Logic notes 3/25/15

Definability

two kinds:
1) if M is a structure, then an element a in M is definable in M if there is a formula \( \varphi(x) \) such that M satisfies \( \varphi(a) \) (M |= \( \varphi(a) \)) and a is the only object satisfying \( \varphi \) in M.

2) if \( A \subseteq M \), then A is a definable class in M if A is the set of all objects that satisfy a certain property for some formula \( \varphi \).

i.e., \( A = \{ a \in M | M \models \varphi(a) \} \) for some formula \( \varphi \)

Eg. Partial orders \( \leq \)

we can talk about a now because it's definable -
we can refer to a language to its unique object satisfying \( \varphi \).

now we can define other things by relation to c:
\( y = b \iff d < y \land \exists x : y < x \)
We can write that without referring directly to \( d \).

\[
\forall z \ (z = d \rightarrow z \leq y \land \exists x \ y < x)
\]

\[
\forall z \ (\forall y \ y \leq z \lor z \leq y) \rightarrow z \leq y \land \exists x \ y < x
\]

So its kosher. (As our kosher we can write it with \( \leq \) and \( \leq \).

\[
U = a \iff b < u \\
V = c \iff d < v \land \forall z a \land b
\]

or \( a \lor z \land \exists u v < u \).

Etc. with the rest. So all these points were definable.

\[\text{Ex 1} \]

Consider partial orders:

(i) \]

(ii) \]

Which elements are definable? Provide definitions. Which elements are indiscernable? Give a justification.

If \( a, b \) are indiscernable, "interchangeable" even if \( u \) is an isomorphism, then neither is definable?
\( \langle N, \langle \rangle \rangle (0, 1, 2, \ldots) \)

all elements are definable. (a stronger property than being Lindernian)

You can say that the are exactly 17 things less than \( x \rightarrow \) has you determine 17.

\[ \exists u_0 \ldots u_{16} \ (u_0 < u_1 < u_2 \ldots u_{16} < x \ \land \ \forall y \ ye x \rightarrow y = u_1 \lor y = u_2 \ldots u_{16}) \]

we could use fact that \( a \leq b \rightarrow a \leq b \).

\( \langle Z, \langle \rangle \rangle (-\infty, 1, 0, 1, 2, \ldots) \)

no definable elements.

all are not all are indiscernable.

\[ \text{all subsets of a would be the set of } b \]

\[ \text{if } a \leq b \text{, then } f(a) \leq f(b) \]

Define \( T(x) = x + k \) where \( k = b - a \).

\[ x \leq y \leftrightarrow x + k \leq y + k \]

\( T \) preserves structure, an automorphism.

\[ M, \ N \]

if \( F \) is a map for \( M \rightarrow N \) (\( F: \mathbb{N} \rightarrow \mathbb{N} \))

- \( F \) is a homomorphism iff \( F \) preserves structure \( (\rightarrow \text{ and } \leftrightarrow \text{ versions}) \) between \( M \rightarrow N \)

- \( F \) is an isomorphism if \( F \) is a homomorphism that is also one-to-one and onto, i.e., bijective, one-to-one.
Let \( M + N \). (i.e. any automorphism model of \( M \) is \( f \) (something of same model))

- \( f \) is an automorphism if \( f \) is an isomorphism of a structure to itself if \( M = N \).

This is why \( \Pi(x) = x + 1 \) in the above structure is an automorphism.

So \( <Z, \cdot> = f(a) \) if \( <Z, \cdot> = f(b) \)

If every object in \( M \) is definable, then \( M \) is Leibnizian.

Q: Is the converse true? Is the ability to discern objects the same as the ability to pick them out uniquely?

No.

E.g. look at the model of infinitely many distinct constants plus one more object.

Language has \( c_1, c_2, c_3, \ldots \). Model interprets those constants as distinct, & then there's one other object. That is \( x \neq c \) of the objects named by the constants, or.

(i) \( M \) is Leibnizian. Easily - if \( x \neq x \) in \( M \), then xor \( x \) or \( x \) is self-consistent \( C \), & this property all implies \( x \) \( \neq y \).
(2) Each $C_n$ is definable, $x = C_n$ defines $C_n$. But $a$ is not definable, because what you want to say is that $a$ is not equal to any of the $C_n$ (i.e., not $C_n$). But such a formula can be of any length, and an $a$ must be finite.

Suppose $C(x)$ is any formula in the language. Since $C$ is finite, it lists any finitely many $C$-variables symbols $C_0, \ldots, C_k$, for some $k$.

Now, consider the reduct of $M$ to this smaller language and any $k$. Consider $C_0, \ldots, C_k$ points. (These used to be $C_1$'s).

This smaller language has an isomorphism:

Swap $a$ with some $C_n$. For $n > k$:

$M_{\text{reduct}} \models C(a)$ if $M_{\text{reduct}} \not\models C(C_n)$

Basically, if $C$ is satisfied by $a$, then any object not explicitly mentioned by that formula will also satisfy $C$.

Is the idea that $M_{\text{reduct}}$ was more clear?

That they must focus on a bounded use of $M_{\text{reduct}}$ here and how we prove things about $M$ with it.
**Exercise 1**
Consider an eq. relation \( \sim \) on a set \( A \) with exactly one eq. class of size 7, one eq. class of size 2, and one of size 3, etc. (Is this possible?)

\[ (A, \sim) \]

What are the definable elements?
What are the definable subsets?
(classies?)

**Exercise 2**
Show that every object in \( \langle \mathbb{N}, + \rangle \)
is definable

\[ \forall x (x + y = x) \text{ defines } \emptyset. \]
get \( x \), the iterate.

Q: is \( \langle \mathbb{N}, +, 0 \rangle \) definable

How do you define a relation?

\( a \leq b \) iff \( \langle \mathbb{N}, +, 0 \rangle \vdash \exists c \ a + c = b \)

\( a \prec b \) iff \( \langle \mathbb{N}, +, 0 \rangle \vdash \exists c \ c \geq 0 \land a + c = b \)

Q: is less than definable in \( \langle \mathbb{Z}, +, 0 \rangle \)?

\( a \prec b \) iff \( \langle \mathbb{Z}, +, 0 \rangle \vdash \)

No. It's impossible.

Let \( f(x) = -x \) \( \in \) automorphism

\( -(x + y) = (-x) + (-y) \)

so \( f \) is an isomorphism of addition. Thus, if
\[ f(0) = 0 \quad \forall \alpha - \alpha = 0 \]

**Theorem:** if \( \langle \mathbb{Z}, +, 0 \rangle \models f(a, b) \)

Then \( \langle \mathbb{Z}, +, 0 \rangle \models f(-a, -b) \)

but \( a < b \) is not equivalent to \( -a < -b \). So therefore, no \( f(a, b) \) can be equivalent to \( a < b \).

i.e. you cannot define \( < \) because \( < \) is not respected by the automorphism on the model.

**Exercise:** Show that \( < \) is not definable in \( \langle \mathbb{Z}, + \rangle \), i.e., there is no formula \( \phi(a, b, c) \) such that \( a < b = c \) iff \( \langle \mathbb{Z}, + \rangle \models \phi(a, b, c) \).

\[ \langle \mathbb{N}, \cdot \rangle \]

**Define:** \( a < b \) (a divides b) iff \( \langle \mathbb{N}, \cdot \rangle \models \phi(a, b, c) \)

**Define:** \( + \) and \( - \) in \( \langle \mathbb{Z}, - \rangle \)

\[ x + y = z \iff \langle \mathbb{Z}, - \rangle \models y = z - x \]

\[ x - y = z \iff \langle \mathbb{Z}, + \rangle \models x = y + z \]

**Peano Structure**

\[ \langle \mathbb{N}, S, 0 \rangle \]

\[ S(0) = 1 \quad S(n) = n, \quad S(x) = x + 1 \]
Some Axioms: 1. \( \forall x \ S(x) \neq 0 \)
2. \( \forall x \forall y (S(x) = S(y) \rightarrow x = y) \)
3. \( \forall x \ (y \neq 0 \rightarrow \exists y \ y = s(x) \)

\[ S(0) \]
\[ S(n) \]
\[ S(\alpha) \]
\[ \emptyset \]

What actually defines election

\[ \langle N, S, \emptyset \rangle \]

\[ x = 1 \text{ means: } \]
\[ \emptyset : x = \emptyset \]
\[ 1: S(0) = x \]
\[ x = 2 \text{ means: } Sx = Ss(0) \]
\[ x = n \text{ means: } x = Ss(\ldots) \]

in object theory,

an object in a category

in metatheory

using this \#,

This quantifier depends on the numbers we use in our metatheory. Matching in some way to the #s that are elements of \( \mathbb{N} \).

Theorem: \( \langle N, S, \emptyset \rangle \) satisfies "elimination of quantifiers" i.e., for any formula \( \phi (x_1, \ldots, x_n) \) in this language, there is a quantifier-free formula \( \psi^* (x_1, \ldots, x_n) \) such that \( \langle N, S, \emptyset \rangle \models (a_1, \ldots, a_n) \rightarrow \]

\( \psi^* (a_1, \ldots, a_n) \) i.e., every formula is equivalent to a quantifier-free formula.
prove this by induction on formulas

(1) The for atomic formulas, b/c the are identical to their quantifier-free selves.
(2) This property is preserved by boolean combinations.

\[ \phi \iff \phi^* \iff (\forall \forall Y) \iff (\phi^* \forall \forall Y) \]

(b) This is reused by quantification lemmas...

-\[ \exists x \phi(x, x, \ldots, x_n) \iff \exists x \phi^*(x_1, \ldots, x_n) \]
-\[ \exists x \phi \iff \text{D.N.F.} \]

-\[ \exists x (A \lor V \lor B) \iff \exists x A \lor \exists x V \lor \exists x B \]

-\[ \exists x \phi \iff \bigwedge \phi \]

-\[ \text{conjunction clause} \]

we can eliminate that \( \exists x \).

\[ \exists x (\exists x \forall = \exists x \forall = \exists x \forall \forall \exists x \forall) \]
-\[ \text{rules to} \]

\[ \exists x \phi \iff \exists x (\exists x \forall = \exists x \forall = \exists x \forall \forall \exists x \forall) \]

ok. So you can eliminate \( x \) from this, but how is this generalizable???
Ex 6 Show "even" and "odd" are definable in \( <\mathbb{N}, +> \), conclude that it is not definable in \( <\mathbb{N}, S> \).

Ex 7 Show "prime" is definable in \( <\mathbb{N}, > \).