To what extent does the existence of large cardinals in V imply the existence of large cardinals in HOD?

Various forcing constructions suggest a generally negative answer.

Meanwhile, a few positive results poke through.

Warming up

Let's begin with an easy case.

Theorem

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Proof idea:

- Start in $V \models \kappa$ is measurable.
- Force to kill measurability (but not permanently dead).
- Code this situation into HOD.
- Finally, resurrect measurability via homogeneous forcing.

Theorem

If κ is a measurable cardinal, then there is a forcing extension in which κ is measurable, but not measurable in HOD.

Proof.

Start with $V \models \kappa$ is measurable, $2^\kappa = \kappa^+$.

Force to add Cohen subsets at every inaccessible $\gamma < \kappa$, but not (yet) at stage κ . It follows κ is not measurable in resulting extension $V[G]$.

Let $H \subseteq \mathbb{R}$ force to code $P(\kappa)^{V[G]}$ into the GCH pattern high up. Finally, add Cohen set $g \subseteq \kappa$ over $V[G][H]$.

The usual lifting arguments show κ is measurable in $V[G][g]$. And hence also in $V[G][H][g]$.

But κ is not measurable in $\text{HOD}^{V[G][H][g]}$, since by homogeneity this is $\subseteq V[G]$ and has $P(\kappa)^{V[G]}$, but κ has no measure there. □

Controlling HOD

In fact, we could have ensured that $\text{HOD}^{V[G][H][g]} = V[G]$ exactly, in the previous argument.

Instead of coding just $P(\kappa)^{V[G]}$ into the GCH pattern, we could have instead let H code all of $V[G]$ into the GCH pattern. For example, use class product in $V[G]$ of lottery sum of GCH or not GCH at successive cardinals.

The end result would be $V[G][H][g]$, where κ is measurable, but $\text{HOD}^{V[G][H][g]} = V[G]$, where κ is not measurable.

Killing more

Lemma (Kunen 1978)

*For κ inaccessible, there is strategically $<\kappa$ -closed forcing \mathbb{S} adding a weakly homogeneous κ -Suslin tree T , such that $\mathbb{S} * \dot{T}$ is equivalent to $\text{Add}(\kappa, 1)$.*

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Application: Start with large cardinal κ , made indestructible by $\text{Add}(\kappa, 1)$. Add the κ -Suslin tree T , so $V[T] \models \kappa$ not weakly compact. Now force with T to add a branch b , so $V[T * b]$ is a Cohen extension by $\text{Add}(\kappa, 1)$, and so κ is large in $V[T][b]$.

So a non-weakly compact cardinal κ can become extremely large after killing a κ -Suslin tree.

Alternative method

For κ Mahlo, consider the forcing that first adds a *stationary non-reflecting set*, a stationary set $S \subseteq \kappa$ of regular cardinals, such that $S \cap \gamma$ is not stationary in γ , for any $\gamma < \kappa$.

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Now, add a club $C \subseteq \kappa$ disjoint from S . The combined forcing $S * C$ is equivalent to $\text{Add}(\kappa, 1)$. The club forcing is locally homogeneous.

Rude error + apology = everything's Fine

In $V[S]$, the cardinal κ is not weakly compact, but it is large again in $V[S][C]$.

Homogeneous forcing

Definition

\mathbb{Q} is *locally homogenous*, if any two conditions p, q have respective strengthenings p', q' with $\mathbb{Q} \upharpoonright p' \cong \mathbb{Q} \upharpoonright q'$.

Lemma (Folklore)

If \mathbb{Q} is a locally homogeneous notion of forcing and $G \subseteq \mathbb{Q}$ is V -generic, then $\text{HOD}^{V[G]} \subseteq \text{HOD}(\mathbb{Q})^V$.

In particular, if $\mathbb{Q} \in \text{HOD}^V$, then $\text{HOD}^{V[G]} \subseteq \text{HOD}^V$.

To prove the lemma, any $\pi : \mathbb{Q} \upharpoonright p \cong \mathbb{Q} \upharpoonright q$ extends to the names, so that $p \Vdash \varphi(\tau)$ iff $\pi(p) \Vdash \varphi(\tau^\pi)$. Since the check-names are invariant, it follows that all conditions must agree on $\varphi(\check{\alpha}, \check{\beta})$. So if $x = \{ \alpha < \gamma \mid \varphi(\alpha, \beta) \}$ in $V[G]$, then $x = \{ \alpha < \gamma \mid \mathbf{1} \Vdash \varphi(\check{\alpha}, \check{\beta}) \}$ in V .

Not weakly compact in HOD

Theorem

If κ is measurable, then there is a forcing extension with κ measurable, but not weakly compact in HOD.

Proof.

Assume κ measurable in V and indestructible by $\text{Add}(\kappa, 1)$.

Force via \mathbb{S} to add homogeneous κ -Suslin tree T . Force over $V[T]$ to code T into the GCH pattern up high. Force over $V[T][H]$ to add a branch b through T . Consider $V[T][H][b]$.

Note κ is measurable in $V[T][b]$, since this is same as $\text{Add}(\kappa, 1)$; hence measurable in $V[T][b][H] = V[T][H][b]$.

But $T \in \text{HOD}^{V[T][H][b]} \subseteq V[T]$, where T has no κ -branch. So κ not weakly compact in HOD. □

Supercompact, not weakly compact in HOD

Theorem

If κ is supercompact, then there is a forcing extension with κ supercompact, but not weakly compact in HOD.

Proof.

Similar. Assume κ supercompact and Laver indestructible in V .

Add homogeneous κ -Suslin tree T , and then code T into GCH pattern at next κ many cardinals. Finally, add a branch b through T . Consider $V[T][H][b] = V[T][b][H]$. This is $<\kappa$ -directed closed, so κ is still supercompact.

But $T \in \text{HOD}^{V[T][H][b]} \subseteq V[T]$, where T has no κ -branch, so κ is not weakly compact in HOD. □

Other large cardinals

Similar methods work with many other large cardinals.

Theorem

If κ has any of the following properties

Local: *weakly compact, indescribable, totally indescribable, Ramsey, strongly Ramsey, measurable, θ -tall, θ -strong, Woodin, θ -supercompact, superstrong, n -superstrong, ω -superstrong, λ -extendible, almost huge, huge, n -huge, rank-into-rank (I_0 , I_1 and I_3);*

Global: *unfoldable, strongly unfoldable, tall, strong, supercompact, superhuge, and many others.*

Then there is a forcing extension preserving this property for κ , but where κ is not weakly compact in HOD.

Proper class of measurable cardinals

Let us turn now to having a proper class of large cardinals.

Theorem

There is a class forcing notion \mathbb{P} such that

- 1 All measurable cardinals of the ground model are preserved and no new measurable cardinals are created.*
- 2 There are no measurable cardinals in the HOD of the extension.*
- 3 The measurable cardinals of the extension are not weakly compact in the HOD of the extension.*

We may also ensure GCH in the extension and its HOD.

Goal: preserve all measurables; make not w. compact in HOD.

Start in V with proper class of measurables. Force to \bar{V} , where every set is coded into GCH pattern at δ^{+++} for $\delta = \beth_\delta$, and all measurables κ indestructible by $\text{Add}(\kappa, 1)$, and $2^\kappa = \kappa^+$.

Let $\mathbb{P} = \prod_{\kappa} \text{Add}(\kappa, 1) * \mathbb{R}(\kappa)$, where $\text{Add}(\kappa, 1)$ is viewed as $\mathbb{S}_\kappa * \dot{T}_\kappa$ and $\mathbb{R}(\kappa) \in V[\dot{T}_\kappa]$ forces to code T_κ into GCH at next κ many cardinals. Consider extension $\bar{V}[G]$.

Every measurable cardinal is preserved. Suffices to see κ is measurable in $\bar{V}[G_\kappa][g_\kappa]$. By indestructibility, it is measurable in $V_1 = \bar{V}[g_\kappa]$. Now use Kunen-Paris argument. Fix $j : V_1 \rightarrow M_1$ with κ not measurable in M_1 ; lift to $j : V_1[G_\kappa] \rightarrow M_1[j(G_\kappa)]$.

No new measurable cardinals are created since this forcing has closure point.

Claim $\text{HOD}^{\bar{V}[G]} = \bar{V}[\vec{T}]$. Inclusion \subseteq by homogeneity. \supseteq since everything is coded. No measurable cardinals in $\bar{V}[\vec{T}]$. Indeed, T_κ is κ -Suslin in $V[\vec{T}]$. Higher forcing does not add branch, and $\prod_{\gamma < \kappa} \mathbb{S}_\gamma$ is productively κ -c.c., so cannot add a branch. \square

Killing all weakly compact cardinals

Note that we arranged merely that the measurable cardinals not themselves weakly compact in HOD, but we did not kill all the weakly compact cardinals of HOD.

In fact, one cannot kill all the weakly compact cardinals of HOD, if there is to be a measurable cardinal.

Supercompacts, not weakly compact in HOD

Theorem

There is a class forcing notion \mathbb{P} forcing that

- 1** *All supercompact cardinals are preserved and no new supercompact cardinals are created.*
- 2** *There are no supercompact cardinals in the HOD of the extension.*
- 3** *The supercompact cardinals of the extension are not weakly compact in the HOD of the extension.*

One may also ensure that the GCH holds in the extension and its HOD.

Proof

Goal: many supercompacts, not weakly compact in HOD.

Proof. Start with proper class of indestructible supercompacts in V . It follows that $V = \text{HOD}$.

Let $\mathbb{P} = \prod_{\kappa} \text{Add}(\kappa, 1) * \mathbb{R}(\kappa)$, as before, but only for κ supercompact. View $\text{Add}(\kappa, 1)$ as $\mathbb{S}_{\kappa} * \dot{T}_{\kappa}$, and $\mathbb{R}(\kappa)$ codes T_{κ} into GCH pattern. Consider extension $V[G]$.

Each $T_{\kappa} \in \text{HOD}^{V[G]} \subseteq V[\vec{T}]$. So no supercompact κ is weakly compact in $\text{HOD}^{V[G]}$.

But every supercompact κ is preserved to $V[G]$. Forcing above is $<_{\kappa}$ -directed closed. Use Kunen-Paris argument to handle forcing up to κ . \square

Strong cardinals

Theorem

There is a class forcing notion \mathbb{P} forcing that

- 1** *All strong cardinals of the ground model are preserved and no new strong cardinals are created.*
- 2** *There are no strong cardinals in the HOD of the extension.*
- 3** *The strong cardinals of the extension are not weakly compact in the HOD of the extension.*

Indeed, the method works with many other kinds of large cardinals, including strongly unfoldable cardinals, strongly Ramsey cardinals, tall cardinals and many others.

Weakly compact

Theorem

There is a class forcing notion \mathbb{P} , preserving all weakly compact cardinals, creating no new weakly compact cardinals, and forcing that there are no weakly compact cardinals in the HOD of the extension.

Questions

- Can there be a strong cardinal (or a proper class), but no measurable cardinal in HOD?
- Can there be an extendible cardinal that is not weakly compact in HOD?
- Can there be a supercompact cardinal, which is strongly compact but not supercompact in HOD?
- Can there be $V \subseteq W$ with a measurable cardinal in W , but no weakly compact cardinals in V ?
- Can there be a supercompact cardinal in a forcing extension $V[G]$, if there are no measurable cardinals in V ?

A positive result pokes through

Theorem (Woodin)

If there is a supercompact cardinal, then there are measurable cardinals in HOD.

Woodin says:

Suppose κ is supercompact, and consider two cases. First, if the HOD conjecture fails, then all sufficiently large regular cardinals are measurable in HOD, witnessed by the club filter of V on a stationary set S in HOD. Second, otherwise, the HOD conjecture holds. In this case, one proves that for each $\alpha < \kappa$ there is a cardinal δ with $\alpha < \delta < \kappa$ that is α -extendible in HOD. Here, one uses the proof of the HOD dichotomy theorem since κ may not be HOD-supercompact in V (if κ is HOD-supercompact in V then κ itself is supercompact in HOD; and every extendible cardinal κ is HOD-supercompact).

Measurable gives weakly compacts in HOD

Theorem (Gitik)

If there is a measurable cardinal, then there are many weakly compact cardinals in HOD.

Proof.

Suppose κ measurable, with $j : V \rightarrow M$. If κ is not weakly compact in HOD^M , then there is κ -tree $T \in \text{HOD}^M$ with no κ -branch. Consider $j(T)$, which has nodes on κ^{th} level, and which are in HOD. Contradiction. □

Actually, it suffices that κ is subtle.

Superstrongly unfoldable cardinals

Definition (Hamkins, Johnstone)

- A cardinal κ is *strongly uplifting* if for every $A \subseteq \kappa$ there are arbitrarily large γ and $A^* \subseteq \gamma$ with $\langle V_\kappa, \in, A \rangle \prec \langle V_\gamma, \in, A^* \rangle$.
- κ is *superstrongly unfoldable*, if $\forall \theta \forall A \in H_{\kappa^+}$ there is $j : M \rightarrow N$ with $A \in M \models \text{ZFC}$, $j(\kappa) \geq \theta$ and $V_{j(\kappa)} \subseteq N$.
- κ is *almost-hugely unfoldable*, if $\forall \theta \forall A \in H_{\kappa^+}$ there is such $j : M \rightarrow N$ with $N^{< j(\kappa)} \subseteq N$.

Theorem (Hamkins, Johnstone)

These large cardinal notions are all equivalent.

Superstrongly unfoldable cardinals in HOD

Theorem

If there is a measurable cardinal (subtle suffices), then there are many superstrongly unfoldable cardinals in HOD.

Proof.

Fix $j : V \rightarrow M$. Consider $\langle V_\kappa^{\text{HOD}}, \in, A \rangle$ where $A \subseteq \kappa$ in HOD^M .
Apply j to conclude

$$\langle V_\kappa^{\text{HOD}^M}, \in, A \rangle = \langle V_\kappa^{\text{HOD}}, \in, A \rangle \prec \langle V_{j(\kappa)}^{\text{HOD}^M}, \in, j(A) \rangle.$$

All initial segments of $j(A)$ are in HOD^M , and therefore
 $\langle V_\kappa^{\text{HOD}^M}, \in, A \rangle \prec \langle V_\gamma^{\text{HOD}^M}, \in, A^* \rangle$ some $\gamma < j(\kappa)$, with
 $A^* \in \text{HOD}^M$. So κ is superstrongly unfoldable in HOD^M , and so
there are many such cardinals in HOD. □

Many further open questions

The original motivating questions were:

- To what extent must a large cardinal in V exhibit its large cardinal properties in HOD?
- To what extent does the existence of large cardinals in V imply the existence of large cardinals in HOD?

The forcing constructions provided many negative answers to the first question.

We've had a few instances of positive answers to the second question.

The boundary between positive and negative, however, remains a bit mysterious...

Thank you.

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Slides and article available on <http://jdh.hamkins.org>