The pluralist perspective on the axiom of constructibility

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Most of this talk is based on my article:

J. D. Hamkins, “A multiverse perpsective on the axiom of constructibility,” in Infinity and Truth, IMS Lecture Note Series, NUS vol. 25, 2013.

The article is also available on

- arxiv.org/abs/1210.6541.
- jdh.hamkins.org/multiverse-perspective-on-constructibility.

Main thesis

My thesis in this talk is that what I shall call the $V \neq L$ via maximize argument relies on a singularist as opposed to pluralist stand on the question whether there is an absolute background concept of ordinal, that is, whether the ordinals can be viewed as forming a unique completed totality.

The argument, therefore, implicitly takes sides in the universe versus multiverse debate, and I shall argue that without that stand, the $V \neq L$ via maximize argument lacks force.

The axiom of constructibility

The *constructible universe* L is constructed in a transfinite process

$$L = \bigcup_{\alpha} L_{\alpha},$$

where at each ordinal stage α one forms $L_{\alpha+1}$ by adding to the collection L_{α} of sets formed at earlier stages all sets definable in the structure $\langle L_{\alpha}, \in \rangle$.

Gödel proved that L is an inner model of ZFC+GCH, thereby establishing the relative consistency of both the axiom of choice and the continuum hypothesis.

Some arguments against $V = L$

$V = L$ is seen as an artificial limitation on set-theoretic possibility.

Shelah asks, “Why the hell should it be true?”.

It is a mistaken principle to assume all structure is definable.

$V = L$ too often settles mathematical issues in the ‘wrong’ way, making L the land of pathological counterexamples. Compare the situation of descriptive set theory in L, for example, with the elegant smoothness of that theory under $\mathrm{AD}^{L(\mathbb{R})}$.

The deal-breaking objection

More troublesome than any issue of definabilism or consequentialism, however, is that $V = L$ is inconsistent with the largest large cardinals.

To block access to that realm is intolerably limiting, and this is the real end-of-line, deal-breaking objection.

Set theorists simply cannot accept an axiom that prevents access to their best and strongest theories, the large cardinal hypotheses, which encapsulate our dreams of what set theory can achieve and express.

Steel sees $V = L$ as limiting

John Steel argues that, "$V = L$ is restrictive, in that adopting it limits the interpretative power of our language."

He points out that the large cardinal set theorist can still understand the $V = L$ believer by means of the translation

$$\varphi \mapsto \varphi^L,$$

but, "there is no translation in the other direction" and "adding $V = L$... just prevents us from asking as many questions!"

The $V \neq L$ via maximize argument

Let me refer to this general line of reasoning as the $V \neq L$ via maximize argument.

Maddy explains:

> *The idea is simply this: there are things like $0^\sharp$ that are not in L. And not only is $0^\sharp$ not in L; its existence implies the existence of an isomorphism type that is not realized by anything in L.... So it seems that $\mathrm{ZFC} + V{=}L$ is restrictive because it rules out the extra isomorphism types available from $\mathrm{ZFC} + \exists 0^\sharp$.*
> *(1997, Naturalism in Mathematics)*

Giving the argument legs, she presents a formal account of what it means for a theory to be 'restrictive'.

Maddy's formal account of 'restrictive'

A theory T shows φ *is an inner model*, if it proves φ defines a transitive class $M \models \mathrm{ZFC}$, and T proves $\mathrm{Ord} \subseteq M$ or T proves $\exists\kappa$ inaccessible, with $\kappa \subseteq M$.

A formula φ is a *fair interpretation* of T in $T' \supseteq \mathrm{ZFC}$, if T' shows φ is an inner model satisfying T.

A theory T' *maximizes* over T, if there is a fair interpretation φ of T in T', and T' proves that this inner model is not everything.

A theory T' *strongly maximizes* over T, if they contradict, T' maximizes over T and no consistent $T'' \supseteq T$ properly maximizes over T'.

Finally, T is *restrictive*, if there is a consistent theory T' that strongly maximizes over it.

The idea

The axiom of constructibility $V = L$ comes out as formally restrictive.

The main purpose is to give precise mathematical substance to the intuitive idea that some set theories seem restrictive in a way that others do not. We view $V = L$ and 'there is a largest inaccessible cardinal' as limiting, while 'there are unboundedly many inaccessible cardinals' seems open-ended and unrestrictive.

A quibble

The syntactic form of 'shows φ is an inner model' requires T to settle the question whether $\mathrm{Ord} \subseteq M$ or $\exists\kappa$ inaccessible $\kappa \subseteq M$, rather than merely to prove that one or the other holds.

The distinction is between $(T \vdash A) \vee (T \vdash B)$ and $T \vdash A \vee B$.

Consider the theories:

- $\text{Inacc} = \mathrm{ZFC} + \exists$unboundedly many inaccessible cardinals
- $T = \mathrm{ZFC} +$ there is a Mahlo cardinal in V or unboundedly many inaccessible cardinals in L.

Every model of T has an inner model of Inacc, either by going to L or by truncating at a Mahlo. But T cannot decide which.

So Inacc is actually NOT fairly interpreted in T, contrary to what we might wish.

Another quibble

In the second case, with the ordinals up to an inaccessible κ, the official definition requires only $\kappa \subseteq \mathrm{Ord}^M$, rather than $\kappa = \mathrm{Ord}^M$, which might have been the intent.

Requiring $\kappa = \mathrm{Ord}^M$ would seem to conform more closely with the idea of "truncation at an inaccessible cardinal," the phrase often mentioned in connection with Maddy's proposal.

Otherwise, the process would more accurately be described by the slogan, "truncating at, *or somewhere above*, an inaccessible cardinal."

The fairly-interpreted-in relation is not transitive

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$$\begin{aligned} R &= \text{ZFC} + V{=}L + \text{there is no inaccessible cardinal} \\ S &= \text{ZFC} + V{=}L + \text{there is an inaccessible cardinal} \\ T &= \text{ZFC} + \omega_1 \text{ is inaccessible in } L \end{aligned}$$

R is fairly intrepreted in S by truncating at the first inaccessible; S is fairly interpreted in T by going to L; but R has no fair interpretation in T, since T is consistent with the lack of inaccessible cardinals.

Essence of example: first theory is fairly interpreted in the second only by truncating, and the second is fairly interpreted in the third only by going to an inner model with all ordinals.

Maximizing-over relation is not transitive

The same example shows also that the maximizing-over relation is not transitive.

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T maximizes over S and S maximizes over R, but T does not maximize over R, since R has no fair interpretation in T.

Similarly, the properly-maximizes-over and the strongly-maximizes-over relations also are not transitive.

False positives

Maddy presents some examples (several due to Steel) of 'false positives', theories deemed formally restrictive that intuitively do not feel restrictive.

Let me give a few additional examples, having more attractive maximizing theories, which skirt some of the objections to those examples.

Inacc is restrictive

Recall the theory Inacc, asserting $\mathrm{ZFC} + \exists$unboundedly many inaccessible cardinals.

Let $T = \mathrm{ZFC}+$there are unboundedly many inaccessible cardinals in L, but no worldly cardinals in V. (A cardinal κ is *worldly* just in case $V_\kappa \models \mathrm{ZFC}$.)

The theory Inacc is fairly interpreted in T, by going to L, and so T maximizes over Inacc.

But no strengthening of Inacc properly maximizes over T, because if Inacc holds, then every transitive class M containing all the ordinals will have an inaccessible cardinal; and every transitive set containing all ordinals up to an inaccessible will contain a worldly cardinal, violating T in either case.

So Inacc is strongly maximized by T, and so Inacc is restrictive.

Another example

Let MC^* be the theory $\text{ZFC}+$there is a measurable cardinal with no worldly cardinals above it.

By truncating at a measurable, we produce a model of Inacc, so MC^* maximizes over Inacc.

But no consistent strengthening of Inacc can maximize over MC^* (with modified truncation rule), since if $V \models \text{Inacc}$ and $M \models \text{MC}^*$ is inner model, then M cannot contain all the ordinals of V, since the inaccessible cardinals would be worldly in M, and neither can the height of M be inaccessible in V, since this would cause worldly cardinals in M.

So MC^* strongly maximizes over Inacc, and thus Inacc is restrictive.

The same idea works with many other large cardinal theories.

$V = L$ is compatible with strength

In order to support my main thesis, let me next explain a series of mathematical results, some from mathematical folklore and others more recently proved, which reveal various senses in which the axiom of constructibility $V = L$ is compatible with strength in set theory, particularly if one has in mind the possibility of moving from one universe of set theory to a much larger one.

V and L agree on consistency

Since L and V have the same proof objects, they agree on the consistency of any given theory.

Easy Observation

The constructible universe L and V agree on the consistency of any constructible theory. They have models of the same constructible theories.

In particular, $V = L$ is compatible with whatever kind of large cardinal consistency assertion we might like to make in V.

Transitive models of same theories

Theorem (folklore)

V and L have transitive models of the same theories.

This is simply because the assertion,

"there is a transitive model of T"

is a Σ^1_2 assertion about T, and hence absolute between V and L by the Shoenfield absoluteness theorem.

So L has transitive models with fake $0^\sharp$'s, or transitive models of ZFC+supercompact, or ZFC+there is a proper class of Woodin cardinals, or what have you.

Levy-Shoenfield absoluteness

Levy-Shoenfield absoluteness theorem

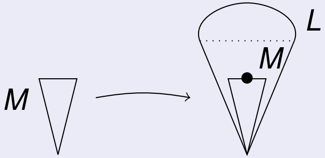
L and V satisfy the same Σ_1 sentences, with parameters hereditarily countable in L. Indeed, $L_{\omega_1^L}$ and V satisfy the same such sentences.

One can deduce this as a consequence of the transitive model theorem, or conversely.

Every structure is compatible with $V = L$

Theorem

Every countable transitive set is a countable transitive set in the well-founded part of an ω-model of $V = L$.



Proof.

Statement is true in L; has complexity Π^1_2; hence, true. □

Can also make the L-model pointwise definable.

A few curious examples

Countable $M \models \mathrm{ZFC}+$large cardinals. So there is an ω-model $N \models V = L$ with M as an element. The ordinals of N must be well-founded beyond height of M.

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Force $L \subseteq L[g]$, collapsing ω_1. Find $N \models V = L$ with $g \in N$. So N is well-founded beyond ω_1^L, thinks true ω_1^L is countable and thinks g is constructible, at some (nonstandard) countable stage.

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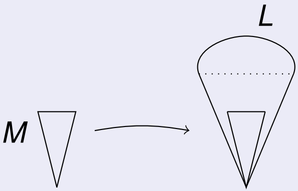
Suppose $0^\sharp$ exists. We may find $N \models V = L$ with true $0^\sharp$ existing as a real in N. So true $0^\sharp$ sits unrecognized inside a model of $V = L$.

The point: the isomorphism type of $0^\sharp$ is not fundamentally incompatible with $V = L$.

The beautiful Barwise theorem

Theorem (Barwise)

Every countable model of ZF *has an end-extension to a model of* ZFC $+$ $V{=}L$.

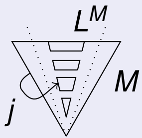


One can prove most instances of this using soft arguments as in previous results; use Barwise compactness for the general case. See my article for further general discussion.

The embedding theorem

Theorem (Hamkins)

Every countable model of set theory $\langle M, \in^M \rangle$, including every transitive model, is isomorphic to a submodel of its constructible universe $\langle L^M, \in^M \rangle$. Thus, there is an $\in$-embedding $j : M \to L^M$.



$$x \in y \longleftrightarrow j(x) \in j(y)$$

If we lived inside M, then by adding new sets and elements, our universe could be transformed into a copy of the constructible universe L^M.

Comparability of countable models of set theory

Theorem (Hamkins)

The countable models of set theory are linearly pre-ordered, and indeed, pre-well-ordered, by embeddability.

Given any two countable models $\langle M, \in^M\rangle$ and $\langle N, \in^N\rangle$ of set theory, one of them is isomorphic to a submodel of the other.

The order agrees with the "height" order, namely, $\mathrm{Ord}^M \hookrightarrow \mathrm{Ord}^N$ if and only if $M \hookrightarrow N$.

The central discovery in set theory

To my way of thinking, the central set-theoretic discovery of the last fifty years—and the one most in need of explanation—is the discovery that there is an enormous diversity of set-theoretic worlds.

A principal set-theoretic activity, nowadays, is the construction of yet another set-theoretic world, using the methods of forcing, ultrapowers, relative definability and all our other methods. We often build these worlds to exhibit specific desirable or unusual features or to have certain specific relations to other worlds.

Set theory now is this kind of model theory of set theory, and this is the main process by which we gain set-theoretic insight.

The main set-theoretic activity is the exploration of these possible worlds.

The analogy with geometry

The analogy with geometry is strong.

In geometry, the classical notions of point, line, plane, space, which were once thought to describe unique absolute geometrical concepts, have splintered into an array of thousands of distinct yet related geometrical concepts, each giving rise to a corresponding Euclidean or non-Euclidean geometrical world.

In set theory, what was thought to be a unique absolute background concept of set, has splintered into a diverse array of distinct yet closely related set-theoretic concepts, each giving rise to a corresponding set-theoretic world.

The multiverse view on CH

On the multiverse view, the Continuum Hypothesis is a settled question. It is incorrect to describe CH as an open question.

CH is solved not by the independence result, but rather solved by the extensive knowledge we have gained about how CH behaves in the multiverse, about how we may force CH or $\neg$CH while preserving this or that other desirable feature, and about how we may or may not find CH in an inner model of a certain kind.

Of course, not every question about CH is solved, but the main and most important facts about CH are deeply understood.

Those facts are what constitute the answer to the question of CH.

Support for $V \neq L$ therefore weakens

These diverse forcing and model-construction arguments are valued because they provide us with deeper set-theoretic insight; they enlarge us, revealing the set-theoretic situations to which we might aspire.

Much of the value of these arguments is provided already when they are undertaken with countable models of ZFC. We gain genuine set-theoretic insight working with countable models of set theory.

This perspective, historically, was at the heart of how forcing was commonly formalized.

On this account, L is not so limiting, and this undermines the argument against $V = L$.

A small step up

Any set theorist entertaining a hypothesis ψ is generally also willing to entertain the hypothesis that ψ holds in a transitive model of set theory.

This is a small step up in consistency strength, but a step that accords with common underlying philosophical justifications. This step up is the mathematical content of the reflection perspective.

$$\psi \quad \mapsto \quad \text{there is a transitive model of } \mathrm{ZFC} + \psi$$

The new statement has essentially the same explanatory force in terms of exhibiting the range of set-theoretic possibility, but is now compatible with $V = L$.

A translation for Steel

Steel had claimed, 'there is no translation' from the large cardinal realm to the $V = L$ context. But the translation

$$\psi \quad \mapsto \quad \text{there is a transitive model of } \mathrm{ZFC} + \psi$$

allows the $V = L$ believer to converse meaningfully with any large cardinal set-theorist, simply by imagining that the large cardinal set-theorist is living inside a countable transitive model.

What is more, if the large cardinal set-theorist believes the large cardinal axiom on account of the reflection arguments, then he or she furthermore agrees with the truth of the translation, which simply reinforces the accuracy with which the $V = L$ believer has captured the situation.

Set-theoretic experience

Set theorists have had a truly rich experience exploring diverse set-theoretic worlds, investigating how they relate to one another and studying the methods used to construct them.

These diverse worlds instantiate various concepts of set, which we have come to know as robust and legitimate, each suitable as a foundation for the mathematics that takes place inside of the corresponding world.

The traditional Universe perspective in set theory has largely failed to explain this fundamental phenomenon, the phenomenon of diverse set-theoretic possibility.

The set-theoretic multiverse

The multiverse perspective, in constrast, embraces this mathematical experience, and takes the phenomenon as a fundamental aspect of mathematical ontology.

On the multiverse view, we do not have a single absolute concept of set, but rather many distinct and often closely related concepts of set.

On the multiverse view, there are many set-theoretic worlds.

Each may be regarded as just as "real" as the universe is on the universe view.

Thus, the multiverse perspective separates the issue of realism from the issue of uniqueness of the universe.

Toy multiverse

On the multiverse view, our current universe V might become a countable model inside another better universe $\overline{V}$.

Our current universe V and its forcing extensions appear amongst the countable models of set theory inside $\overline{V}$.

So by studying the countable models of set theory, we find a toy version of the real multiverse, from which we may gain insight. The real multiverse is reflected in the toy simulacrum, which is amenable to mathematical analysis.

Just as every countable model has its forcing extensions, we expect actual forcing extensions of V. Just as every countable model of set theory has end-extensions to models of $V = L$, we expect V to have such extensions. And so on.

Upwardly extensible concept of set

The multiverse vision leads to an upwardly extensible concept of set, where any current set-theoretic universe may be extended to a taller, better universe. The current universe becomes countable in a larger universe, which has still larger extensions, some with large cardinals, some without, some with the continuum hypothesis, some without, some with $V = L$ and some without, in a series of extensions continuing longer than we can imagine.

Models with $0^\sharp$ are extended to larger models with new ordinals, revealing it as false. Any given set-theoretic situation is fundamentally compatible with $V = L$, if one only moves to a better, taller universe. Every set, every universe of sets, becomes countable and constructible, if we wait long enough.

L as rewarder of the patient

The constructible universe L thus becomes a *rewarder of the patient*, revealing hidden constructibility structure for any given mathematical object or universe, if one should only extend the ordinals far enough beyond one's current set-theoretic universe.

This perspective turns the $V \neq L$ via maximize argument on its head, for by maximizing the ordinals, we seem able to recover $V = L$ as often as we like, extending our current universe to larger and taller universes in diverse ways, attaining $V = L$ and destroying it in an on-again, off-again pattern, upward densely in the set-theoretic multiverse, as the ordinals build eternally upward, eventually exceeding any particular conception of them.

Thank you.

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Slides and article available on http://jdh.hamkins.org