COMPLETE SETS OF CONNECTIVES FOR GENERALIZED ŁUKASIEWICZ LOGICS

KAMERYN J WILLIAMS

CUNY Graduate Center
365 Fifth Avenue
New York, NY 10016 USA

Abstract. While $\wedge, \vee, \neg$ form a complete set of connectives for classical propositional logic, this does not hold for Łukasiewicz's three-valued propositional logic, nor its generalization to n-valued logic. We define a unary connective $\mathfrak{r}$ so that $\wedge, \vee, \neg, \mathfrak{r}$ form a complete set of connectives for n-valued Łukasiewicz logic. We discuss generalizations of this to infinitary logics. If we allow infinite conjunctions and disjunctions of arbitrary size, this provides a complete set of connectives for real-valued Łukasiewicz logic. Restricting to countable conjunctions and disjunctions, the truth functions expressible with these connectives are precisely the Borel functions.

1. Introduction

In [3, 4] Łukasiewicz defined a three-valued propositional logic. His logic extends classical propositional logic, adding a truth value intermediate between truth and falsity. This third truth value, which he interpreted as "possibility",[1] was introduced to address some (perceived) shortcomings in Aristotelian logic. In this article, we set aside the philosophical questions surrounding multi-valued logic to look at some of their algebraic properties. We will investigate what is necessary to extend the original connectives of Łukasiewicz's logic to be able to express any truth function as a logical formula.

The definitions of Łukasiewicz's connectives can be found in in Figure 1. He later generalized the definition of these connectives to define $\wedge$, $\vee$, $\neg$, $\Rightarrow$, and $\Leftrightarrow$ for n-valued logic, with n any integer > 2. The connectives are defined to be as close as possible to the classical, two-valued connectives. Indeed, when restricting their input to classical truth values, these connectives are precisely their classical counterparts. As a consequence of this, not every truth function $\{0, \frac{1}{2}, 1\}^\ell \to \{0, \frac{1}{2}, 1\}$ can be represented as a formula in Łukasiewicz's logic; any formula must send classical truth values to classical truth values. The same result holds for the n-valued logics.

This stands in contrast to classical propositional logic, which is well-known to have a complete set of connectives. Every truth function $f : \{0, 1\}^\ell \to \{0, 1\}$ can

E-mail address: `kwilliams4@gradcenter.cuny.edu`.

[1]Note that although Łukasiewicz was a contemporary of Lewis, he developed his three-valued logic before Lewis's work on modal logic had the influence it enjoys today. In particular, Lewis's and Langford's *Symbolic Logic* [2] would not be published for another decade.

Figure 1. Connectives for Łukasiewicz's three-valued logic.

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\Rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\Leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

be expressed by a propositional formula in disjunctive normal form. Moreover, as conjunction, disjunction, and negation are expressible in terms of nand—$a \uparrow b = \neg(a \wedge b)$—every classical truth function is expressible as a formula using a single connective.

The main result of this article is a generalization of this result to n-valued logic. We will define a unary connective $\mathfrak{r}$ and see that $\{\uparrow, \mathfrak{r}\}$ is a complete set of connectives for n-valued logic. There are also some results for infinitary propositional logics. Allowing infinite conjunctions and disjunctions of arbitrary size, by a similar argument can be seen how to produce a complete set of connectives for real-valued propositional logic. Allowing only countably conjunctions and disjunctions, the truth functions expressible as a formula with these connectives are precisely the Borel functions.

2. Notation and definitions

$[n]$ will be used to denote the set $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$.

Definition 1. $Ł_n$ will denote the generalized Łukasiewicz logic with n many truth values. That is, truth values for $Ł_n$ are elements of $[n]$ and the logical connectives are defined as follows:

- $a \wedge b = \min(a, b)$;
- $a \vee b = \max(a, b)$; and
- $\neg a = 1 - a$.

$Ł_3$ is Łukasiewicz's original three-valued logic. $Ł_2$ is classical propositional logic.

As we are only looking at complete sets of connectives, the question of how to define implication is not of interest to us. However, it is worth noting that none of the standard choices of implication for three-valued logic would yield a complete set of connectives when added to $\wedge$, $\vee$, and $\neg$. Kleene [1] defined implication in his three-valued logic K_3 so that $a \Rightarrow b$ is equivalent to $\neg a \vee b$. Thus, no new truth functions can be represented by adding it. Łukasiewicz's implication (see Figure 1) allows the representation of new truth functions—such as the constant 1 function—but does not suffice to allow all truth functions to be represented. Łukasiewicz's $\Rightarrow$ and $\Leftrightarrow$ send classical truth values to classical truth values. Thus the argument of Proposition 3 (that $\{\wedge, \vee, \neg\}$ is not a complete set of connectives for $Ł_n$) goes through using $\{\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow\}$ as the set of connectives.

Definition 2. Let K be a propositional logic with T as its set of truth values. A function $f : T^\ell \to T$ will be called a *K-truth function*. If there is a formula φ with connectives from C so that for all $\bar{a}$, $f(\bar{a}) = \varphi(\bar{a})$, we say that f is *representable as a K, C-formula*. If every K-truth function is representable as a K, C-formula, we say that C is a *complete set of connectives for K*.

$(Ł_2, \wedge, \vee, \neg)$ embeds into the logic $(Ł_n, \wedge, \vee, \neg)$ under the identity map. Further, if m is even, then $(Ł_m, \wedge, \vee, \neg)$ embeds into $(Ł_n, \wedge, \vee, \neg)$ for all $n \geq m$ and if m is odd, then $(Ł_m, \wedge, \vee, \neg)$ embeds into $(Ł_n, \wedge, \vee, \neg)$ for all odd $n \geq m$. This is witnessed by the embedding

$$f_{m,n}(a) = \begin{cases} \frac{a(m-1)}{n-1} & \text{if } a < \frac{1}{2} \\ 1 - \frac{(1-a)(m-1)}{n-1} & \text{if } a > \frac{1}{2} \\ \frac{1}{2} & \text{if } a = \frac{1}{2} \end{cases}$$

Proposition 3. *$\{\wedge, \vee, \neg\}$ is not complete for $Ł_n$.*

Proof. Consider $\varphi(\bar{x})$, a formula in $\wedge, \vee, \neg$. If $\bar{a}$ consists only of 0 and 1, then in $Ł_n$, $\varphi(\bar{a}) = f_{2,n}(\varphi(\bar{a})) \in \{0, 1\}^\ell$. Thus, φ cannot represent any truth function which does not map classical truth values to classical truth values. □

This result generalizes: if C is any set of connectives for $Ł_m$ and $Ł_n$, $m < n$, so that $(Ł_m, C)$ embeds into $(Ł_n, C)$, then C is not complete for $Ł_n$. If we wish to define complete sets of connectives for $Ł_n$, we must avoid this.

Definition 4. For $Ł_n$, define the following unary connectives:

- $\mathfrak{r}a = a - \frac{1}{n-1} \pmod 1$. $\mathfrak{r}$ "rotates" the truth values down by one step.
- $\mathfrak{d}a = \max(0, a - \frac{1}{n-1})$. $\mathfrak{d}$ moves the truth values down by one step, fixing 0.
- $\mathfrak{p}1 = 1$ and $\mathfrak{p}a = 0$ for $a < 1$. $\mathfrak{p}$ projects the truth values, sending any truth value less than absolute truth to absolute falsehood.

It is easy to see that if $(Ł_m, \wedge, \vee, \neg)$ embeds into $(Ł_n, \wedge, \vee, \neg)$, then we have $(Ł_m, \wedge, \vee, \neg, \mathfrak{p})$ embeds into $(Ł_n, \wedge, \vee, \neg, \mathfrak{p})$. This does not hold, however, for either $\mathfrak{r}$ or $\mathfrak{d}$. Indeed, for no $m < n$ does $(Ł_m, \mathfrak{d}, \mathfrak{r})$ embed into $(Ł_n, \mathfrak{d}, \mathfrak{r})$. Therefore, the situation of Proposition 3 does not apply here.

From these connectives we can define more connectives. $\mathfrak{r}$ suffices to define rotation by any amount: $\mathfrak{r}^k$ rotates the truth values down by k steps. In particular, $\mathfrak{r}^{-1} = \mathfrak{r}^{n-1}$ rotates the truth values up by 1 step. Using $\neg$, we can define dual connectives for $\mathfrak{d}$ and $\mathfrak{p}$. $\mathfrak{u}a = \neg\mathfrak{d}\neg a$ moves the truth values up by one step, fixing 1. $\mathfrak{b}a = \neg\mathfrak{p}\neg a$ projects the truth values up, sending any truth value more than absolute falsehood to absolute truth.

In addition to finitary generalized Łukasiewicz logics, logics with infinitely many truth values can also be defined. Łukasiewicz and Tarski [5] were the first to do so. Again, we will define only conjunction, disjunction, and negation, setting aside the question of how to define implication.

Definition 5. $Ł_{\aleph_0}$ will denote the generalized Łukasiewicz logic with countably many truth values. Its set of truth values is $\mathbb{Q} \cap [0, 1]$. $Ł_{\mathfrak{c}}$ will denote denote the logic with $\mathfrak{c} = 2^{\aleph_0}$ many truth values. Its set of truth values is $[0, 1]$. For both logics, the basic logical connectives are defined as in the finitary case: $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and $\neg a = 1 - a$.

For any finite n, $Ł_n$ embeds into $Ł_{\aleph_0}$ and $Ł_{\mathfrak{c}}$ via the identity map. Thus, as was the case for finitary generalized Łukasiewicz logic, $\{\wedge, \vee, \neg\}$ is complete for neither $Ł_{\aleph_0}$ nor $Ł_{\mathfrak{c}}$. Further, in order for C to be complete for $Ł_{\aleph_0}$ or $Ł_{\mathfrak{c}}$, C must not respect these embeddings.

We define new unary connectives for these infinitary logics, similar to the ones for $Ł_n$. Unlike with $Ł_n$, however, we will need infinitely many new connectives: in $Ł_n$, $\mathfrak{r}$ generated every possible rotation of $[n]$. For $Ł_{\aleph_0}$ and $Ł_{\mathfrak{c}}$, this clearly cannot be.

Definition 6. For each $x \in (0, 1)$, define the following unary connectives:

- $\mathfrak{r}_x a = a - x \pmod 1$; and
- $\mathfrak{d}_x a = \max(0, a - x)$.

Define $\mathfrak{p}$ as $\mathfrak{p}a = 1$ if $a = 1$ and $\mathfrak{p}a = 0$ otherwise. We also define the dual connectives $\mathfrak{u}_x a = \neg\mathfrak{d}_x\neg a$ and $\mathfrak{b}a = \neg\mathfrak{p}\neg a$.

3. A complete set of connectives for the finitary case

Let $C = \{\wedge, \vee, \neg, \mathfrak{r}, \mathfrak{d}, \mathfrak{p}\}$.

Lemma 7. *For any $\bar{a}$, there is a formula $\varphi_{\bar{a}}(\bar{x})$ in C so that $\varphi_{\bar{a}}(\bar{a}) = 1$ and for all $\bar{b} \neq \bar{a}$, $\varphi_{\bar{a}}(\bar{b}) < 1$.*

Proof. Let

$$\varphi_{\bar{a}}(\bar{x}) = \bigwedge_{0 \le k < n} \bigwedge_{a_i = \frac{k}{n-1}} \mathfrak{r}^{k+1-n} x_i.$$

$\mathfrak{r}^{k+1-n}c$ rotates c up by k steps. Thus, $\mathfrak{r}^{k+1-n}c = 1$ iff $c = \frac{k}{n-1}$ and otherwise $c < 1$. This gives that $\varphi_{\bar{a}}(\bar{a}) = 1$ and $\varphi_{\bar{a}}(\bar{b}) < 1$ for $\bar{b} \neq \bar{a}$. □

Lemma 8. *C is a complete set of connectives for $Ł_n$.*

Proof. Let $g : [n]^\ell \to [n]$ be a truth function. Set

$$\psi(\bar{x}) = \bigvee_{0 \le k < n} \bigvee_{g(\bar{a}) = \frac{k}{n-1}} \mathfrak{d}^{n-1-k}\mathfrak{p}\varphi_{\bar{a}}(\bar{x}),$$

where $\varphi_{\bar{a}}$ is as in the previous lemma. $\mathfrak{p}\varphi_{\bar{a}}(\bar{x})$ is 1 if $\bar{x} = \bar{a}$ and 0 otherwise. $\mathfrak{d}^{n-1-k}$ moves 1 down to $\frac{k}{n-1}$, so $\mathfrak{d}^{n-1-k}\mathfrak{p}\varphi_{\bar{a}}(\bar{x})$ is $\frac{k}{n-1}$ if $\bar{x} = \bar{a}$ and 0 otherwise. Therefore, $\mathfrak{d}^{n-1-k}\mathfrak{p}\varphi_{\bar{x}}(\bar{x}) = g(\bar{x})$. □

Theorem 9. *$\{\uparrow, \mathfrak{r}\}$ is a complete set of connectives for $Ł_n$, where $\uparrow$ is defined as $a \uparrow b = \neg(a \wedge b)$.*

Proof. Exactly as in the classical case, $\wedge$, $\vee$, and $\neg$ are expressible in terms of $\uparrow$. Thus, by Lemma 8, we have only to see that $\mathfrak{d}$ and $\mathfrak{p}$ can be expressed in terms of $\wedge$, $\vee$, $\neg$, and $\mathfrak{r}$. It is easy to see that $\mathfrak{d}a = a \wedge \mathfrak{r}a$; if $a > 0$, then $\mathfrak{r}a = a - \frac{1}{n-1}$ and hence $a \wedge \mathfrak{r}a = a - \frac{1}{n-1}$. Otherwise, if $a = 0$, then $a \wedge \mathfrak{r}a = 0$.

For $\mathfrak{p}$, notice that

$$\bigwedge_{k < n-1} \mathfrak{r}^k a = \begin{cases} \frac{1}{n-1} & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathfrak{p}a = \mathfrak{r}^{-1}\neg\left(\bigwedge_{k<n-1} \mathfrak{r}^k a\right).$$

□

Question 10. *Is there a single binary connective $\star : [n]^2 \to [n]$ so that $\{\star\}$ is complete for $Ł_n$?*

4. The infinitary case

Let us now consider to what extent these results can be generalized to infinitary logics. A simple cardinality argument shows that most truth functions in $Ł_{\aleph_0}$ and $Ł_{\mathfrak{c}}$ cannot be represented by adding $\mathfrak{p}$, $\mathfrak{r}_x$, and $\mathfrak{d}_x$. There are continuum many functions $\mathbb{Q}\cap[0,1] \to \mathbb{Q}\cap[0,1]$ but only countably many formulae in $\{\wedge, \vee, \neg, \mathfrak{p}, \mathfrak{r}_x, \mathfrak{d}_x : x \in \mathbb{Q}\cap(0,1)\}$. There are $2^{2^{\aleph_0}}$ functions $[0,1] \to [0,1]$, but only continuum many formulae in $\{\wedge, \vee, \neg, \mathfrak{p}, \mathfrak{r}_x, \mathfrak{d}_x : x \in (0,1)\}$. Even if we only look at continuous functions $[0,1] \to [0,1]$, of which there are continuum many, a simple induction shows that only piecewise-linear functions can be represented with these connectives.

It is natural to ask whether allowing infinitary connectives will allow us to represent more truth functions. That is, if we define $\bigvee_{i\in I} a_i = \sup\{a_i : i \in I\}$ and $\bigwedge_{i\in I} a_i = \inf\{a_i : i \in I\}$, what truth functions can we represent? For this to make sense, we need the set of truth values for our logic to be closed under suprema and infima. That is, $Ł_{\aleph_0}$ is not an adequate logic for this purpose. We will hereon work solely with $Ł_{\mathfrak{c}}$. For an infinite cardinal κ, let $Ł_{\mathfrak{c}}(\kappa)$ denote that conjunctions and disjunctions of size $< \kappa$ are allowed. $Ł_{\mathfrak{c}}(\infty)$ will denote that conjunctions and disjunctions of arbitrarily large size are allowed. $Ł_{\mathfrak{c}}(\omega_1)$ denotes that only countable conjunctions and disjunctions are allowed, ω_1 being the least uncountable cardinal. As there are only continuum many truth values in $Ł_{\mathfrak{c}}$, having conjunctions or disjunctions of size $> \max(\mathfrak{c}, \kappa)$ is redundant if we only look at formulae involving $\leq \kappa$ variables.

We get an analogue of Theorem 9, using essentially the same argument as before. Let $C = \{\wedge, \vee, \neg, \mathfrak{p}, \mathfrak{r}_x, \mathfrak{d}_x : x \in (0,1)\}$.

Theorem 11. *Every truth function $[0,1]^\kappa \to [0,1]$ is representable as a $Ł_{\mathfrak{c}}(\lambda), C$-formula, where λ is the smallest cardinal $> \max(\kappa, \mathfrak{c})$. Therefore, any $Ł_{\mathfrak{c}}$-truth function, i.e. of arbitrary infinite arity, is representable as a $Ł_{\mathfrak{c}}(\infty), C$-formula.*

We will need the following lemma:

Lemma 12. *For any open interval $(a,b) \subseteq [0,1]$, the characteristic function $\chi_{(a,b)}$ is representable as a $Ł_{\mathfrak{c}}(\omega_1), C$-formula.*

Proof. I claim that if $\varepsilon < b - a$, then $\chi_{(a,b)}(x) = \mathfrak{p}\mathfrak{u}_{b-a-\varepsilon}\mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon\mathfrak{r}_b x$. To see this, it is enough to see that $\mathfrak{u}_{b-a-\varepsilon}\mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon\mathfrak{m}_b x = 1$ iff $a < x < b$. This splits into three cases to check:

- $0 \leq x \leq a$: Then,

$$\begin{array}{ccccc}
1-b & \leq & \mathfrak{r}_b x & \leq & 1-b+a \\
1-b+\varepsilon & \leq & \mathfrak{u}_\varepsilon\mathfrak{r}_b x & \leq & 1-b+a+\varepsilon \\
b-\varepsilon & \geq & \neg\mathfrak{u}_\varepsilon\mathfrak{r}_b x & \leq & b-a-\varepsilon \\
a & \geq & \mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon\mathfrak{r}_b x & \geq & 0 \\
b-\varepsilon & \geq & \mathfrak{u}_{b-a-\varepsilon}\mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon\mathfrak{r}_b x & \geq & b-a-\varepsilon.
\end{array}$$

- $a < x < b$: Then,

$$\begin{array}{ccccc} 1-b+a & < & \mathfrak{r}_b x & < & 1 \\ 1-b+a+\varepsilon & < & \mathfrak{u}_\varepsilon \mathfrak{r}_b x & \leq & 1 \\ b-a-\varepsilon & > & \neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & 0 \\ 1 & > & \mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & 1-b+a+\varepsilon \\ 1 & \geq & \mathfrak{u}_{b-a-\varepsilon}\mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & 1. \end{array}$$

- $b \leq x \leq 1$: Then,

$$\begin{array}{ccccc} 0 & \leq & \mathfrak{r}_b x & \leq & 1-b \\ \varepsilon & \leq & \mathfrak{u}_\varepsilon \mathfrak{r}_b x & \leq & 1-b+\varepsilon \\ 1-\varepsilon & \geq & \neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & b-\varepsilon \\ 1-b+a & \geq & \mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & a \\ 1-\varepsilon & \geq & \mathfrak{u}_{b-a-\varepsilon}\mathfrak{r}_{b-a-\varepsilon}\neg\mathfrak{u}_\varepsilon \mathfrak{r}_b x & \geq & b-\varepsilon. \end{array}$$

See Figure 2 for a graphical depiction of these transformations. □

Proof of Theorem 11. Let $f : [0,1]^\kappa \to [0,1]$ be a fixed truth function. First, note that for every $A \subseteq [0,1]^\kappa$, there is $\sigma_A(\bar{x}) = \chi_A(\bar{x})$. By Lemma 12, if I is an open interval, there is a $Ł_\mathfrak{c}(\omega_1), C$-formula φ_I so that $\chi_I(x) = \varphi_I(x)$. Thus, if $\{I_n\}$ is a countable collection of open intervals whose intersection is $\{a\}$, $\psi_a(x) = \bigwedge_n \varphi_I(x) = \chi_{\{a\}}(x)$. Therefore,

$$\sigma_A(\bar{x}) = \bigvee_{\bar{a}\in A} \bigwedge_{i<\kappa} \psi_{a_i}(x_i) = \chi_A(\bar{x}).$$

By an argument similar to the one in Theorem 9, f is represented by

$$\bigvee_{y\in[0,1]} \mathfrak{d}_{1-y}\sigma_{f^{-1}(y)}(\bar{x}).$$ □

We now turn to the question of which $Ł_\mathfrak{c}$ truth functions are representable when only countable disjunctions and conjunctions are allowed.

Definition 13. $f : [0,1]^\ell \to [0,1]$ is *Borel* if the pre-image under f of any open set is Borel.

Definition 14. $s : [0,1]^\ell \to [0,1]$ is a *step function* if there are $A_1, \ldots, A_k \subseteq [0,1]^\ell$ and $a_1, \ldots, a_k \in [0,1]$ so that $s = \sum_{i\leq k} a_i \chi_{A_i}$. If each A_i is Borel, we say that s is a *Borel step function.*

Recall the following standard facts about Borel functions:

Proposition 15.

- $f : [0,1]^\ell \to [0,1]$ *is Borel iff there is an increasing sequence* $\langle s_i \rangle$ *of Borel step functions whose limit is* f.
- *If* $f_i : [0,1]^\ell \to [0,1]$ *are Borel,* $i \in \mathbb{N}$, *then* $\sup f_i$ *and* $\inf f_i$ *are Borel.*
- *The composition of Borel functions of Borel.*

Arguments for these can be found in e.g. [6, Theorems 1.12d, 1.14, 1.17].

It is easy to see that every truth function represented as a $Ł_\mathfrak{c}(\omega_1), C$-formula is Borel. The unary connectives are all Borel and being Borel is preserved by countable suprema and infima, i.e. by countable disjunctions and conjunctions. The content of the following theorem is that the reverse implication is also true.

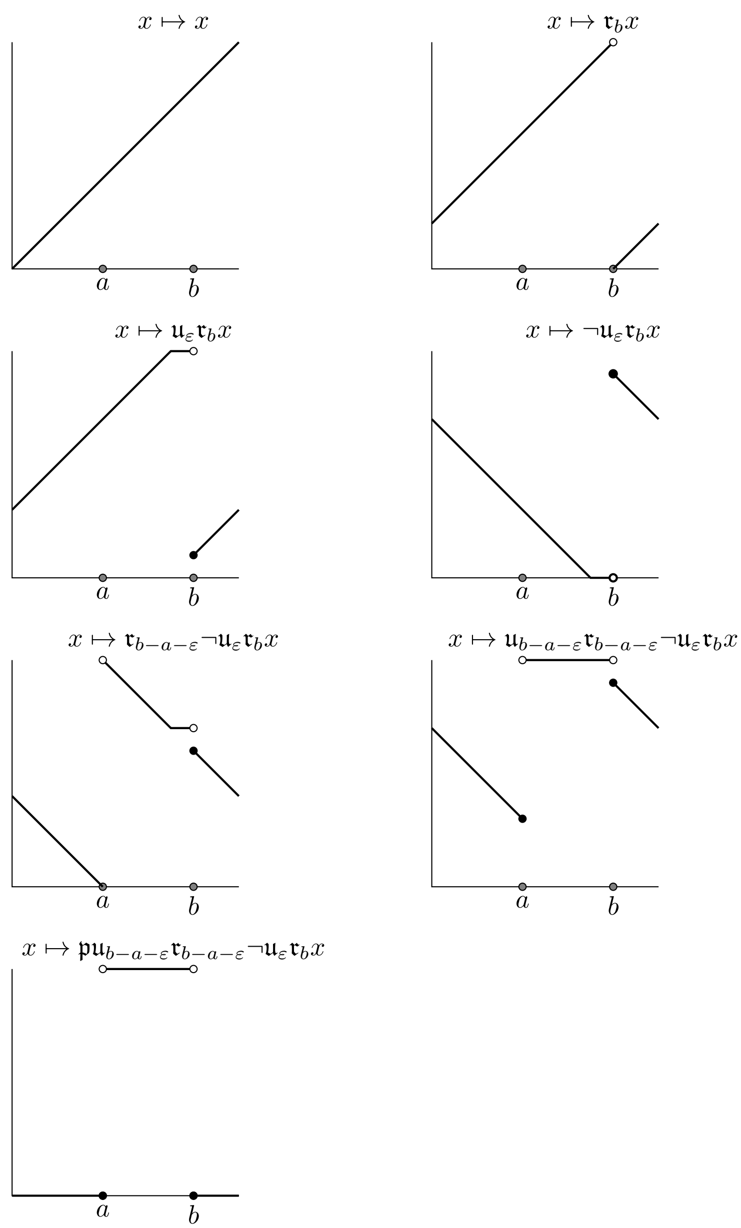

FIGURE 2. Representing $\chi_{(a,b)}$ as a $Ł_\mathfrak{c}$ truth function.

Theorem 16. *The class of truth functions* $[0,1]^\ell \to [0,1]$ *expressible as* $Ł_\mathfrak{c}(\omega_1), C$*-formulae is the class of Borel functions.*

Lemma 17. *For every open* $G \subseteq [0,1]^\ell$, χ_G *is representable as a* $Ł_\mathfrak{c}(\omega_1), C$*-formula.*

Proof. G is a countable union of cubes $Q = \prod_{i \leq \ell}(a_i, b_i)$. Lemma 12 gives that χ_Q is representable as a $Ł_\mathfrak{c}(\omega_1), C$-formula: if φ_i represents $\chi_{(a_i,b_i)}$, then $\psi_Q = \bigwedge_{i\leq \ell} \varphi_i$

represents χ_Q. To represent χ_G, take the disjunction of ψ_Q for these countably many Q. □

Lemma 18. *For every Borel $B \subseteq [0,1]^\ell$, χ_B is representable as a $Ł_{\mathfrak{c}}(\omega_1), C$-formula.*

Proof. The family of sets whose characteristic functions can be represented as $Ł_{\mathfrak{c}}(\omega_1), C$-formulae is closed under countable unions and complements. The former fact is because $\chi_{\bigcup A_n} = \bigvee \chi_{A_n}$. The latter fact is because $\chi_{[0,1]^\ell \setminus A} = \neg \chi_A$. Therefore, this class of sets contains all the Borel sets. □

Proof of Theorem 16. We have already seen one direction of the argument. For the other direction, fix Borel $f : [0,1]^\ell \to [0,1]$ and increasing sequence $\langle s_i \rangle$ whose supremum is f. It suffices to show that each s_i is representable as a $Ł_{\mathfrak{c}}(\omega_1), C$-formula. Let $s_i = \sum_{j \le k} a_j \chi_{A_j}$, where A_j is Borel. We may assume without loss that the A_j are disjoint. By the previous lemma, χ_{A_j} is representable as $\varphi_j(\bar{x})$. Thus, $a_j \chi_{A_j}$ is representable as $\mathfrak{d}_{1-a_j} \varphi_j(\bar{x})$. Therefore, s_i is representable as $\bigwedge_{j \le k} \mathfrak{d}_{1-a_j} \varphi_j(\bar{x})$. □

Finally, let us consider to what extent when can get this result with a smaller set of connectives. First, observe that $\wedge$, $\vee$, and $\neg$ can be defined in terms of $\uparrow$, as in the case of $Ł_n$. For the rotations $\mathfrak{r}_x$, note that they can be replaced with constants—i.e. 0-ary connectives—and a single binary connective. Let $x \ominus y = x - y \pmod 1$. Then $r_x a = a \ominus x$.

The definition in $Ł_n$ of $\mathfrak{d}$ in terms of $\wedge$ and $\mathfrak{r}$ can be generalized to $Ł_{\mathfrak{c}}$:

$$\mathfrak{d}_x a = \bigwedge_{y < x} \mathfrak{r}_y a.$$

If $a \ge x$, this is $\mathfrak{r}_x a = \mathfrak{d}_x a$. Else, this is $0 = \mathfrak{d}_x a$. Moreover, this conjunction can be taken over any set dense in $(0, x)$. This allows us to define $\mathfrak{d}_x$ as a countable conjunction of rotations.

On the other hand, the definition in $Ł_n$ of $\mathfrak{p}$ does not carry over to the infinite case. That definition used essentially that the set of truth values for $Ł_n$ is discrete. This is not so bad, however, as $\mathfrak{p}$ is just one connective, much less than the continuum many we already have included.

Putting these remarks together, along with Theorems 11 and 16, yields:

Theorem 19. *Let $D = \{\uparrow, \ominus, \mathfrak{p}, x : x \in (0,1)\}$.*

(1) *Every truth functions $[0,1]^\kappa \to [0,1]$ is representable as a $Ł_{\mathfrak{c}}(\lambda), D$-formula, where λ is the least cardinal $> \max(\kappa, \mathfrak{c})$. Therefore, any $Ł_{\mathfrak{c}}$-truth function, i.e. of arbitrary infinite arity, is representable as a $Ł_{\mathfrak{c}}(\infty), D$-formula.*

(2) *The class of functions $[0,1]^\ell \to [0,1]$ representable as $Ł_{\mathfrak{c}}(\omega_1), D$-formulae is the class of Borel functions.*

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