The weakly compact embedding property

Joel David Hamkins

New York University
Philosophy (visiting 2015)

City University of New York

College of Staten Island
Mathematics

CUNY Graduate Center
Mathematics, Philosophy, Computer Science

MathOverflow :)

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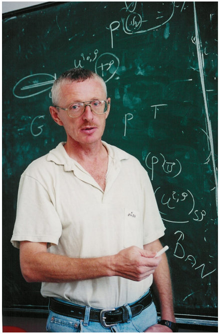
Best wishes on your 60th birthdays!

Arthur W. Apter

Moti Gitik



Joint work in progress

This is joint work in progress by Brent Cody, Sean Cox, myself and Thomas Johnstone.

The paper will be available on the ArXiv and on my blog when it is complete.

Weakly compact cardinals

I'd like to speak about the weakly compact embedding property.

There are an enormous number of equivalent characterizations of the weakly compact cardinals.

Many of these diverse characterizations of weak compactness arise from disparate parts of logic and set theory.

Characterizations of weak compactness

An inaccessible cardinal κ is weakly compact just in case it has any of the following properties.

Compactness for $L_{\kappa,\kappa}$ languages of size κ

Every κ-satisfiable theory in an $L_{\kappa,\kappa}$-language of size at most κ is satisfiable.

Tree property

Every κ-tree has a κ-branch.

Partition property $\kappa \to (\kappa)^2_2$

Every coloring of pairs from κ with two colors admits a monochromatic subset of size κ.

More characterizations of weak compactness

Π^1_1-indescribability

If $A \subseteq V_\kappa$ and $\langle V_\kappa, \in, A\rangle \models \varphi$, where φ has complexity Π^1_1, then there is $\delta < \kappa$ with $\langle V_\delta, \in, A \cap V_\delta\rangle \models \varphi$.

Filter property

For any family of κ many subsets of κ, there is a κ-complete nonprincipal filter measuring them.

Extension property

For every $A \subseteq V_\kappa$, there is a transitive structure W properly extending V_κ and $A^* \subseteq W$ such that $\langle V_\kappa, \in, A\rangle \prec \langle W, \in A^*\rangle$.

The weakly compact embedding property

One more equivalent characterization...

Weakly compact embedding property

For every transitive set M of size κ with $\kappa \in M$ there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ.

If κ is inaccessible (actually $2^{<\kappa} = \kappa$ suffices), then this is equivalent to κ being weakly compact.

In the inaccessible case, the embedding property is extremely robust, and one can insist that $M^{<\kappa} \subseteq M$ or that $M \prec H_{\kappa^+}$ or that $M \models \mathrm{ZFC}$ or many other requirements.

Main theme

What if κ is not necessarily inaccessible?

I want to investigate the embedding property at cardinals that are not necessarily inaccessible.

Weakly compact embedding property for κ

For every transitive set M of size κ with $\kappa \in M$ there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ.

The embedding property at small cardinals

First realization: the weakly compact embedding property can arise at cardinals below the continuum!

Observation (Hamkins)

It is relatively consistent with ZFC that there is a cardinal $\kappa < 2^\omega$ with the embedding property.

Indeed, we may simply force to add at least κ^+ many Cohen reals over a model in which κ is weakly compact.

Observation (Hamkins)

If κ weakly compact and $G \subseteq \mathrm{Add}(\omega, \kappa^+)$ is V-generic, then κ has the embedding property in $V[G]$.

Proof.

A simple lifting argument.

Fix any $A \subseteq \kappa$ in $V[G]$, with nice name $\dot{A} \in V$. Since the forcing is c.c.c., may assume without loss by automorphism that $\dot{A}$ is an $\mathrm{Add}(\omega, \kappa)$-name, so $A = \dot{A}_{G \restriction \kappa}$. Place $\dot{A} \in M$ and get weakly compact embedding $j : M \to N$ with critical point κ and $|N| = \kappa$.

We may lift the embedding to $j : M[G \restriction \kappa] \to N[G \restriction j(\kappa)]$. Since $\dot{A} \in M$, it follows that $A \in M[G \restriction \kappa]$.

If A codes transitive set M_0, then $M_0 \in M[G]$ and $j \restriction M_0 \to j(M_0)$ verifies this instance of the embedding property in $V[G]$. □

Embedding property implies tree property

Theorem

If κ has the weakly compact embedding property, then κ has the tree property.

Proof.

A familiar argument. If T is a κ-tree, place it inside a transitive $M \prec H_{\kappa^+}$ and let $j : M \to N$ be an elementary embedding into a transitive set N, with critical point κ. So $j(T)$ is a tree of height $j(\kappa)$, and agrees with T on the levels below κ. It follows that if $b \in j(T)$ is any node on level κ of $j(T)$, the predecessors of b in $j(T)$ form a κ-branch through T. □

But the embedding property is not equivalent to the tree property...

Embedding property implies weakly Mahlo

Theorem

If κ has the embedding property, then κ is weakly Mahlo.

Weakly Mahlo = regular plus REG is stationary.

Proof.

(Regular) $j : M \to N$ with critical point κ is impossible if M has a short cofinal sequence.

(Weakly Mahlo) If $C \subseteq \kappa$ is club, place into $M \prec H_{\kappa^+}$ and consider $j : M \to N$. Since $\kappa \in j(C)$, the club contains regular cardinals in M, which is correct. □

Embedding property implies non-power

Theorem

If κ has the weakly compact embedding property, then $\kappa \neq 2^\delta$ for any cardinal δ.

Proof.

If $2^\delta = \kappa$, then enumerate $\vec{x} = \langle x_\alpha \mid \alpha < \kappa\rangle$ all the subsets of δ without repetition. Place into transitive $M \prec H_{\kappa^+}$ and consider $j : M \to N$. So $j(\vec{x})(\kappa)$ is a subset of δ not on the original list, a contradiction. □

Corollary

If κ has the embedding property, then either $\delta < \kappa < 2^\delta$ for some $\delta < \kappa$, or κ is weakly compact.

Embedding property implies stationary reflection

Theorem

If κ has the weakly compact embedding property, then it has stationary reflection: for every stationary $S \subseteq \kappa$, there is $\gamma < \kappa$ with $S \cap \gamma$ stationary in γ.

Proof.

Asssume $S \subseteq \kappa$ is stationary; place $S \in M \prec H_{\kappa^+}$ and get $j : M \to N$. Since $S = j(S) \cap \kappa$ is stationary in V, it is also stationary in N, and so $j(S)$ reflects in N. So S reflects in M, and M is correct. □

Equivalent variations of the embedding property

Embedding property

For every transitive set M of size κ with $\kappa \in M$, there is a transitive set N and elementary embedding $j : M \to N$ with critical point κ.

Embedding property, variation

For every $A \subseteq \kappa$, there are transitive $M, N \models \mathrm{ZFC}^-$ with $A, \kappa \in M$ and elementary $j : M \to N$ with critical point κ.

Embedding property, Hauser variation

For every $A \subseteq \kappa$, there are such $j : M \to N$ with $M, j \in N$.

Proof of equivalents

Theorem

If κ has the weakly compact embedding property, then it has the Hauser version of the embedding property.

Proof.

Fix $A \subseteq \kappa$ and place $A \in M \prec H_{\kappa^+}$. Code $\langle M, \in\rangle \cong \langle \kappa, E\rangle$, place $E \in \overline{M} \prec H_{\kappa^+}$, and get $j : \overline{M} \to \overline{N}$. Since $E \in \overline{M}$ it follows $M \in \overline{M}$ and actually $M \prec \overline{M}$. Thus, $j \restriction M : M \to \overline{N}$ is elementary. Since $E = j(E) \restriction \kappa$, we get $M \in \overline{N}$. Also, if α codes a with respect to E, then α codes $j(a)$ wrt $j(E)$, and so $j \restriction M \in \overline{N}$ also, fulfilling the Hauser property. □

Embedding property $\iff$ Extension properties

Extension property

For every transitive set $W \subseteq H_\kappa$ of size κ and every $A \subseteq W$, there is a proper elementary extension $\langle W, \in, A\rangle \prec \langle W', \in, A'\rangle$ to a transitive set W' with $A' \subseteq W'$.

Extension property, variation

For every $A \subseteq \kappa$, there is some $\gamma > \kappa$ and $A' \subseteq \gamma$ such that $\langle L_\kappa[A], \in, A\rangle \prec \langle L_\gamma[A'], \in, A'\rangle$.

Extension property, variation

For every $A \subseteq \kappa$ and any regular $\delta \leq \kappa$, there is some $\gamma > \kappa$ with $\operatorname{cof}(\gamma) = \delta$ and $A' \subseteq \gamma$ such that $\langle L_\kappa[A], \in, A\rangle \prec \langle L_\gamma[A'], \in, A'\rangle$.

Proof

Embedding property $\to$ extension property

If $W \subseteq H_\kappa$ transitive, size κ and $A \subseteq W$, place $W, A \in M \prec H_{\kappa^+}$ and get $j : M \to N$. It follows that $\langle W, \in, A\rangle \prec \langle j(W), \in, j(A)\rangle$.

Extension property $\to$ constructible variation

If $A \subseteq \kappa$, let $W = L_\kappa[A]$. Note $W \subseteq H_\kappa$ size κ. By the extension property, get $\langle L_\kappa[A], \in, A\rangle \prec \langle W', \in, A'\rangle$. By elementarity, $W' = L_\gamma[A']$ for some ordinal $\gamma > \kappa$.

Constructible variation $\to$ cofinality variation

Let $C \subseteq \kappa$ be club of $\alpha < \kappa$ with $L_\alpha[A \cap \alpha] \prec L_\kappa[A]$. Get A' and C' by extension property. Let γ be the δ^{th} element of C' above κ, so $L_\kappa[A] \prec L_\gamma[A' \cap \gamma]$ and $\mathrm{cof}(\gamma) = \delta$.

Completing the circle

Extension property $\to$ embedding property

Suppose M transitive size κ. So $\langle M, \in\rangle \cong \langle \kappa, E\rangle$. By the extension property, get $\langle L_\kappa[E], \in, E\rangle \prec \langle L_\gamma[E'], \in, E'\rangle$, where $\gamma > \kappa$ has uncountable cofinality.

Since E is well-founded, $L_\kappa[E]$ has ordinal ranking functions for $E \cap \alpha$ every $\alpha < \kappa$. Thus, $E' \cap \alpha$ has such ranking functions every $\alpha < \gamma$. Since $\mathrm{cof}(\gamma)$ uncountable, this implies E' is well-founded.

Let M' be Mostowski collapse of $\langle \gamma, E'\rangle$, and define $j : M \to M'$ by: if a is coded by α via E, then $j(a)$ is coded by α via E'. $\square$

Thus, all embedding and extension properties are equivalent.

Equiconsistent with weak compactness

Theorem

If κ has the weakly compact embedding property, then it is weakly compact in L.

Proof.

If κ has the embedding property, then it is weakly inaccessible and hence inaccessible in L. Also, if T is a κ-tree in L, then $T \in L_\alpha$ some $\alpha < \kappa^+$. By the embedding property, get $j : L_\alpha \to L_\beta$ with critical point κ, so T has a branch in L. So κ has the tree property in L and therefore must be weakly compact in L. □

Embedding property implies κ-compactness

Theorem

If κ has the weakly compact embedding property, then every κ-satisfiable $L_{\kappa,\kappa}$ theory of size κ is satisfiable.

Proof.

Suppose that T is a κ-satisfiable $L_{\kappa,\kappa}$ theory of size κ. Place $T \in M \prec H_{\kappa^+}$ and get $j : M \to N$. So $T = j(T) \restriction \kappa$ is satisfiable in N, and hence it is satisfiable. □

Note subtle distinction: size of *theory* versus size of *language*.

- Compactness for language size $\kappa \iff$ weakly compact.
- Compactness for theories size $\kappa \iff$ embedding property.

Embedding property equivalent to κ-compactness

Theorem

A cardinal κ has the weakly compact embedding property $\iff$ every κ-satisfiable $L_{\kappa,\kappa}$ theory of size κ is satisfiable.

Proof.

($\leftarrow$) Given $\langle W, \in, A\rangle$ with $A \subseteq W \subseteq H_\kappa$, let T be the theory:

- The elementary diagram of $\langle W, \in, A\rangle$.
- The assertion: $\hat{\in}$ is well-founded.
- $c \neq \hat{a}$, for every $a \in W$, where c is new constant.
- $\forall x(x \mathrel{\hat{\in}} \hat{a} \iff \bigvee_{b \in a} x = \hat{b})$, for $a \in W$.

This $L_{\kappa,\kappa}$ theory is κ-satisfiable in $\langle W, \in, A\rangle$, and has size κ. So it is satisfiable, and any model provides a proper extension $\langle W, \in, A\rangle \prec \langle W', \in, A'\rangle$. □

Equivalents of the embedding property

To summarize, we have diverse equivalent formulations of the embedding property.

- The embedding property.
- Variations of the embedding property.
- The extension property.
- Variations of the extension property.
- The κ-compactness property.

Weakly compact filter property

Weakly compact filter property at κ

For every κ many subsets $A_\alpha \subseteq \kappa$ $(\alpha < \kappa)$, there is κ-complete uniform filter F on κ measuring every A_α. Equivalently, for every transitive $M \models \mathrm{ZFC}^-$ size κ, there is κ-complete M-measure F.

δ-complete filter property, for uncountable $\delta < \kappa$

Require only that F is δ-complete (in V) and (M, κ)-complete.

Weakly σ-complete filter property

For every M there is (M, κ)-complete filter F that is weakly σ-complete (countable intersections from F are nonempty).

Filter property implies embedding property

Theorem

Each of the filter properties implies the embedding property.

Proof.

Assume the weak σ-complete filter property. Fix any transitive $M \prec H_{\kappa^+}$ size κ, and let F be a weakly σ-complete (M, κ)-complete uniform filter. It follows that $\mathrm{Ult}(M, F)$ is well-founded, and we have $j : M \to \mathrm{Ult}(M, F)$, witnessing the embedding property. □

Filter property equivalents

Theorem

For uncountable $\delta \leq \kappa$, the following are equivalent.

1. *κ has the δ-complete filter property.*
2. *κ has the weakly δ-complete filter property.*
3. *κ has the embedding property and $\kappa^{<\delta} = \kappa$.*
4. *For every $A \subseteq \kappa$ there is transitive $M \prec H_{\kappa^+}$ with $A \in M \supseteq M^{<\delta}$ and $j : M \to N$ with critical point κ.*

Corollary

1. weakly compact filter property $\iff$ weakly compact.
2. Weakly σ-complete filter property is strictly stronger than the embedding property.

Proof of filter equivalents

Weak δ-complete filter property $\to$ embedding property $+ \kappa^{<\delta} = \kappa$.

Proof.

Assume that κ has the weakly δ-complete filter property, $\delta < \kappa$. So we have well-founded ultrapowers and hence the embedding property. Fix $M \prec H_{\kappa^+}$ and let F be weakly δ-complete M, κ-complete uniform filter on κ. Must show $\kappa^{<\delta} = \kappa$. Claim that $\lambda^\eta < \kappa$, if $\lambda < \kappa$ and $\eta < \delta$. If not, then fix $\langle x_\alpha \mid \alpha < \kappa \rangle \in M$ distinct $x_\alpha \in {}^\eta\lambda$. Let $X_{\xi,\nu} = \{\, \alpha \mid x_\alpha(\xi) = \nu \,\}$ for $\xi < \eta$, $\nu < \lambda$. Since $\kappa = \sqcup_{\nu<\lambda} X_{\xi,\nu} \in M$, it follows by (M, κ)-completeness that for each $\xi < \eta$, there is ν_ξ with $X_{\xi,\nu_\xi} \in F$. By weak δ-completeness, it follows that $\bigcap_{\xi<\eta} X_{\xi,\nu_\xi} \neq \emptyset$. But this intersection can have at most one α, since $x_\alpha(\xi) = \nu_\xi$ is determined. By omitting an initial segment, can make this empty, a contradiction. So $\lambda^\eta < \kappa$, and consequently $\lambda^{<\delta} < \kappa$ and so also $\kappa^{<\delta} = \kappa$. □

Embedding property weaker than weak filter property

Corollary

It is relatively consistent with ZFC that a cardinal κ has the weakly compact embedding property, but not the weakly σ-complete filter property.

Proof.

Assume κ weakly compact and then force $2^\omega > \kappa$. □

Can similarly separate the δ-complete filter property from the weak δ^+-complete filter property.

A strict hierarchy between the tree property and weak compactness

weak compactness

various δ-complete filter properties $\quad \delta \leq \kappa$

weak σ-complete filter property

embedding property $\iff$ extension $\iff$ compactness

tree property

Bounding number does not have embedding property

Theorem

The bounding number $\mathfrak{b}$ does not have the embedding property.

Proof.

Suppose $B = \langle f_\alpha \mid \alpha < \mathfrak{b}\rangle$ enumerates an unbounded family in ω^ω, but no initial segment of B is unbounded. Place $B \in M \prec H_{\mathfrak{b}^+}$ and get $j : M \to N$ with critical point $\mathfrak{b}$. But $B = j(B) \upharpoonright \mathfrak{b}$ is an initial segment of $j(B)$ that is unbounded in $(\omega^\omega)^N$, a contradiction. □

Other cardinals do not have embedding property

Similar arguments establish the following:

Theorem

None of the following cardinals have the embedding property.

1. *The almost disjointness number,* $\mathfrak{a}$.
2. *The bounding number,* $\mathfrak{b}$.
3. *The dominating number,* $\mathfrak{d}$.
4. *The reaping number,* $\mathfrak{r}$.
5. *The splitting number,* $\mathfrak{s}$.
6. *The ultrafilter number,* $\mathfrak{u}$.

Summary

- Joint with Brent Cody, Sean Cox, myself, Thomas Johnstone.
- We studied the weakly compact embedding property.
- For inaccessible cardinals, it characterizes weak compactness.
- But can hold at other cardinals, even below the continuum.
- Embedding property is a strong form of the tree property
- The embedding property has diverse equivalents in terms of extension property, compactness property.
- The embedding property sits amidst a proper hierarchy of concepts: from the tree property, the embedding property, the filter properties up to weak compactness.
- Connections with cardinal characteristics.
- It will be interesting to investigate further instances.

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Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
New York University
City University of New York