

# The absolute truth about non-absolute truth

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This is joint work with Ruizhi Yang of Fudan University, Shanghai.

Our paper is available on my blog:

*J. D. Hamkins, R. Yang, "Satisfaction is not absolute," manuscript under review.*

# My favorite situation

A philosophical concern...

leads to interesting mathematical questions...

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# The philosophical concern

## Question

To what extent does definiteness of mathematical objects lead to definiteness of our theory of mathematical truth about those objects?

Many mathematicians express a commitment to the definiteness of the natural numbers  $0, 1, 2, \dots$  and the structure  $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ . Does this commit us to the definiteness of arithmetic truth?

Feferman, for example, defends a view whereby arithmetic truth has a definite character, whereas set-theoretic truth, at the level of  $P(\mathbb{N})$  and above, is less so. “CH is vague”

Solomon Feferman (EFI 2013):

In my view, the conception [of the bare structure of the natural numbers] is *completely clear*, and thence *all arithmetical statements are definite*.

It is Feferman's 'thence' to which I call attention.

Donald Martin (EFI 2012):

What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers.

# The mathematical question

## Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?



# The mathematical question

## Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The answer is yes, and indeed, this is pervasive.

- Models of set theory can have the same  $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ , yet disagree on arithmetic truth.
- Models of set theory can have the same reals, yet disagree on projective truth.
- Models of set theory can have  $V_\delta$  in common, yet disagree about whether it is a model of ZFC.
- and many more...

A word about the folklore nature of these results. Several of the examples follow from established techniques in models of arithmetic, such as resplendency.

Nevertheless, I will give soft proofs, using classical methods.

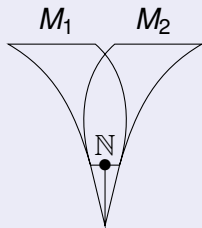
Meanwhile, I find the examples to be striking and I think they deserve to be more widely known.

## Theorem

If ZFC is consistent, then there are  $M_1, M_2 \models \text{ZFC}$  which agree on the structure of arithmetic

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



$$M_1, M_2 \models \text{ZFC} \quad \mathbb{N}^{M_1} = \mathbb{N}^{M_2}$$

There is a sentence  $\sigma$  in  $M_1$  and  $M_2$

$M_1$  believes  $\mathbb{N} \models \sigma$

$M_2$  believes  $\mathbb{N} \models \neg\sigma$

## Standard models

We all know THE standard model of arithmetic:  $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ .

A model of arithmetic is A standard model, or *ZFC-standard*, if it is the standard model of arithmetic  $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^M$  extracted from some model of set theory  $M \models \text{ZFC}$ .

So a standard model of arithmetic is precisely one that is thought to be the standard model of arithmetic from the perspective of a model of set theory.

If  $M \models \text{ZFC}$ , we may extract from it  $\mathbb{N}^M$  as well as  $\text{TA}^M = \{ \sigma \mid M \models \mathbb{N} \models \sigma \}$ .

## Satisfaction class

A *truth predicate* on a model of arithmetic  $\mathcal{N} = \langle N, +, \cdot, 0, 1, < \rangle$  is a class  $\text{Tr} \subseteq N$  for which:

- For  $\sigma$  atomic,  $\sigma \in \text{Tr}$  just in case  $\mathcal{N}$  thinks  $\sigma$  is true.
- $\sigma \wedge \tau \in \text{Tr}$  iff  $\sigma \in \text{Tr}$  and  $\tau \in \text{Tr}$ .
- $\neg\sigma \in \text{Tr}$  if  $\sigma \notin \text{Tr}$ .
- $\exists x \varphi(x) \in \text{Tr}$  iff  $\varphi(\underbrace{1 + \dots + 1}_n) \in \text{Tr}$  some  $n \in N$ .

Tarski: No such truth predicate is definable in  $\mathcal{N}$ .

Every ZFC-standard model of arithmetic has an *inductive* truth predicate, meaning  $\langle N, +, \cdot, 0, 1, <, \text{Tr} \rangle \models \text{PA}(\text{Tr})$ .

# Incompatible truth predicates

## Theorem (Krajewski 1974)

*There are models of arithmetic with different incompatible inductive truth predicates.*

## Proof.

Suppose  $\mathcal{N}_0 = \langle N_0, +, \cdot, 0, 1, < \rangle$  has an inductive truth predicate. Let  $T$  be the elementary diagram  $\Delta(\mathcal{N}_0)$ , plus “Tr is an inductive truth predicate.” This is consistent. Any model of  $T$  provides an elementary extension of  $\mathcal{N}_0$ . If they all have a unique truth predicate, then Tr would be implicitly definable in the sense of Beth’s implicit definability theorem, and hence by Beth must be explicitly definable, which contradicts Tarski’s theorem. So there is an elementary extension  $\mathcal{N}$  of  $\mathcal{N}_0$  with at least two different truth predicates. □

## Theorem (Enayat)

*The following are equivalent for countable nonstandard  $\mathcal{N}$ :*

- 1**  $\mathcal{N}$  is ZFC-standard. That is,  $\mathcal{N} = \mathbb{N}^M$  some  $M \models \text{ZFC}$ .
- 2**  $\mathcal{N}$  is a computably saturated model of  $\text{Th}(\mathbb{N})^{\text{ZFC}}$ .

### Proof.

(1  $\rightarrow$  2) If  $\mathcal{N}$  is ZFC-standard, then  $\mathcal{N} \models \text{Th}(\mathbb{N})^{\text{ZFC}}$ , and inductive truth predicate implies computably saturated by overspill argument.

(2  $\rightarrow$  1) Suppose  $\mathcal{N} \models \text{Th}(\mathbb{N})^{\text{ZFC}}$  is countable computably saturated. Let  $T = \text{ZFC} + \sigma^{\mathbb{N}}$  for all  $\sigma \in \text{Th}(\mathcal{N})$ . Consistent. By computable saturation,  $T$  is coded in  $\mathcal{N}$ ; so get nonstandard finite consistent  $t \in \mathcal{N}$  extending  $T$ . Build the Henkin completion of  $t$  in  $\mathcal{N}$ , with Henkin model  $M$ . So  $M \models \text{ZFC}$  and  $\mathbb{N}^M \cong \mathcal{N}$ . Further,  $a \mapsto (1 + \dots + 1)^a$  maps  $\mathcal{N}$  to an initial segment of  $\mathbb{N}^M$ , so same standard system. By back-and-forth, they are isomorphic, and so  $\mathcal{N}$  is ZFC-standard.  $\square$

## Characterizing ZFC-standard arithmetic truth

Similarly, let  $\text{Th}(\mathbb{N}, \text{TA})^{\text{ZFC}}$  be the ZFC-provable assertions about the structure  $\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle$ .

### Theorem

*The following are equivalent for any countable nonstandard model of arithmetic  $\mathcal{N}$  with a truth predicate  $\text{Tr}$ .*

- 1**  $\langle \mathcal{N}, \text{Tr} \rangle$  is a ZFC-standard model of arithmetic and arithmetic truth. That is,  $\mathcal{N} = \mathbb{N}^M = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^M$  for some  $M \models \text{ZFC}$  in which  $\text{Tr} = \text{TA}^M$  is the theory of true arithmetic.
- 2**  $\langle \mathcal{N}, \text{Tr} \rangle$  is a computably saturated model of  $\text{Th}(\mathbb{N}, \text{TA})^{\text{ZFC}}$ .

One may use this with Beth's theorem to prove the non-absolute truth theorem.



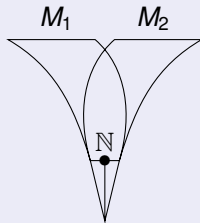
# Satisfaction is not absolute

## Theorem

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but which disagree on arithmetic truth.



$$M_1, M_2 \models \text{ZFC} \quad \mathbb{N}^{M_1} = \mathbb{N}^{M_2}$$

There is a sentence  $\sigma$  in  $M_1$  and  $M_2$

$M_1$  believes  $\mathbb{N} \models \sigma$

$M_2$  believes  $\mathbb{N} \models \neg\sigma$

## Proof

Fix any countable  $M_1 \models \text{ZFC}$  with  $\langle \mathbb{N}, \text{TA} \rangle^{M_1}$  computably saturated.

Claim there are sentences  $\sigma, \tau$  with same 1-type in  $\mathbb{N}^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1}$ , but  $M_1$  thinks  $\sigma$  is true and  $\tau$  false in  $\mathbb{N}^{M_1}$ . (Proof: consider the type  $p(s, t)$  containing  $\varphi(s) \iff \varphi(t)$  and  $s \in \text{TA}$  and  $t \notin \text{TA}$ ; this is finitely realized, since TA is not definable.)

By back-and-forth, there is automorphism  $\pi : \mathbb{N}^{M_1} \rightarrow \mathbb{N}^{M_1}$  with  $\pi(\tau) = \sigma$ .

Build a copy  $M_2$  of  $M_1$  so that  $\pi$  extends to an isomorphism  $\pi^* : M_1 \rightarrow M_2$ . So  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ . But  $M_1$  thinks  $\sigma$  is true, yet  $M_2$  thinks  $\sigma = \pi^*(\tau)$  is false.  $\square$

## A generalization

### Theorem

*For any countable  $M \models \text{ZFC}$ , any structure  $\mathcal{N} \in M$  finite language, any  $S \subseteq N$  in  $M$  not definable in  $\mathcal{N}$ . Then there are  $M \prec M_1$  and  $M \prec M_2$  with  $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$ , yet  $S^{M_1} \neq S^{M_2}$ .*

Note  $S^{M_1}$  and  $S^{M_2}$  share all properties of  $S$  in  $M$ .

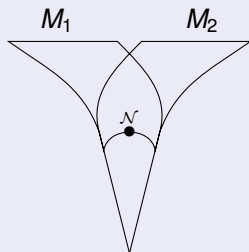
### Proof.

Fix  $M \prec M_1$  countable computably saturated. So  $\langle \mathcal{N}, S \rangle^{M_1}$  is computably saturated. Since  $S$  not definable, there are  $a, b \in \mathcal{N}^{M_1}$  with same 1-type in  $\mathcal{N}^{M_1}$ , but  $a \in S, b \notin S$ . So  $\exists \pi : \mathcal{N}^{M_1} \cong \mathcal{N}^{M_1}$  with  $\pi(b) = a$ . Extend  $\pi$  to  $\pi^* : M_1 \cong M_2$ . So  $a \in S^{M_1}$  but  $a = \pi(b) \notin S^{M_2}$ . □

# Satisfaction is not absolute

## Corollary

*If  $M$  is a countable model of set theory and  $\mathcal{N}$  is (sufficiently robust) structure in  $M$ , in a finite language. Then there are  $M \prec M_1$  and  $M \prec M_2$ , which agree on the natural numbers  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$  and on  $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$ , yet disagree on satisfaction  $\mathcal{N} \models \sigma[\vec{a}]$  for this structure.*



$M \prec M_1, M_2 \models \text{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2}, \quad \mathcal{N}^{M_1} = \mathcal{N}^{M_2}$

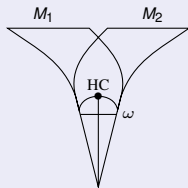
*there are  $\sigma$  and  $\vec{a}$  for which*

*$M_1$  believes  $\mathcal{N} \models \sigma[\vec{a}]$*

*$M_2$  believes  $\mathcal{N} \models \neg\sigma[\vec{a}]$*

## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which agree on their natural numbers  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ , their reals  $\mathbb{R}^{M_1} = \mathbb{R}^{M_2}$  and their hereditarily countable sets  $\langle \text{HC}, \in \rangle^{M_1} = \langle \text{HC}, \in \rangle^{M_2}$ , but which disagree on their theories of projective truth.*



$M_1, M_2 \models \text{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2} \quad \mathbb{R}^{M_1} = \mathbb{R}^{M_2}$

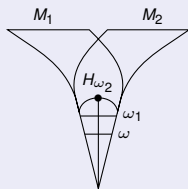
$\langle \text{HC}, \in \rangle^{M_1} = \langle \text{HC}, \in \rangle^{M_2}$

$M_1$  believes  $\text{HC} \models \sigma$

$M_2$  believes  $\text{HC} \models \neg \sigma$

## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which agree on the structure  $\langle H_{\omega_2}, \in \rangle^{M_1} = \langle H_{\omega_2}, \in \rangle^{M_2}$ , but which disagree on truth in this structure.*



$M_1, M_2 \models \text{ZFC}$

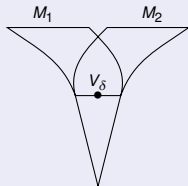
$\langle H_{\omega_2}, \in \rangle^{M_1} = \langle H_{\omega_2}, \in \rangle^{M_2}$

$M_1$  believes  $H_{\omega_2} \models \sigma$

$M_2$  believes  $H_{\omega_2} \models \neg\sigma$

## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which have a transitive rank-initial segment  $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$  in common, but which disagree on truth in this structure.*



$M_1, M_2 \models \text{ZFC}$

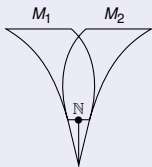
$V_\delta^{M_1} = V_\delta^{M_2} \models \text{ZFC}$

$M_1$  believes  $V_\delta \models \sigma$

$M_2$  believes  $V_\delta \models \neg\sigma$

## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which agree on their natural numbers with successor, addition and order  $\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$ , but which disagree on natural-number multiplication, so that  $M_1$  thinks  $a \cdot b = c$  for some particular natural numbers, but  $M_2$  disagrees.*



$M_1, M_2 \models \text{ZFC}$

$\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$

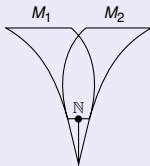
$M_1$  believes  $\mathbb{N} \models a \cdot b = c$

$M_2$  believes  $\mathbb{N} \models a \cdot b \neq c$



## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which agree on their natural numbers with successor and order  $\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$ , but which disagree on the even numbers, the prime numbers and the powers of two, so that  $M_1$  thinks some  $n$  is a large odd prime number, but  $M_2$  thinks it is a large power of 2.*



$M_1, M_2 \models \text{ZFC}$

$\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$

$M_1$  believes  $\mathbb{N} \models n$  is an odd prime

$M_2$  believes  $\mathbb{N} \models n = 2^k$  for some  $k$

## Non-isomorphic non-absoluteness

Roman Kossak pointed out that our models  $M_1$  and  $M_2$  with  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$  and disagreeing on truth, are nevertheless isomorphic. He asked whether we can produce non-isomorphic examples.

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}, \text{ but}$$

$$\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle^{M_1} \not\cong \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle^{M_2}.$$

Jim Schmerl observed that indeed this occurs: the theory  $T = \text{TA} + \text{Th}(\mathbb{N}, \text{TA})^{\text{ZFC}}$ , in the language with  $\text{Tr}$ , is computable from  $0^{(\omega)}$ , but the theory of  $\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{TA} \rangle$  has degree  $0^{(\omega+\omega)}$ , and so  $T$  is not complete. Thus, there are computably saturated models  $\langle \mathcal{N}, \text{Tr}_1 \rangle$  and  $\langle \mathcal{N}, \text{Tr}_2 \rangle$  of  $T$ , which have the same reduct to arithmetic, but which are not elementarily equivalent in the full language and hence not isomorphic. These models are ZFC-standard, as desired. ◻

## Iterated truth predicates

Begin with standard model  $\mathbb{N}_0 = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ .

Consider  $\mathbb{N}_1 = \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{Tr}_0 \rangle$ , with a truth predicate  $\text{Tr}_0$  for arithmetic assertions.

Continue with  $\mathbb{N}_2 = \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{Tr}_0, \text{Tr}_1 \rangle$ , by appending an additional truth predicate  $\text{Tr}_1$  in the expanded language.

Continue with  $\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{Tr}_0, \dots, \text{Tr}_n \rangle$  and so on...

# Truth about truth is not absolute

## Corollary

*For every countable model of set theory  $M$  and any natural number  $n$ , there are  $M \prec M_1$  and  $M \prec M_2$  with  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$  and same iterated truths up to  $n$*

$$\langle \mathbb{N}, +, \cdot, 0, 1, <, \text{Tr}_0, \dots, \text{Tr}_n \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, <, \text{Tr}_0, \dots, \text{Tr}_n \rangle^{M_2}$$

*but which disagree on the next order of truth  $\text{Tr}_{n+1}$ .*

The point:  $\text{Tr}_{n+1}$  is not definable in  $\langle \mathbb{N}, +, \cdot, 0, 1, \text{Tr}_0, \dots, \text{Tr}_n \rangle$ .

# Disagreement on the Church-Kleene ordinal

## Corollary

*Every countable model of set theory  $M$  has elementary extensions  $M \prec M_1$  and  $M \prec M_2$ , which agree on their standard model of arithmetic  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$  and have a computable linear order  $\triangleleft$  on  $\mathbb{N}$  in common, yet  $M_1$  thinks  $\langle \mathbb{N}, \triangleleft \rangle$  is a well-order and  $M_2$  does not.*

## Proof.

Being the computable index of a well-order is  $\Pi_1^1$ -complete and hence not definable in  $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ . □

## Further disagreement

### Corollary

*Every countable model of set theory  $M$  has  $M \prec M_1$  and  $M \prec M_2$ , which agree on their standard model of arithmetic  $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ , yet disagree on Kleene's  $\mathcal{O}$ , with  $\mathcal{O}^{M_1} \neq \mathcal{O}^{M_2}$ .*

### Corollary

*Every countable model of set theory  $M$  which thinks  $0^\sharp$  exists has elementary extensions  $M_1$  and  $M_2$ , which agree on the ordinals up to any desired uncountable cardinal  $\kappa \in M$ , on the constructible universe  $L_\kappa^{M_1} = L_\kappa^{M_2}$  up to  $\kappa$  and on the facts that  $\kappa$  is an uncountable cardinal and  $0^\sharp$  exists, yet disagree on which ordinals below  $\kappa$  are the Silver indiscernibles. Similarly, we may ensure that they disagree on  $0^\sharp$  itself, so that  $(0^\sharp)^{M_1} \neq (0^\sharp)^{M_2}$ .*

## Disagreement on definability

### Theorem

*Every countable model of set theory  $M$  has  $M \prec M_1$  and  $M \prec M_2$ , which agree on*

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$$

*and which have a set  $A \subseteq \mathbb{N}$  in common, yet  $M_1$  thinks  $A$  is first-order definable in  $\mathbb{N}$  and  $M_2$  thinks it is not.*

The proof relies on the non-absoluteness theorem, plus:

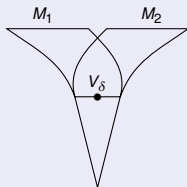
### Lemma (Andrew Marks)

There is  $B \subseteq \mathbb{N} \times \mathbb{N}$ , such that  $\{n \in \mathbb{N} \mid B_n \text{ is arithmetic}\}$  is not definable in the structure  $\langle \mathbb{N}, +, \cdot, 0, 1, <, B \rangle$ .

# Precise violation of ZFC

## Theorem

*Every countable model of set theory  $M \models \text{ZFC}$  has elementary extensions  $M_1$  and  $M_2$ , with a transitive rank-initial segment  $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$  in common, such that  $M_1$  thinks that the least natural number  $n$  for which  $V_\delta$  violates  $\Sigma_n$ -collection is even, but  $M_2$  thinks it is odd.*



$M_1, M_2 \models \text{ZFC}$

$V_\delta^{M_1} = V_\delta^{M_2}$

$n$  is least with  $\neg \Sigma_n$ -collection in  $V_\delta$

$M_1$  believes  $n$  is even

$M_2$  believes  $n$  is odd



## Proof

Suppose that  $M \models \text{ZFC}$ , and let

$$T_1 = \Delta(M) + V_\delta \prec V + \{m \in V_\delta \mid m \in M\}$$

+ the least  $n$  such that  $V_\delta \not\models \Sigma_n$ -collection is even,

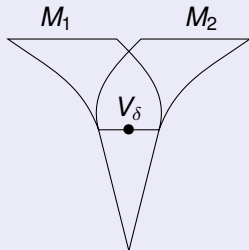
Consistent via the reflection theorem. Similar with theory  $T_2$ , where we assert  $n$  is odd.

Let  $\langle M_1, M_2 \rangle$  be a computably saturated model pair, with  $M_1 \models T_1$  and  $M_2 \models T_2$ . It follows that  $\langle V_\delta^{M_1}, V_\delta^{M_2} \rangle$  is a computably saturated model pair of elementary extensions of  $\langle M, \in^M \rangle$ , which are therefore elementarily equivalent in the language of set theory with constants for elements of  $M$ , and hence isomorphic by an isomorphism respecting those constants. So without loss  $\langle M, \in^M \rangle \prec \langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$ . Meanwhile,  $M_1$  thinks that this  $V_\delta$  violates  $\Sigma_n$ -collection first at an even  $n$  and  $M_2$  thinks it does so first for an odd  $n$ , as desired.  $\square$

# Disagreement about whether $V_\delta \models \text{ZFC}$

## Theorem

*If  $M$  is a countable model of set theory in which the worldly cardinals form a stationary proper class, then there are  $M \prec M_1$  and  $M \prec M_2$  with  $\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$ , but  $M_1$  thinks  $V_\delta \models \text{ZFC}$  and  $M_2$  thinks  $V_\delta \not\models \text{ZFC}$ .*



$M_1, M_2 \models \text{ZFC}$

$\langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$

$M_1$  believes  $V_\delta \models \text{ZFC}$

$M_2$  believes  $V_\delta \not\models \text{ZFC}$

# Proof

Fix any countable  $M \models \text{ZFC}$ , with worldly cardinals a stationary proper class. Let  $T_1 = \Delta(M) + V_\delta \prec V$ , plus  $a \in V_\delta$  for each  $a \in M$ , plus “ $\delta$  is worldly.” Every finite subtheory is consistent, because for any standard  $n$  there is a club of  $\Sigma_n$ -correct cardinals, and so one of them is worldly in  $M$ . So  $T_1$  is consistent.

Similarly, let  $T_2$  assert the same, except instead “ $\delta$  is not worldly.” This theory is also finitely consistent and hence consistent.

Let  $\langle M_1, M_2 \rangle$  be a computably saturated model pair, where  $M_1 \models T_1$  and  $M_2 \models T_2$ . So  $\langle V_\delta^{M_1}, V_\delta^{M_2} \rangle$  is computably saturated model pair, both with the diagram of  $M$ . So they are isomorphic, preserving  $M$ . So we may assume  $\langle M, \in^M \rangle \prec \langle V_\delta, \in \rangle^{M_1} = \langle V_\delta, \in \rangle^{M_2}$ . The theories  $T_1$  and  $T_2$  ensure that  $V_\delta \prec V$  in both  $M_1$  and  $M_2$ , and that  $M_1 \models \delta$  is worldly and  $M_2 \models \delta$  is not worldly, or in other words,  $M_1 \models (V_\delta \models \text{ZFC})$ , but  $M_2 \models (V_\delta \not\models \text{ZFC})$ , as desired.

## Models of ZFC inside models of ZFC

I learned this gem from Brice Halimi:

### Theorem

*Every model of ZFC has an element that is a model of ZFC. Specifically, if  $\langle M, \in^M \rangle \models \text{ZFC}$ , then there is  $\langle m, E \rangle$  in  $\mathcal{M}$ , which when extracted as an actual structure, satisfies ZFC.*

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## Proof.

If  $\mathcal{M}$  is  $\omega$ -nonstandard, then by the reflection theorem plus overspill, there is some  $\langle V_\delta^M, \in \rangle^{\mathcal{M}}$  satisfying a nonstandard fragment of ZFC, and hence satisfies the actual ZFC externally.

If  $\mathcal{M}$  is  $\omega$ -standard, then it must satisfy  $\text{Con}(\text{ZFC})$  and so one has the Henkin model inside  $\mathcal{M}$ . □

## Definiteness of truth

Let's briefly revisit the philosophical motivation.

The question was whether we may infer definiteness in our theory of mathematical truth as a consequence of the definiteness of our mathematical objects?

Many mathematicians and mathematical philosophers appear to do so.

Meanwhile, the mathematical results appear to undermine that conclusion.

After all, perhaps our world is like  $M_1$ , where the natural numbers  $\mathbb{N}^{M_1}$  are definite, yet have different truths in another world  $M_2$ .

Our thesis is that the definiteness of the theory of truth for a structure does not follow as a consequence of the definiteness the structure in which that truth resides. Rather, it must be seen as a separate additional higher-order claim.

Please see the article for further extended discussion.

Thank you.

Slides and articles available on <http://jdh.hamkins.org>.

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