

Logic 2/18/15

Predicate Logic

Prop. logic is a 'baby language' b/c the worlds of prop logic consist of propositions with T-values. But sometimes what we need to reason about a richer context in which we have objects & individuals standing in various relations to each other

Q: Couldn't you express all that with propositions?

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## Relational Logic / Theory of relations

Context: domain of objects

various relations on that domain

e.g.  $a S b$  (a is a sibling of b)

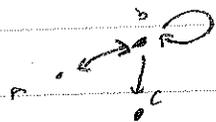
$\Psi \equiv \phi$   $\Psi$  is logically eq. to  $\phi$  in

classical logic (domain of discourse here is prop. assertions using some connectives)

Q: for each relation, you need to define a domain?

$X \perp Y$  the 'never true' / empty relation

$X T Y$ , the 'always true' relation



This is a picture of the relations that may hold between some objects

prefix notation nicer for non-binary relations. (e.g. ternary  $A(x, y, z)$ ) ('n-ary')

relations are (always?) directionally/asymmetrical.

Such thing as a 'unary relation'? It's really a property of individuals.  $A(x)$ .

Such thing as a 0-ary relation? Yes, in a way - it's a logical constant - True or False. (still different from  $\perp$  &  $\top$  even though they, too, are always T or F.)

It is convention to have a fixed 'arity' per relation (# of inputs)  $\rightarrow$  used to distinguish different relations?

Binary relation  $R$  on domain  $A$  is reflexive if  $aRa$  for all  $a \in A$

Is  $\perp$  reflexive? Only in the ~~domain~~ <sup>empty</sup> domain. Then it's reflexive vacuously.

"There is ~~no~~ only one relation in the empty domain" (interesting!)  
b/c we define relations externally

classical logic context

$R$  is symmetric if  $a R b \iff b R a$   
 $R$  is transitive if  $(a R b \wedge b R c) \rightarrow a R c$

what the domain is determines what  
some  $R$  is symmetric, transitive, etc.

Q: how can we talk about the empty  
relation?

The empty relation is vacuously symmetric  
and transitive.

empty domain + empty relation can  
help test our concepts for accuracy? tightness?

[Ex 1] Are all 8 combinations of  
reflexivity, symmetry, and transitivity  
possible? Provide instances.

Support or  
[Ex 2] / criticize the following argument:

"If  $R$  is symmetric and transitive,  
then it is reflexive, since if  $a R b$ ,  
 $b R a$  by symmetry, and so  $a R a$   
by transitivity."

If this is not correct, what is a  
similar statement proved by a similar argument?

Reflexive + trans. but not Symmetric :  
a subset of b. ( $a \subseteq b$ )

Reflex + Sym +  $\neg$  trans:  
having a class in common (domain =  
all enrolled students)

trans + Sym +  $\neg$  reflex:  
a  $\perp$  b on the non-empty domain.  
a  $\cap$  b (neither a nor b is Ivan Malcovich)

R is irreflexive if  $\neg aRa$  (for all a in domain)

on the empty domain, anything is  
both reflexive and irreflexive.

So irreflexive (R)  $\nrightarrow$  not reflexive (R),  
but this does work on any non-empty  
domain

Standard  
but not  
universal  
usages

R is asymmetric if  $aRb \rightarrow \neg(bRa)$   
R is antisymmetric if  $(aRb \wedge bRa) \rightarrow a=b$

asymmetric (R)  $\rightarrow$  antisymmetric (R).  
(vacuously)

= is antisymmetric, but not asymmetric  
(Same as  $\leq, \geq$  args)

(all relations are reflexive on the empty domain)

Ex 3 how are the 3 properties (relations?)

- symmetry
- non-symmetry
- antisymmetry

related? which imply which?  
which of the 8 combinations are possible? Give examples.

$R$  is anti-transitive if  $aRb \wedge bRc \rightarrow \neg (aRc)$   
e.g. being 1 less than

Brother-sister relation is both transitive and anti-transitive (vacuously)

A binary relation ' $\sim$ ' is an equivalence relation if it is reflexive, symmetric, and transitive.  
e.g.  $=$ ,  $\equiv$ ,

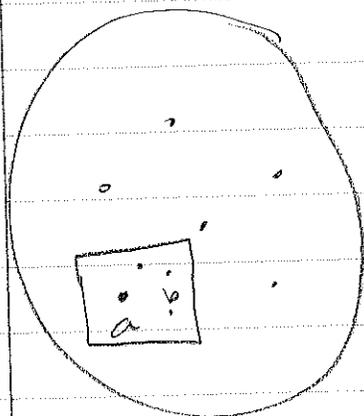
Reflexivity is domain-sensitive, but being symmetric and transitive are not.

So you haven't referred to a relation definitively if you haven't specified the domain

a relation consists in (a) a domain and (b) information about whether the relation holds for each set of domain members of the right arity.

'Same height' is another equivalence relation.

$A$  is the domain  
 $\sim$  is an eq. relation on that domain.



$[a]_{\sim}$  equivalence class of  $a$   
 $= \{b \mid b \sim a\}$

e.g.  $[JDH]_{\text{same height}}$   
is the set of all people  
w/ the same height of you.

(?) there is an eq. class for each item  
in the ~~set~~ domain?

every item is a member of its own eq. class;  $\therefore$  eq. classes never empty. ~~two~~

Two eq. classes are  $= \iff a \sim b$   
 $[a] = [b] \iff a \sim b$

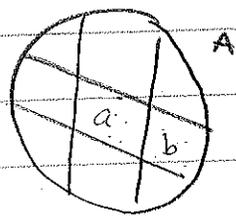
eq. classes turn  $\sim$  into identity

eq. is the same in a sense, when you take an eq. class, it becomes actually identity.

IF  $[a] \overset{\text{overlap}}{\cap} [b]$ ; Then  $[a] = [b]$   
 $c \in [a] \cap [b]$

b/c if  $c \in [a]$  and  $c \in [b]$ ,  
then  $c \sim a$ ,  $\exists c \sim b$ , and  $\therefore a \sim b$ .

eq. relations give rise to a partition of the domain



each piece of the partition is  $[a]$  for some  $a \in A$ .  
pieces are nonempty and don't overlap. each item in set is in one of the pieces.

Conversely, if we have a set  $A$  w/ a partition, then we can define  ~~$a \sim b$~~   $a \sim b \iff a$  and  $b$  are in the same piece of the partition.

$\therefore$  eq. relations are essentially the same as having a partition. To define an eq. relation is to define a partition. Weird!

draw picture refinements.

### Exercise 4

How many eq. relations are there on a 4-element set? Draw a picture of the entire hierarchy.

least fine relation  $\rightarrow$   always the relation

There's 1 eq. relation where they're all eq. to each other, another where the first 3 are, etc.



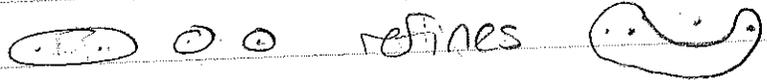
At bottom, 

$\uparrow$  finest

$\uparrow$  I.D. relation

① One relation  $F$  refines another,  $G$ , if  $a \in b \rightarrow a \in Fb$  relations contained in other relations???

if  $R$  relation holds, then  $Q$  holds  $\rightarrow$   
 ~~$R$  refines  $Q$~~   $R$  refines  $Q$ .

e.g.  refines 

e.g. 'living in same city as' refines 'living in same country'.  
the e.g. ~~is~~ classes for the 'living in same city as' relation and the cities.

w/ idealizing assumptions

### Exercise 5

Discuss the precise issues undermining the above example and ways in which it might not be true.

e.g. person outside ACA, or a person lives in more than one city, or a city is in more than one country, or, say what exact part of claim is going wrong for each nonidealized consideration. (see website)

"ways in which it could be vague" ?  
e.g. if same lives in 2 cities, then messes up transitivity.

## orders

a partial order on a set is a reflexive, transitive, anti-symmetric relation

e.g.  $\leq$      $a \leq a$ ,  $a \leq b \wedge a \leq c \rightarrow a \leq c$ ,  
 $a \leq b \wedge b \leq a \rightarrow a = b$

## examples

Usual order  $x \leq y$  on real numbers;  
Subset<sup>of</sup> relation  $a \leq a$ ,  $a \leq b$ ,  $a \leq b \rightarrow a \leq c$

"is a factor of"  $\equiv$   $a \leq b \wedge a \leq c \rightarrow a \leq b$

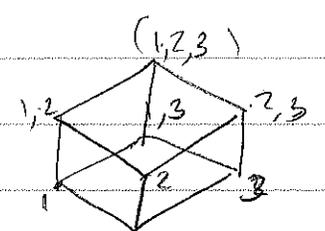
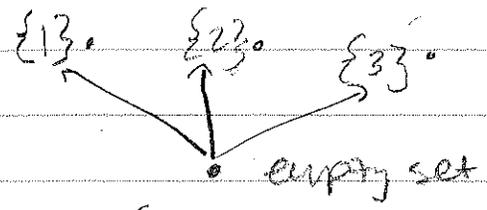
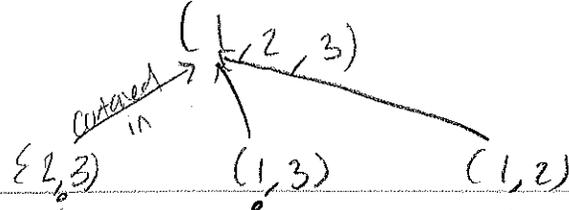
$\frac{\mathbb{N}}{\neq}$  "divides" relation ( $n \mid m$ )

( $\because$   $m$  is a multiple of  $n$  /  $m$  divides  $n$ )

is a partial order ~~on~~ ~~the~~ set of natural #s. (but not on integers)

⊛ is there a way of talking about relations that are equivalent, such as 'divides' & 'is a factor of'? Are they just  $\neq$  identical? what about domains.

From  $P(\{1, 2, 3\}) =$  Pow set of  $\{1, 2, 3\}$  - set of all subsets (8 points)



1 & 2 are not comparable (2,3)

each point is also a subset of itself

⚠ Skeleton of the order (doesn't include all lines, but they follow from relations' Properties

'being on the same level' is an eq. relation

Define:  $a$  is comparable to  $b$  in a partial order  $\leq$  if either  $a \leq b$  or  $b \leq a$ , and otherwise incomparable.

Exercise 6

Is comparability an equivalence relation? Consider each of the properties we've considered (reflexivity, irreflexivity, symmetry... etc.) and argue whether comparability always has that property or give a counterexample