Same structure, different truth

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This is joint work with Ruizhi Yang of Fudan University, Shanghai.

Our paper is available on my blog:

> *J. D. Hamkins, R. Yang, "Satisfaction is not absolute," to appear in the Review of Symbolic Logic.*

My favorite situation

A philosophical concern...

leads to interesting mathematical questions...

whose answers illuminate the philosophical issue.

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The philosophical question

Question

To what extent does definiteness of mathematical objects lead to definiteness of our theory of mathematical truth about those objects?

Many mathematicians express a commitment to the definiteness of the natural numbers $0, 1, 2, \ldots$ and the structure $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Does this also commit us to the definiteness of arithmetic truth?

Definiteness of truth

For example, Feferman and others have defended a view whereby arithmetic truth has a definite character, while higher-order truth, such as set-theoretic assertions at the level of $P(\mathbb{N})$ and above, are less definite.

For example, on such a view you might view the continuum hypothesis as a vague mathematical assertion, not capable of genuine resolution.

From structure to truth

Solomon Feferman (EFI 2013):

> In my view, the conception [of the bare structure of the natural numbers] is completely clear, and thence all arithmetical statements are definite.

It is Feferman's 'thence' to which I call attention.

Donald Martin (EFI 2012):

> What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers.

Structure to truth

So we are interested in the philosophical question concerning the extent one gets definiteness of the theory of truth for a structure merely from the definiteness of the objects and relations of the structure itself.

Or is the definiteness of the theory of truth for a structure a kind of higher-order ontological commitment requiring its own justification?

The mathematical question

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

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Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

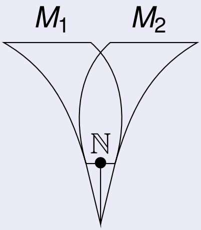
The answer is yes, and indeed, this is pervasive.

Theorem

If ZFC is consistent, then there are models $M_1, M_2 \models \mathrm{ZFC}$ which have the same arithmetic structure

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



There is a sentence σ in M_1 and M_2

M_1 believes $\mathbb{N} \models \sigma$

M_2 believes $\mathbb{N} \models \neg\sigma$

Standard models

We all know *the* standard model of arithmetic: $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

A model of arithmetic is *a* standard model, or ZFC-*standard*, if it is the standard model of arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^M$ extracted from some model of set theory $M \models \mathrm{ZFC}$.

So a standard model of arithmetic is precisely one that is thought to be the standard model of arithmetic from the perspective of a model of set theory.

If $M \models \mathrm{ZFC}$, we may also extract its version of the theory of true arithmetic, $\mathrm{TA}^M = \{\, \sigma \mid M \models \mathbb{N} \models \sigma \,\}$.

Satisfaction class

A *truth predicate* on a model of arithmetic $\mathcal{N} = \langle N, +, \cdot, 0, 1, < \rangle$ is a class $\mathrm{Tr} \subseteq N$ satisfying the Tarskian recursion:

- For σ atomic, $\sigma \in \mathrm{Tr}$ just in case $\mathcal{N}$ thinks σ is true.
- $\sigma \wedge \tau \in \mathrm{Tr}$ iff $\sigma \in \mathrm{Tr}$ and $\tau \in \mathrm{Tr}$.
- $\neg\sigma \in \mathrm{Tr}$ if $\sigma \notin \mathrm{Tr}$.
- $\exists x\, \varphi(x) \in \mathrm{Tr}$ iff $\varphi(\underbrace{1 + \cdots + 1}_{n}) \in \mathrm{Tr}$ some $n \in N$.

Tarski: No such truth predicate is definable in $\mathcal{N}$.

Every ZFC-standard model of arithmetic has an *inductive* truth predicate, meaning $\langle N, +, \cdot, 0, 1, <, \mathrm{Tr} \rangle \models \mathrm{PA}(\mathrm{Tr})$.

Incompatible truth predicates

Theorem (Krajewski 1974)

There are models of arithmetic with different incompatible inductive truth predicates.

Proof.

Suppose $\mathcal{N}_0 = \langle N_0, +, \cdot, 0, 1, < \rangle$ has an inductive truth predicate. Let T be the elementary diagram $\Delta(\mathcal{N}_0)$, plus "Tr is an inductive truth predicate." This is consistent. Any model of T provides an elementary extension of $\mathcal{N}_0$. If they all have a unique truth predicate, then Tr would be implicitly definable in the sense of Beth's implicit definability theorem, and hence by Beth must be explicitly definable, which contradicts Tarski's theorem. So there is an elementary extension $\mathcal{N}$ of $\mathcal{N}_0$ with at least two different truth predicates. □

Theorem (Enayat)

The following are equivalent for countable nonstandard $\mathcal{N}$:

1. *$\mathcal{N}$ is ZFC-standard. That is, $\mathcal{N} = \mathbb{N}^M$ some $M \models \mathrm{ZFC}$.*
2. *$\mathcal{N}$ is a computably saturated model of $\mathrm{Th}(\mathbb{N})^{\mathrm{ZFC}}$.*

Proof.

($1 \to 2$) If $\mathcal{N}$ is ZFC-standard, then $\mathcal{N} \models \mathrm{Th}(\mathbb{N})^{\mathrm{ZFC}}$, and inductive truth predicate implies computably saturated by overspill argument.

($2 \to 1$) Suppose $\mathcal{N} \models \mathrm{Th}(\mathbb{N})^{\mathrm{ZFC}}$ is countable computably saturated. Let $T = \mathrm{ZFC} + \sigma^{\mathbb{N}}$ for all $\sigma \in \mathrm{Th}(\mathcal{N})$. Consistent. By computable saturation, T is coded in $\mathcal{N}$; so get nonstandard finite consistent $t \in \mathcal{N}$ extending T. Build the Henkin completion of t in $\mathcal{N}$, with Henkin model M. So $M \models \mathrm{ZFC}$ and $\mathbb{N}^M \equiv \mathcal{N}$. Further, $a \mapsto (1 + \cdots + 1)^a$ maps $\mathcal{N}$ to an initial segment of $\mathbb{N}^M$, so same standard system. By back-and-forth, they are isomorphic, and so $\mathcal{N}$ is ZFC-standard. □

Characterizing ZFC-standard arithmetic truth

Similarly, let $\mathrm{Th}(\mathbb{N}, \mathrm{TA})^{\mathrm{ZFC}}$ be the ZFC-provable assertions about the structure $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{TA} \rangle$.

Theorem

The following are equivalent for any countable nonstandard model of arithmetic $\mathcal{N}$ with a truth predicate Tr.

1. $\langle \mathcal{N}, \mathrm{Tr} \rangle$ *is a ZFC-standard model of arithmetic and arithmetic truth. That is,* $\mathcal{N} = \mathbb{N}^M = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^M$ *for some* $M \models \mathrm{ZFC}$ *in which* $\mathrm{Tr} = \mathrm{TA}^M$ *is the theory of true arithmetic.*
2. $\langle \mathcal{N}, \mathrm{Tr} \rangle$ *is a computably saturated model of* $\mathrm{Th}(\mathbb{N}, \mathrm{TA})^{\mathrm{ZFC}}$*.*

One may use this with Beth's theorem to prove the non-absolute truth theorem.

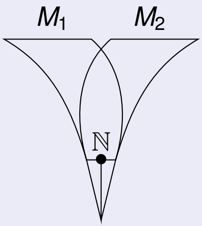
Satisfaction is not absolute

Theorem

If ZFC is consistent, then there are $M_1, M_2 \models \mathrm{ZFC}$ which have the same natural numbers and arithmetic structure

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



There is a sentence σ in M_1 and M_2

M_1 believes $\mathbb{N} \models \sigma$

M_2 believes $\mathbb{N} \models \neg\sigma$

Proof

Fix any countable $M_1 \models \mathrm{ZFC}$ with $\langle \mathbb{N}, \mathrm{TA} \rangle^{M_1}$ computably saturated.

Claim there are sentences σ, τ with same 1-type in $\mathbb{N}^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1}$, but M_1 thinks σ is true and τ false in $\mathbb{N}^{M_1}$. (Proof: consider the type $p(s,t)$ containing $\varphi(s) \iff \varphi(t)$ and $s \in \mathrm{TA}$ and $t \notin \mathrm{TA}$; this is finitely realized, since TA is not definable.)

By back-and-forth, there is automorphism $\pi : \mathbb{N}^{M_1} \to \mathbb{N}^{M_1}$ with $\pi(\tau) = \sigma$.

Build a copy M_2 of M_1 so that π extends to an isomorphism $\pi^* : M_1 \to M_2$. So $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. But M_1 thinks σ is true, yet M_2 thinks $\sigma = \pi^*(\tau)$ is false. □

A generalization

Theorem

For any countable $M \models \mathrm{ZFC}$, any structure $\mathcal{N} \in M$ finite language, any $S \subseteq N$ in M not definable in $\mathcal{N}$. Then there are $M \prec M_1$ and $M \prec M_2$ with $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$, yet $S^{M_1} \neq S^{M_2}$.

Note S^{M_1} and S^{M_2} share all properties of S in M.

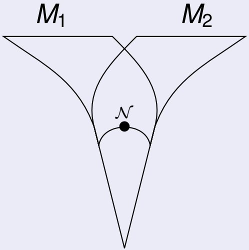
Proof.

Fix $M \prec M_1$ countable computably saturated. So $\langle \mathcal{N}, S\rangle^{M_1}$ is computably saturated. Since S not definable, there are $a, b \in \mathcal{N}^{M_1}$ with same 1-type in $\mathcal{N}^{M_1}$, but $a \in S, b \notin S$. So $\exists \pi : \mathcal{N}^{M_1} \cong \mathcal{N}^{M_1}$ with $\pi(b) = a$. Extend π to $\pi^* : M_1 \cong M_2$. So $a \in S^{M_1}$ but $a = \pi(b) \notin S^{M_2}$. □

Satisfaction is not absolute

Corollary

If M is a countable model of set theory and $\mathcal{N}$ is (sufficiently robust) structure in M, in a finite language. Then there are $M \prec M_1$ and $M \prec M_2$, which agree on the natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and on $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$, yet disagree on satisfaction $\mathcal{N} \models \sigma[\vec{a}]$ for this structure.



$M \prec M_1, M_2 \models \mathrm{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2}, \qquad \mathcal{N}^{M_1} = \mathcal{N}^{M_2}$

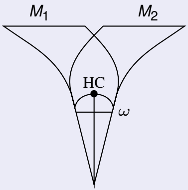
there are σ and $\vec{a}$ for which

M_1 believes $\mathcal{N} \models \sigma[\vec{a}]$

M_2 believes $\mathcal{N} \models \neg\sigma[\vec{a}]$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, their reals $\mathbb{R}^{M_1} = \mathbb{R}^{M_2}$ and their hereditarily countable sets $\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$, but which disagree on their theories of projective truth.



$M_1, M_2 \models \mathrm{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2} \qquad \mathbb{R}^{M_1} = \mathbb{R}^{M_2}$

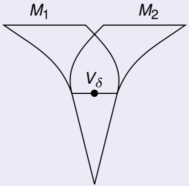
$\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$

M_1 *believes* $\mathrm{HC} \models \sigma$

M_2 *believes* $\mathrm{HC} \models \neg\sigma$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which have a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, but which disagree on truth in this structure.



$M_1, M_2 \models \mathrm{ZFC}$

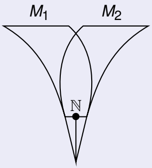
$V_\delta^{M_1} = V_\delta^{M_2} \models \mathrm{ZFC}$

M_1 *believes* $V_\delta \models \sigma$

M_2 *believes* $V_\delta \models \neg\sigma$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers with successor, addition and order $\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$, but which disagree on natural-number multiplication, so that M_1 thinks $a \cdot b = c$ for some particular natural numbers, but M_2 disagrees.



$M_1, M_2 \models \mathrm{ZFC}$

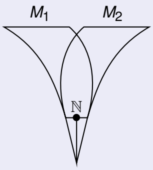
$\langle \mathbb{N}, S, +, < \rangle^{M_1} = \langle \mathbb{N}, S, +, < \rangle^{M_2}$

M_1 *believes* $\mathbb{N} \models a \cdot b = c$

M_2 *believes* $\mathbb{N} \models a \cdot b \neq c$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers with successor and order $\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$, but which disagree on the even numbers, the prime numbers and the powers of two, so that M_1 thinks some n is a large odd prime number, but M_2 thinks it is a large power of 2.



$M_1, M_2 \models \mathrm{ZFC}$

$\langle \mathbb{N}, S, < \rangle^{M_1} = \langle \mathbb{N}, S, < \rangle^{M_2}$

M_1 *believes* $\mathbb{N} \models n$ *is an odd prime*

M_2 *believes* $\mathbb{N} \models n = 2^k$ *for some* k

Iterated truth predicates

Begin with the standard model $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Add a truth predicate $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0 \rangle$, where Tr_0 is a truth predicate for arithmetic assertions.

Add a truth predicate for that structure, $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle$, where Tr_1 is a truth predicate for assertions in the language with Tr_0.

And so on $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$...

Truth about truth is not absolute

Corollary

For every countable model of set theory M and any natural number n, there are $M \prec M_1$ and $M \prec M_2$ with $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and same iterated truths up to n

$$\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_2}$$

but which disagree on the next order of truth Tr_{n+1}.

The point: Tr_{n+1} is not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$.

Disagreement on the Church-Kleene ordinal

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and have a computable linear order $\lhd$ on $\mathbb{N}$ in common, yet M_1 thinks $\langle \mathbb{N}, \lhd \rangle$ is a well-order and M_2 does not.

Proof.

Being the computable index of a well-order is Π^1_1-complete and hence not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. □

Disagreement on definability

Theorem

Every countable model of set theory M has $M \prec M_1$ and $M \prec M_2$, which agree on

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$$

and which have a set $A \subseteq \mathbb{N}$ in common, yet M_1 thinks A is first-order definable in $\mathbb{N}$ and M_2 thinks it is not.

The proof relies on the non-absoluteness theorem, plus:

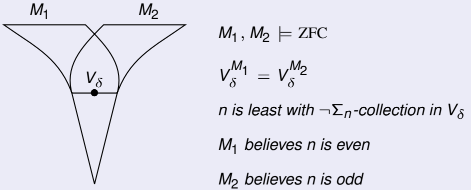
Lemma (Andrew Marks)

There is $B \subseteq \mathbb{N} \times \mathbb{N}$, such that $\{\, n \in \mathbb{N} \mid B_n \text{ is arithmetic} \,\}$ is not definable in the structure $\langle \mathbb{N}, +, \cdot, 0, 1, <, B \rangle$.

Precise violation of ZFC

Theorem

Every countable model of set theory $M \models \mathrm{ZFC}$ has elementary extensions M_1 and M_2, with a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, such that M_1 thinks that the least natural number n for which V_δ violates Σ_n-collection is even, but M_2 thinks it is odd.



Proof

Suppose that $M \models \mathrm{ZFC}$, and let

$$T_1 = \Delta(M) + \ V_\delta \prec V \ + \{\, m \in V_\delta \mid m \in M \,\} \\ + \text{ the least } n \text{ such that } V_\delta \not\models \Sigma_n\text{-collection is even,}$$

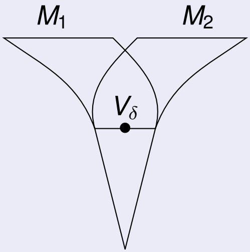
Consistent via the reflection theorem. Similar with theory T_2, where we assert n is odd.

Let $\langle M_1, M_2\rangle$ be a computably saturated model pair, with $M_1 \models T_1$ and $M_2 \models T_2$. It follows that $\langle V_\delta^{M_1}, V_\delta^{M_2}\rangle$ is a computably saturated model pair of elementary extensions of $\langle M, \in^M\rangle$, which are therefore elementarily equivalent in the language of set theory with constants for elements of M, and hence isomorphic by an isomorphism respecting those constants. So without loss $\langle M, \in^M\rangle \prec \langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$. Meanwhile, M_1 thinks that this V_δ violates Σ_n-collection first at an even n and M_2 thinks it does so first for an odd n, as desired. □

Disagreement about whether $V_\delta \models \mathrm{ZFC}$

Theorem

If M is a countable model of set theory in which the worldly cardinals form a stationary proper class, then there are $M \prec M_1$ and $M \prec M_2$ with $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$, but M_1 thinks $V_\delta \models \mathrm{ZFC}$ and M_2 thinks $V_\delta \not\models \mathrm{ZFC}$.



$M_1, M_2 \models \mathrm{ZFC}$

$\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$

M_1 *believes* $V_\delta \models \mathrm{ZFC}$

M_2 *believes* $V_\delta \not\models \mathrm{ZFC}$

Proof

Fix any countable $M \models \mathrm{ZFC}$, with worldly cardinals a stationary proper class. Let $T_1 = \Delta(M) + V_\delta \prec V$, plus $a \in V_\delta$ for each $a \in M$, plus "δ is worldly." Every finite subtheory is consistent, because for any standard n there is a club of Σ_n-correct cardinals, and so one of them is worldly in M. So T_1 is consistent.

Similarly, let T_2 assert the same, except instead "δ is not worldly." This theory is also finitely consistent and hence consistent.

Let $\langle M_1, M_2\rangle$ be a computably saturated model pair, where $M_1 \models T_1$ and $M_2 \models T_2$. So $\langle V_\delta^{M_1}, V_\delta^{M_2}\rangle$ is computably saturated model pair, both with the diagram of M. So they are isomorphic, preserving M. So we may assume $\langle M, \in^M\rangle \prec \langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$. The theories T_1 and T_2 ensure that $V_\delta \prec V$ in both M_1 and M_2, and that $M_1 \models \delta$ *is worldly* and $M_2 \models \delta$ *is not worldly*, or in other words, $M_1 \models (V_\delta \models \mathrm{ZFC})$, but $M_2 \models (V_\delta \not\models \mathrm{ZFC})$, as desired.

Models of ZFC inside models of ZFC

I learned this gem from Brice Halimi:

Theorem

Every model of ZFC *has an element that is a model of* ZFC*. Specifically, if* $\langle M, \in^M \rangle \models \mathrm{ZFC}$*, then there is* $\langle m, E \rangle$ *in* M*, which when extracted as an actual structure, satisfies* ZFC*.*

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Proof.

If M is ω-nonstandard, then by the reflection theorem plus overspill, there is some $\langle V_\delta^M, \in \rangle^{\mathcal{M}}$ satisfying a nonstandard fragment of ZFC, and hence satisfies the actual ZFC externally.

If M is ω-standard, then it must satisfy Con(ZFC) and so one has the Henkin model inside M. □

A universal algorithm

Theorem

There is a Turing machine program p, such that for any function $f : \mathbb{N} \to \mathbb{N}$*, there is a model* $M \models \mathrm{ZFC}$ *(assuming consistency), such that program p inside M computes f on standard input.*

Inside M, on standard input n, the program p with output $f(n)$.

Thus, the program is a *universal algorithm*, capable in principle of computing any desired function, if it is run in the right universe.

Theorem

There is a universal program p, which can compute any f $: \mathbb{N} \to \mathbb{N}$ *on standard input, in the right universe.*

Proof.

My first proof proceeded via the Rosser tree: searching for proofs validating Rosser assertions computes any desired path.

Vadim Kosoy made a suggestion on my blog leading to an alternative elegant proof:

By the recursion theorem, there is a program p that searches for a proof that program p disagrees with a certain finite list of function input/output values. For the first such proof that is found, output the correspond output.

The theory cannot prove that the function disagrees with any particular value, since then it wouldn't. So for any function f, it is consistent with the theory that p computes exactly $f(n)$ on input n. □

Definiteness of truth

Let's briefly revisit the philosophical motivation.

The question was whether we may infer definiteness in our theory of mathematical truth as a consequence of the definiteness of our mathematical objects?

Many mathematicians and mathematical philosophers appear to do so.

Meanwhile, the mathematical results appear to undermine that conclusion.

After all, perhaps our world is like M_1, where the natural numbers $\mathbb{N}^{M_1}$ are definite, yet have different truths in another world M_2.

Our thesis is that the definiteness of the theory of truth for a structure does not follow as a consequence of the definiteness the structure in which that truth resides. Rather, it must be seen as a separate additional higher-order claim.

Please see the article for further extended discussion.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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