Set-theoretic potentialism

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In memory of

Yuzuru Kakuda

During 1998, I had the pleasure to live in Japan and work at Kobe University with the excellent research group in logic there. It was a formative experience, and I was very glad to have had the chance to interact with Kakuda-sensei and the other Kobe researchers. I have many fond memories of that time.

Joint work with Øystein Linnebo, University of Oslo, Norway.

Classical potentialism

The classical debate concerning potentialism goes back to Archimedes.

potential infinity vs. actual infinity.

According to the potentialist, the natural numbers $\mathbb{N}$ are merely *potentially* infinite; but they are not *actually* infinite. We can never have all of them at once as a completed totality.

Set-theoretic potentialism

A similar distinction arises in the philosophy of set theory.

Set-theoretic **potentialism** is the view that the set-theoretic universe itself is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us.

On this view, even though we may have some actual infinities, nevertheless the upper or outer reaches of the set-theoretic universe have a merely potential character.

Kinds of set-theoretic potentialism

Height-potentialism (+ width actualism)

The universe grows taller as new ordinals are formed, but power sets are actual.

Width-potentialism (+ height-actualism)

The universe grows wider as one adds new subsets to infinite sets, such as by forcing. But the ordinals are completed.

Height- and width-potentialism

The universe can be made both taller and wider.

Project goals

- To provide precise accounts of the various kinds of potentialism.
- To investigate the *modal* commitments of the various potentialist perspectives.

Potentialism and modal logic

The first realization is that indeed potentialism exhibits an essentially *modal* character.

$\Diamond\varphi$	φ is possible
$\Box\varphi$	φ is necessary

Model-theoretic potentialism

We can provide a general model-theoretic account of potentialism.

A *potentialist system* is:

- A collection $\mathcal{W}$ of structures in a common language $\mathcal{L}$, plus
- a reflexive transitive accessibility relation on those structures, such that
- whenever U accesses W, then the domain of U is contained in W.

So this is a Kripke model, with a corresponding semantics.

Semantics of potentialism

Suppose $\mathcal{W}$ is a potentialist system of $\mathcal{L}$-structures.

Language $\mathcal{L}^\Diamond$ augments $\mathcal{L}$ with modal operators $\Diamond, \Box$.

Define satisfaction via Kripke/Tarski semantics

$$W \models_{\mathcal{W}} \varphi(a)$$

- Atomic, Boolean combinations φ are defined as by Tarski.
- Quantifiers are interpreted in the current world W.
  $W \models_{\mathcal{W}} \exists x\, \varphi(x, a)$ means
  $W \models_{\mathcal{W}} \varphi(x, a)$ for some $x \in W$.
- Modal operators use the accessibility relation.
  $\Diamond \varphi$ means φ is true in some accessible world
  $\Box \varphi$ means φ is true in all accessible worlds.

Coherent potentialist systems

A potentialist system $\mathcal{W}$ is *coherent*, with limit M, if

- Every world in $\mathcal{W}$ is a substructure of M.
- Every world in $\mathcal{W}$ can be extended so as to accommodate any desired individual of M.

This is a weak form of directedness.

Examples:

- finite (or finitely generated) substructures of a given structure.
- countable substructures of a fixed uncountable structure.

The Potentialist translation

For every ψ in $\mathcal{L}$, form the *potentialist translation* $\psi^\Diamond$ by

replace $\exists x$ with $\Diamond\,\exists x$; replace $\forall x$ with $\Box\,\forall x$.

Theorem

If potentialist system $\mathcal{W}$ has limit M, then

$$M \models \psi(a) \qquad \longleftrightarrow \qquad W \models_{\mathcal{W}} \psi^\Diamond(a),$$

for any world $W \in \mathcal{W}$ in which the individual a exists.

Thus, actual truth in the limit structure amounts to potentialist truth in the approximating structures. So the potentialist can in effect refer to actual truth.

Proved by simple induction on formulas.

Potentialist Validities

A modal assertion $\varphi(p_0, \ldots, p_n)$ is *valid* at world W in potentialist system $\mathcal{W}$ if

$$W \models_{\mathcal{W}} \varphi(\psi_0, \ldots, \psi_n)$$

for all assertions ψ_i from $\mathcal{L}^\diamond$ (or sometimes $\mathcal{L}$, possibly parameters from W allowed).

Each validity is really a scheme of truth assertions.

In some cases, it matters whether one considers only $\mathcal{L}$-instances or $\mathcal{L}^\diamond$ or whether parameters are allowed.

Easy validities

It is easy to see that S4 is valid at every world W in any potentialist system.

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Dual	$\neg\Diamond\varphi \longleftrightarrow \Box\neg\varphi$
S	$\Box\varphi \to \varphi$
4	$\Box\varphi \to \Box\Box\varphi$.

S4 is obtained by closing under modus ponens and necessitation.

Potentialist validities

Similarly, the *converse Barcan* formula is valid at every world in every potentialist system.

$$\Box\forall x\,\psi(x) \implies \forall x\,\Box\,\psi(x).$$

If $\forall x\,\psi(x)$ is true in all worlds accessible from W, then for any $x \in W$, we must have $\psi(x)$ in all further worlds, since this x still exists in those worlds.

Further validities

If potentialist system $\mathcal{W}$ is directed, then S4.2 is valid at every world.

.2 $\quad \Diamond\Box\varphi \to \Box\Diamond\varphi$

If $\mathcal{W}$ is linearly ordered, then S4.3 is valid at every world.

.3 $\quad (\Diamond\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\psi \wedge \Diamond\varphi)$

The above validities hold for all assertions in $\mathcal{L}^{\Diamond}$, with parameters, and more.

So far, we've only mentioned *lower* bounds on the modal validities of a potentialist system.

Upper bounds

We would like to identify exactly the modal validities of various potentialist systems.

In particular, we need a way to recognize *upper* bounds on the validities of a world.

In order to do so, we shall make use of tools—control statements such as buttons, switches, dials and ratchets—which arose in my work with Benedikt Löwe on the modal logic of forcing.

Switches

A *switch* in a Kripke model is a statement s that can always be turned on or off by accessing another world.

Thus, $\Diamond s$ and $\Diamond \neg s$ are true at every world.

A family of switches $s_0, \ldots, s_n$ is *independent*, if every world can access a world realizing any given finite truth pattern.

Switches → S5

Theorem

If Kripke model $\mathcal{W}$ has arbitrarily large families of independent switches, then the validities of each world are within S5.

Proof.

If φ is not in S5, then it fails in a propositional Kripke model M with a finite frame in which every world accesses all others. Associate each world w in M with a switch pattern Φ_w. For each propositional variable p, let

$$\psi_p = \bigvee \{\, \Phi_w \mid p \text{ is true in } w \,\}.$$

M is simulated inside $\mathcal{W}$ via

$$U \models_{\mathcal{W}} \phi(\psi_{p_0}, \ldots, \psi_{p_n}) \qquad \longleftrightarrow \qquad (M, u) \models \phi(p_0, \ldots, p_n),$$

when U satisfies Φ_u. This instance shows φ is not valid in $\mathcal{W}$. $\square$

Dials

A *dial* is a list of statements $d_0, d_1, d_2, \ldots$, such that every world in $\mathcal{W}$ satisfies exactly one of them, and each is possible from any world.

Theorem

A Kripke model has arbitrarily large independent switches iff it has arbitrarily large dials.

Each dial d_r asserts a switch pattern. Each switch asserts a binary digit of the dial index d_r.

Buttons

A *button* is a statement b such that $\Diamond\Box b$ is true at every world.

The button is *pushed* if $\Box b$, and otherwise unpushed.

A *pure* button is one for which $\Box(b \to \Box b)$.

A family of buttons and switches is *independent* if you can control them as desired: push any button without pushing others, and set the switches as desired.

Buttons $\to$ S4.2

Theorem

If Kripke model $\mathcal{W}$ has (arbitrarily many) independent buttons and switches (or buttons and a dial), then the validities of any world where the buttons are not yet pushed are contained within S4.2.

Proof.

As in the modal logic of forcing (Hamkins,Löwe).

Using buttons and switches, can simulate any Kripke model built on a finite pre-Boolean algebra frame, which is complete for S4.2. $\square$

Ratchets

A *ratchet* is a sequence of buttons $r_1, \ldots, r_n$, such that each implies all the earlier, and each can be pushed without pushing the next.

So a ratchet has one-way operation: the ratchet volume can only go up.

Ratchets $\to$ S4.3

Theorem

If a world in Kripke model $\mathcal{W}$ has arbitrarily large ratchets + independent switches (or a dial), then the validities are within S4.3.

The proof similarly is to simulate the Kripke models with finite linear pre-order frames inside $\mathcal{W}$.

Long ratchets

In a model of set theory, a *long ratchet* is a formula $\varphi(\alpha)$ with ordinal parameter α, which form a ratchet.

With a long ratchet, we don't need the independent switches, since we can simulate them by the position within an ω-block.

So any model of set theory with a long ratchet has its validities contained within S4.3.

Set-theoretic potentialism

Let us now turn to investigate various specific cases of set-theoretic potentialism, using the tools we just provided in order to analyze the modal validities.

Rank-potentialism

First, consider set-theoretic rank-potentialism.

Rank-potentialism arises from the potentialist system consisting of the sets V_β, the rank-initial segments of the cumulative hiearchy.

In this system, $\Diamond\varphi$ is true at some V_β, if there is a larger V_δ in which φ is true.

Modal validities of rank-potentialism

Theorem

For set-theoretic rank-potentialism,

1. *Every S4.3 assertion is valid in every V_β for any $\mathcal{L}_\in^\Diamond$ assertion with parameters from V_β.*
2. *Some worlds validate only the S4.3 assertions.*
3. *Validities at any world are within S5.*

Proof.

The V_β are linearly ordered, so S4.3 is valid.

Long ratchet: "$\aleph_\alpha$ exists." □

The potentialist maximality principle

Meanwhile, some V_δ can exhibit additional validities.

5 $\quad \Diamond\Box\varphi \to \varphi$

Theorem

The following are equivalent for any ordinal δ:

1. *S5 is valid in V_δ, for $\mathcal{L}_\in$-assertions with parameters.*
2. *δ is Σ_3-correct. That is, $V_\delta \prec_{\Sigma_3} V$.*

Proof.

(2 $\implies$ 1) Assume δ is Σ_3-correct and $V_\delta \models \Diamond\Box\varphi(a)$. So $\exists\lambda\geq\delta\ \forall\theta\geq\lambda\ V_\theta \models \varphi(a)$. This is Σ_3. It follows that $V_\delta \models \varphi(a)$.

(1 $\implies$ 2) If S5 is valid at V_δ, then $\delta = \beth_\delta$. If $\exists x\forall\beta\ V_\beta \models \varphi(a)$, then $V_\delta \models \Diamond\Box\exists x\forall\beta\ V_\beta \models \varphi(a)$. By S5, it is true in V_δ. $\Box$

The language matters

Allowing assertions from the potentialist language is strictly stronger.

Theorem

The following schemes are equivalent:

1. V_δ *validates S5 for* $\mathcal{L}_\in^\Diamond$*-assertions with parameters.*
2. δ *is a correct cardinal,* $V_\delta \prec V$.

The point is that the modal operators $\Diamond$ and $\exists$ in V_δ work essentially as quantifiers in V, by the potentialist translation.

ZFC proves Σ_3-correct cardinals exist, but it doesn't prove that fully correct cardinals exist.

Variations on rank-potentialism

One can refine the potentialist system by allowing only certain V_β, for β in some class A.

These are still linearly ordered, so S4.3 remains valid.

And one can still make a long ratchet: "there are at least α many elements in A." So some worlds have exactly S4.3.

S5 is valid at V_δ iff δ is $\Sigma_3(A)$-correct.

Grothendieck-Zermelo universes

The potentialist perspective is well illustrated in current mathematical practice by the use of Grothendieck-Zermelo universes in category theory: V_κ for inaccessible cardinal κ.

Category-theorists use these universes in a potentialist manner. Work inside one universe V_κ, but if needed, move to a higher one.

Zermelo also had this perspective explicitly (1930).

> *What appears as an 'ultrafinite non- or super-set' [a proper class] in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain.*

Grothendieck-Zermelo potentialism

Assume the Grothendieck universe axiom. Then:

S4.3 is valid at every GZ-universe V_κ.

Some GZ-universes have only S4.3 as valid.

S5 is valid at GZ-universe V_κ, for $\mathcal{L}_\in$-assertions with parameters, if and only if κ is Σ_3-reflecting.

S5 is valid at V_κ, for $\mathcal{L}_\in^\Diamond$-assertions with parameters, iff κ is fully reflecting.

Transitive-set potentialism

Consider next the potentialist system of all transitive sets

$$\mathcal{T} = \{\, W \mid W \text{ is transitive} \,\}.$$

So $\Diamond\,\psi$ is true at W if there is a larger transitive set with ψ.

This system exhibits potentialism both with respect to height and width.

But width can eventually stabilize. For example, every set x eventually gets its full power set, containing not only all subsets, but all potential subsets.

$$\forall x\ \Diamond\,\exists y\ \Box\, y = P(x)$$

Modal logic of transitive set potentialism

Theorem

The propositional modal validities of transitive-set-potentialism are exactly the assertions of S4.2.

1. *S4.2 is valid in every world, for assertions in $\mathcal{L}_{\in}^{\Diamond}$ with parameters.*
2. *Some worlds validate only S4.2.*
3. *For any particular world, validities are within S5.*

Proof.

Upward directed, so S4.2 is valid.

Provide independent buttons and switches to get exactly S4.2. □

Maximality principle S5

Theorem

The following are equivalent in transitive-set potentialism.

1. *S5 is valid at M for $\mathcal{L}_\in$-assertions with parameters.*
2. *$M = V_\delta$, for some Σ_2-correct cardinal δ.*

Proof.

($2 \implies 1$) Suppose δ is Σ_2-correct, and assume $\Diamond\Box\varphi(a)$ holds at V_δ. So there is transitive set $N \supseteq V_\delta$ with all $U \supseteq N$ having $\varphi(a)$. This is Σ_2. So already such N inside V_δ. So $V_\delta \models \varphi(a)$.

($1 \implies 2$) Assume S5 at M. Show M is correct about power sets. Similar argument shows $M = V_\delta$ some δ. Use $\Diamond\Box\varphi(a) \to \varphi(a)$ to conclude δ is Σ_2-correct. □

Strong maximality principle

If you want S5 for assertions in the potentialist language $\mathcal{L}_{\in}^{\Diamond}$, then it is stronger.

Theorem

The following are equivalent in transitive-set potentialism.

1. *S5 is valid at world M for $\mathcal{L}_{\in}^{\Diamond}$-assertions with parameters.*
2. *$M \prec V$. In other words, $M = V_\delta$ for a correct cardinal δ.*

Variations on transitive set potentialism

It is natural to want only transitive models of a particular nice theory T.

Consider this as a potentialist system, and assume every $x \in V$ is an element of such a model. Then:

S4.2 is valid at every world, for $\mathcal{L}_{\in}^{\Diamond}$-assertions with parameters.

Examples show that some worlds can exhibit exactly S4.2, or exactly S4.3, or exactly some intermediate theory, depending on the theory T and the set-theoretic background.

Solovay's modalities

Solovay had studied the modalities of "true in all transitive sets" and "true in all V_κ for inaccessible κ."

It might seem at first that this is the same thing we are doing with potentialism.

But it is not the same.

Solovay's modalities are not potentialist, since in effect they are oriented downward, rather than upward. For Solovay, $\Diamond\Box\varphi$ is true at V_κ if there is a *smaller* V_β such that φ is true inside all still smaller V_δ.

In contrast, potentialism is upward oriented.

Modal logic of forcing

Consider next the set-theoretic universe V in the potentialist context of all its forcing extensions.

This is width-potentialism, height-actualism.

Benedikt Löwe and I studied the modal validities that arise in this system.

Modal logic of forcing

Theorem (Hamkins,Löwe)

In the potentialist system of all forcing extensions of a fixed countable model of ZFC*,*

1. *S4.2 is valid at every world, for $\mathcal{L}_{\in}^{\Diamond}$-assertions with parameters.*
2. *The validities of any particular world are within S5.*
3. *Some models have exactly S4.2 as their set of validities.*
4. *Depending on the original model, some models have S5 valid for sentences.*

For 3, find a model with independent buttons and switches.

S5 can be forced, but you cannot allow uncountable parameters, since $\Diamond\,\Box(x \text{ is countable})$ is true for any particular set x.

Generic multiverse potentialism

The *generic multiverse* of a model M of set theory is obtained by closing under the operations of forcing extension and ground.

This forms a natural potentialist system.

The modal validities are identical to that in the modal logic of forcing.

Generic multiverse rank-potentialism

It is interesting to combine rank-potentialism with generic-multiverse-potentialism.

Consider a model of set theory M in the context of its generic multiverse. Form the potentialist system of all V_β^W, where W is in the generic multiverse of M.

So this is height-and-width-potentialism, since we can always force outward, adding more subsets, and we can add more ordinals on top.

Validities of generic-multiverse rank-potentialism

Theorem

For generic-multiverse rank-potentialism over a fixed countable model of ZFC*.*

1. *S4.2 is valid at every world for* $\mathcal{L}_{\in}^{\Diamond}$*-assertions with parameters.*
2. *The validities of any particular world are contained within S5, even when restricted to the sentences of set theory.*
3. *If* ZFC *is consistent, then examples show some worlds validate only S4.2.*

CTM potentialism

Consider the collection of countable transitive models of ZFC as a potentialist system.

This system exhibits both height- and width-potentialism.

CTM potentialist validities

Assume every real is in a countable transitive model of ZFC (a weak large cardinal axiom). Then:

- Collection of countable transitive models of ZFC provides a potentialist account of H_{ω_1}.
- S4.2 is valid at every world, any language, with parameters.
- Some worlds validate only S4.2.
- Validities are always within S5.
- Some worlds validate S5 for sentences, no parameters.

For 3, use the Shephardson-Cohen model, which has buttons and switches.

With parameters, all worlds have exactly S4.2 being valid.

Potentialist maximality principle

Let's elucidate the validity of S5, what we call the *potentialist maximality principle*.

$$\Diamond\Box\varphi \to \varphi$$

Theorem

If every real is in a countable transitive model of ZFC*, then every world* $U \in \mathcal{C}$ *can be extended to a world* $W \in \mathcal{C}$ *validating S5 in any countable language extending* $\mathcal{L}_{\in}^{\Diamond}$ *(interpreted in every model of* $\mathcal{C}$*) with real parameters from* U*.*

Proof.

The CTMs are upward σ-closed. Countably many instances of S5 to fulfill. Build a tower of models, achieving $\Diamond\Box\varphi_n$ at stage n, if possible. Any model above the tower has S5. □

V=L and maximize

Although $V = L$ is often viewed as limiting, nevertheless in the potentialist system of CTMs, it is possible that $V = L$ is always recoverable by moving to a taller model, even when there are CTMs satisfying ZFC plus many large cardinals.

This perspective undercuts the view of $V = L$ as necessarily limiting.

Countable models of ZFC

Lastly, let us consider the potentialist system consisting of the countable models of ZFC, under the substructure relation.

This includes the nonstandard models of set theory.

Countable models of set theory

Theorem

Consider the potentialist system of all countable models of ZFC*, under the substructure relation.*

1. *S4.3 is valid at every world W for $\mathcal{L}_\in^\Diamond$ assertions using parameters from W.*
2. *The validities of any particular world W are contained with S4.3, when restricted to $\mathcal{L}_\in$-assertions with parameters.*
3. *The validities of any particular world are contained within S5, when restricted to sentences in the language of set theory.*
4. *S5 is valid at every countable nonstandard model W of* ZFC *for $\mathcal{L}_\in^\Diamond$ sentences.*

S4.3 follows from my embedding theorem: the countable models of set theory are linearly pre-ordered by embeddability.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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