Logic 1/28/15

Propositional /sentential logic

Formal language consists of propositional variables + logical connectives + punctuation (parentheses, etc.)

Truth tables define the meaning of connectives (punctuation?) - "Truth functional semantics" - meanings defined by conditions (in terms of truth of prop.s) under which expressions are true.

Truth tables like graphs of functions (b/c maps values of prop variables to values of expression?)

Material implication

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

has to be defined this way in order to be unique.

Why ∧, ∨, ¬, →, ⟺? (aren't ⟺ + → reducible to ∧ ∨ ¬ anyway?)
Are there other possible connectives?

a new connective = a unique column of truth values. ~~[illegible]~~ so far 2 variables there are <u>16</u> possible ~~[illegible]~~ connectives.

? There are more w/ more variables?

"nand" = $\uparrow$ = "not both"
+ "nor" = $\downarrow$ = "not or"

P	Q	$P \uparrow Q$	$P \downarrow Q$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

? "We can think of $\neg$ as a function of two arguments, we just don't use one of them?"

$\nrightarrow$ $\nleftrightarrow$ Just negated forms of $\rightarrow$ + $\leftrightarrow$

P	Q	$(P \rightarrow Q)$	$\vee$	$(Q \rightarrow P)$
T	T	T	$\boxed{T}$	T
T	F	F	$\boxed{T}$	T
F	T	T	$\boxed{T}$	F
F	F	T	$\boxed{T}$	T

$\uparrow$

all T's = a tautology!

well formed formulas

If φ, ψ are legit expressions, so are $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \leftrightarrow \psi)$ etc.

a well formed formula is any expression you can get combining prop variables w/ connectives w/ right syntax
(syntax v.s. semantics (truth tables))

(put parentheses around all $\wedge$ + $\vee$ expressions)

We don't really need ( ) around $\wedge$ expressions b/c $\wedge$ is associative, but this is semantics, and so from a syntax perspective we don't know that?

For every wff there is a "construction history" (tree):

$(((P \wedge Q) \wedge R) \vee ((P \wedge Q) \to R))$

$\vee$

$((P \wedge Q) \wedge R)$ $\qquad$ $((P \wedge Q) \to R)$

$\wedge$ $\qquad$ $\to$

$(P \wedge Q)$ $\quad R$ $\qquad$ $(P \wedge Q)$ $\quad R$

$\wedge$ $\qquad$ $\wedge$

$P \quad Q$ $\qquad$ $P \quad Q$

(greek letters for complex (maybe) expressions.

This construction enables (?)
The argument method: Induction on formulas: If all prop variables have a common property and this property is preserved by the form. building operations, then all formulas have that property.

(Principle):

infix v. prefix notation
$(((A \wedge B) \wedge C) \rightarrow D)$ infix
(?) $\rightarrow \wedge \wedge A B C D$ prefix

you can't just add parentheses and have it still be well formed, even though it doesn't make sense from a semantic perspective.

Tautology testing

Truth tables not always feasible - e.g.:
is $((A \wedge B) \rightarrow C) \rightarrow (((A \wedge \neg A) \rightarrow C) \vee ((B \wedge \neg A) \rightarrow C)))$ a tautology?

1) whenever C is true, the whole thing is true, b/c consequent will be true.

2) So what if C is false? Then in order for the consequent to turn out true and the antecedent false, we'd need $(A \wedge B)$ to be true + $A \wedge \neg A$ or $B \wedge \neg A$ to be true, $A \wedge \neg A$ obv. is false, so we need ~~AA~~ $A \wedge B$ + $B \wedge \neg A$ → impossible. So it's a tautology!

Logical equivalence

φ is logically eq. to ψ iff they have same truth tables. Same as: φ is logically equiv. to ψ iff $\varphi \leftrightarrow \psi$ is a tautology.

If we have n variables, how many diff. possible truth functional connectives (binary...?) $(2^2)^n$

~~Ex~~ every truth function w/ any # of variables can be expressed using just $\neg$, $\wedge$, + $\vee$. - can be expressed as a disjunction of conjunctions of literals (either a prop. variable or the negation of a prop variable) i.e. can be expressed in Disjunctive Normal form.

disjunction of "conjunction clauses"

α: truth-function

? Proof: Every row in an expression's
? ? truth-table is expressable as
a conjunctive clause exactly
? ? detailing the truth or falsity of
each P_i

$P_1 \quad P_2 \quad P_3 \ \ldots \ P_n$

? ? ? $\boxed{\text{Sorry!}}$

"conjunctive normal form" - conjunction of disjuncts of literals

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Complete sets of connectives
a set of connectives is complete
iff we can express any truth function expressing them

$$P \to Q \equiv \neg(P \wedge \neg Q) \equiv \neg P \vee Q$$
$$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$$
$\therefore \{\vee, \wedge, \neg\}$ is complete

$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$ (Demorgan law)
$\therefore \{\neg, \vee\}$ is complete
$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$ (Demorgan law)
$\therefore \{\neg, \wedge\}$ is complete

Is there a single binary truth function that's complete? Yes! Nand!

$$P \uparrow Q \equiv \neg(P \wedge Q)$$

$$P \uparrow P \equiv \neg P$$

$$\neg(P \uparrow Q) \equiv P \wedge Q \equiv (A \uparrow B) \uparrow (A \uparrow B)$$

also, nor!

$$\neg A \equiv A \downarrow A$$

$$A \vee B \equiv \neg(A \downarrow B) \equiv (A \downarrow B) \downarrow (A \downarrow B)$$

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a formula φ is satisfiable iff some row of its truth table is T (there's a possible world in which it's true)

	P	Q	R	P V Q	R → Q	etc...
	T	T	T	T	T	← "Theorem's true/or false in ~~a w~~ that world"
a world →	T	T	F	T	T	
	T	F	T	T	F	
	⋮	⋮	⋮	⋮		

Satisfiability problem: How can we determine if φ is satisfiable? (w/out making + checking a truth table) ⟹ no feasible algorithm known!

? P vs. NP Problem: it's easy to tell whether φ is satisfiable if you know which row to look at.... ?