Logic 4129/15

Incompleteness

Hilbert's program (reached its set-theoretic peak: proving a self-contained foundation for arithmetic by proving a finite list of axioms for arithmetics that are consistent and complete (using only the axioms, ideally)).

Peano arithmetic was a proposed axiomization: \((\mathbb{N},+,\cdot,0,1,<)\) axioms: familiar successor plus "the induction scheme" (\(\forall x. P(0) \land \forall x. P(x) \Rightarrow P(x+1)\) \(\Rightarrow \forall x. P(x)\)) (very powerful)

Hilbert wanted to prove that P.A. was complete and \(\omega\)-consistent (though it's not finite)?

If there were a finite, complete, consistent set of axioms for arithmetic, then they'd eventually settle all questions systematically.

If not, two are questions not settled by the axioms: we'd have to justify or axiomatize some others away if we can justify it at all.
**Theorem** (Proof by Kleene, cf.)

If $T$ is any true, computably axiomatizable theory of arithmetic, then $T$ does not prove all true statements (i.e., it is incomplete).

Suppose $T$ is the one and has a complete list of axioms. Generate all proofs from those axioms.

Then we can solve the halting problem, i.e., given a Turing machine $P$ and an input $k$, we can check to see if $P$ halts on $k$.

Q: But is $T$ expressible in the language?

Only if so, is this a reduction?

But we know that the halting problem is not decidable.

Five words of $K$ (KG): If $T$ is any true, computably axiomatizable theory, then there is some TM or some input $k$ such that $P$ does not halt on $k$, but $T$ does not prove this.

Proof: If $T$ proves all instances of the halting problem, then we can solve the halting problem.
Given $P, k$, half-time run $P(k)$, hence time code for a proof that $P(k)$ doesn't halt. So we solve the halting problem $P$.

So there must be some formula that does not halt and the thing does not prove it.

**Theorem 2:** For any computable and time-true $P$ in the language of arithmetic, there is a polynomial $P(x, \ldots, x_n)$ with integer coefficients such that $P(x, \ldots, x_n) = 0$ has no solution in the integers, but it does not prove this.

(Consequence of MRDP Theorem)

**Gödel's Incompleteness Theorem:**

Key idea: Arithmeticization of syntax $\rightarrow$

All finite combinatorial mathematical objects can be represented as numbers.

In, we can extract all information of any object from its numerical representation.

**Gödel coding:**

- Formal language of arithmetic ($+, \leq, \vdash;\quad 0, \exists, \forall, \ldots$ etc.)

Assign each symbol a Gödel code:

A number (e.g., $+/2$, $0/3$... etc.)
Now, sequences of symbols have their own Gödel code.

\[ \text{e.g. } \forall x \forall y \exists z \]

the \( n_0, n_1, n_2, \ldots, n_k \) might be the individual codes of the symbols in \( \forall \), you can give a single number that represents that sequence; \( \text{e.g. } 2^{n_0} \cdot 3^{n_1} \cdot 5^{n_2} \cdot \ldots \cdot p_k^{n_k} \) which definitely represents \( \forall \).

So: all of the syntactic operations on formulas are representable in the language of arithmetic, via Gödel coding.

Thus, whether a given program holds on an input is expressible in arithmetic, b/c it can be expressed as a number (I don’t really understand why we care, should we think this?)

So, syntactic properties are expressible too.

\[ z \text{ is the code of an axiom of } T \]

From definition of \( T \), \( T \vdash F(x) \) and \( \forall x : T \vdash F(x) \text{ provable in } T \)

\[ \text{and } \forall x : \text{a syntactic property} \]
Also expressible in arithmetic, but yay! We just saying "that \( \forall y. y \) is the code of a proof of \( \bot \) in \( \mathcal{T} \)". And that's expressible by our notion of proof in syntactic.

\[
\text{Contradiction ("\( \mathcal{T} \) is consistent") is expressible,}
\]

\[
\text{since it just says: "it's not the case that \( \mathcal{T} \) proves a \( \bot \), i.e., \( \neg \text{Pr}_T(\text{\#0} = \text{\#0}) \).}
\]

So assertions about numbers can be assertions about arithmetic.

**Fixed pt. Lemma (Carnap):**

For any formula \( \psi(x) \) in the language of arithmetic, there is a sentence \( \phi \) such that \( \text{PA} + \psi \iff \phi \). \( \psi \) says that its code has property \( \phi \).

\[\psi \iff \phi(n) \text{ when } n \text{ is the code of } \psi.\]

**Proof:**

Define \( \text{Sub}_{1}E(x)\).

Let \( \theta(x) = \phi(\text{Sub}_{1}(x, x)) \)

Let \( \psi = \theta(x) \)

Let \( \phi = \theta(n) \)
\[ PA \vdash \psi \iff \Theta(n) \]
\[ \iff \psi \iff \Theta(n) \]
\[ \iff \psi \iff \Theta(n) \]
\[ \iff \psi \iff \Theta(n) \]
\[ \iff \psi \iff \Theta(n) \]
\[ \iff \psi \iff \Theta(n) \]

So this allows us to have self-reference — indeed, it's everywhere.

\[ \psi \text{ says: } "\psi \text{ holds if the Gödel code of } \psi" \]

\[ \psi \text{ asserts that } \psi \text{ holds of its Gödel code.} \]

From Gödel's First Incompleteness Theorem

If \( T \) is a true, representable / computable, axiomatizable, Theory of arithmetic, then \( T \) is incomplete.

**Proof:**

By the fixed point lemma, there is a sentence \( \psi \) such that

\[ PA \vdash \psi \iff \neg \Pr(\ulcorner \psi \urcorner) \]

\[ \psi \iff \"\psi \text{ is not } \neg \Pr(\ulcorner \psi \urcorner)\" \]

If \( T \vdash \psi \), then \( T \vdash \Pr(\ulcorner \psi \urcorner) \), so

\[ T \vdash \Pr(T(\ulcorner \psi \urcorner)) \land \neg \Pr(\ulcorner \psi \urcorner) \]

So \( T \) is inconsistent.

So \( T \) does not prove \( \psi \). So \( \psi \) is true!
Gold's 2nd incompleteness theorem.

If $T$ is true, then $\text{PA}$ is consistent and $\text{PA}$-yields consistency ($T$):

Proof: We argued before that if $T$ is consistent, then $\text{PA}$ is consistent, $T$ is consistent. Thus $\text{PA} \vdash \neg \text{yields consistency}(T)$.

Now $\text{PA} \vdash \text{yields consistency}(T) \rightarrow \neg T$, i.e.

Fix $\phi$ as before. $\text{PA} \vdash \neg \text{yields consistency}(\phi)$.

Proof: We argued previously that $\text{PA}$ yields consistency ($\phi$).

Hilbert–Bernays derivability conditions (basic logic):

$b): \text{IT} \vdash \text{PA} \vdash \text{yields consistency}(T)$, which implies $\text{PA}$.

So $\text{IT} \vdash \text{PA} \vdash \text{yields consistency}(T)$, but otherwise $\text{PA}$ consistent. $\text{PA} \vdash \neg \text{yields consistency}(T)$.
Löb's Theorem:

IF $T \vdash \text{Pr}_T(\langle \varphi \rangle) \rightarrow \varphi$

Then $T \vdash \varphi$.

3) is my easy. 1) is easy as in how (2)?

(2):

- Formalize the previous implication instead

From (1), (2), and (2):

$$4 \leftrightarrow \neg \text{Pr}_T(\langle \varphi \rangle)$$

(1) $T \vdash \text{Pr}_T(\langle \neg \varphi \rangle) \rightarrow \neg \text{Pr}_T(\langle \neg \varphi \rangle)$

(3) $T \vdash \text{Pr}_T(\langle \varphi \rangle) \rightarrow \text{Pr}_T(\langle \neg \text{Pr}_T(\langle \varphi \rangle) \rangle)$

By (2). $T \vdash \text{Pr}_T(\langle \varphi \rangle) \rightarrow \text{Pr}_T(\langle \neg \text{Pr}_T(\langle \varphi \rangle) \rangle)$

So $T \vdash \text{Pr}_T(\langle \varphi \rangle) \rightarrow \text{Pr}_T(\langle \neg \text{Pr}_T(\langle \varphi \rangle) \rangle)$; so you're proven. Thus

If 4 is provable, I can prove

1, which means that $\neg 4 \rightarrow$

$\neg \text{con} (T)$. So $T \vdash \text{con} \rightarrow 4$.

$$T \vdash \neg 4 \rightarrow \text{con} \neg$$

$$T \vdash \text{con} \rightarrow 4$$

Given assume that $T$ was true but he
only needed to assume it was co-consistent.

Theorem (Gödel/Post correspondence)
Every consistent representable theory $T$
containing PA is incomplete.
Proof: Use Rosser sentence (Ω)

PA ⊢ Ω → "for any proof of 50 ⊢ Ω,
that is a shorter proof of 50 ⊢ ¬Ω!"

(In proof has at least k steps)

If T ⊢ Ω, then T must prove that there is a shorter proof of 50 ⊢ ¬Ω of length < k.

But then T ⊢ ¬Ω. L. T is consistent.

If T ⊢ ¬Ω, then this is a proof of length k

T ⊢ Ω. That is a proof of Ω or no shorter proof of ¬Ω. So the whatever proof of Ω has to be shorter than our proof of ¬Ω.

So T ⊢ Ω. L.

So T does not prove Ω or ¬Ω, hence Ω.

So we cannot find compact axiomatizations of arithmetic. So we cannot expect a recursive independence phenomenon.

∀T there is an Ω such that Ω is not provable by T.

ζ(Ω) ∈ FC, assuming continuum hypothesis, and certain others.
Since you can never prove the consistency of T & then it is not the case that T then no consistent they can be complete.

**Goldsten Sequences**

\[ a_2 = 4^2 = \frac{2^{2^3+1}}{2^{2+1} + 2} \quad \text{(correct base 2)} \]

\[ a_3 = \frac{3^{3^2} + 1}{3^{3+1} + 3} - 1 = \frac{3^{3^2} + 1}{3^{3+1} + 3} - 1 \]

\[ a_4 = \frac{4^{4^{4+1}} + 4^{4+1} + 4}{3} \]

\[ a_5 = \ldots \]

\[ a_6 = \frac{3^{3^{3^{3+1}}} + 3^{3^{3+1}}} - 1 = \ldots \]

**Goldstein's Theorem**

\[ \forall a_2 \in \mathbb{N}, a_3 = 0 \]

**Theorem (Curry & Paris)**

Goldsteins' Theorem is not provable in PA. It is true but not provable in PA.

It is provable in set theory but not arithmetic.

**Hydra Theorem**

1. Every strategy succedes in killing the hydra, even!

2. The existence of (2) is not provable in PA. (But in set theory, PA is not at all enough for that.)