Recent advances in set-theoretic geology

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Set-theoretic geology

In set theory, forcing is often naturally viewed as a method of building *outer* as opposed to *inner* models of set theory.

Set-theoretic geology inverts this perspective by studying how the set-theoretic universe V was obtained by forcing, by studying the fundamental structure of the grounds of V.

The structure of grounds

A transitive class model $W \subseteq V$ of ZFC is a *ground* if the universe was obtained by set forcing over W. That is, W is a ground, if $V = W[G]$ for some W-generic filter $G \subseteq \mathbb{P} \in W$.

In set-theoretic geology, we study the collection of grounds of the set-theoretic universe V and its forcing extensions.

Natural context: the generic multiverse

Set-theoretic geology takes place naturally in the context of the generic multiverse, the collection of models that one can reach by successively moving to forcing extensions or grounds.

Two models of set theory M, N are in the same generic multiverse, if there is a zig-zag path of forcing extensions and grounds moving from M to N.

Set-theoretic geology is focussed on the lower cones of the generic multiverse, looking only downwards.

Sample Question

Suppose that M and N are in the same generic multiverse and it happens that $M \subseteq N$. Must M be a ground of N?

In other words, does the inclusion relation coincide with the ground-model-of relation in the generic multiverse?

You get from M to N in a possibly very long zig-zag path of forcing extensions and grounds, and it happens at the end that $M \subseteq N$. Must it be a ground?

This was open until very recently—I'll provide the answer by the end.

Bedrock

Another fundamental question.

Define that $W \subseteq V$ is a *bedrock*, if it is a minimal ground.

Question (Reitz 2006)

If there is a bedrock, must it be unique?

This question leads naturally to the issue of directedness.

Directedness

Question (Reitz 2006)

Suppose that the set-theoretic universe V was obtained by forcing over two different grounds $V = M[G] = N[H]$. Must there be a common deeper ground?

$$\begin{tikzcd} & V \arrow[dl, dash] \arrow[dr, dash] & \\ M & & N \end{tikzcd}$$

In other words, are the grounds downward directed?

Directedness

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Suppose that the set-theoretic universe V was obtained by forcing over two different grounds $V = M[G] = N[H]$. Must there be a common deeper ground?

$$\begin{tikzcd}[column sep=small] & V \arrow[dl, dash] \arrow[dr, dash] & \\ M \arrow[dr, dash] & & N \arrow[dl, dash] \\ & W? & \end{tikzcd}$$

In other words, are the grounds downward directed?

Downward directedness

Downward-directedness expresses a fundamental property of the structure of grounds.

The question could have been asked 50 years ago.

I've been asking this question of set theorists for a decade.

To my way of thinking, we could not claim to have a deep undertanding of forcing, lacking the knowledge whether or not the forcing-theoretic ground models exhibit this basic, elementary feature.

Some previous knowledge

We knew many instances where directedness was true.

- True in every model where we were able to determine answer.
- True in the *bottomless* models of Reitz—no bedrock.
- (Fuchs,JDH,Reitz) True in any model of the form $L[A]$, where A is a set.

Meanwhile, there were attempts to find a counterexample.

- Woodin proposed a candidate counterexample model (for set-directedness), based on inner model considerations.
- S. Friedman proposed a candidate counterexample, built by a modified coding-the-universe forcing.

It turns out, however, that there is no counterexample.

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Usuba's breakthrough

The question is now answered by Toshimichi Usuba.

Theorem (Usuba)

The grounds of V are downward-directed. Any two grounds have a common deeper ground.

Indeed, any set-indexed family of grounds have a common deeper ground.

I should like to explain the proof, and the numerous consequences.

To my way of thinking, this is one of the most important advances in set theory in recent years.

Let's start with a brief survey of some ideas in set-theoretic geology.

Ground-model definability theorem

Set-theoretic geology begins with the following theorem.

Ground-model definability theorem (Laver, Woodin)

If $V \subseteq V[G]$ is a forcing extension by set-sized forcing, then V is a definable class in $V[G]$.

Laver had contacted me about the theorem, and I was struck immediately by the fundamental importance of the result. I had pointed out to him that what he had made as a minor remark (in the early drafts) was the most important result in his paper.

My student Jonas Reitz had been investigating the idea of "undoing" forcing, leading to the ground axiom, and the ground-model definability theorem was immediately applicable.

Ground-model definability theorem

Laver had contacted me because he had noticed a similarity in his proof of ground-model definability to a certain back-and-forth cover argument, which Woodin and I had used and which appears in my work on the approximation and cover properties.

I don't know Laver's original proof, but I sent him my proof of the ground-model definability theorem based on the approximation and cover properties.

He said, "that is clearly the right way to do it," and adopted my proof in his paper.

Pseudo-ground model definability

The approximation and cover property argument leads to a stronger version of the theorem.

Pseudo-ground definability theorem (Hamkins)

If $W \subseteq V$ has the δ-approximation and δ-cover properties, then W is a definable class in V.

This theorem immediately implies the ground-model definability theorem, and applies to many natural instances of class forcing.

GCH forcing; progressively closed iterations; global Laver preparation.

Other improvements

The pseudo-ground ideas also allow for improvements in the size of the parameter used to define V in $V[G]$.

If this extension has the δ-approximation and δ-cover properties, then parameter $r = (2^{<\delta})^V$ suffices.

This can be significantly smaller than the size of the forcing.

The ground-model enumeration theorem

The uniformity of the ground model definition leads to:

Ground-model enumeration theorem

There is a parameterized family $\{ W_r \mid r \in V \}$ such that

1. Every W_r is a ground of V and $r \in W_r$.
2. Every ground of V is W_r for some r.
3. The relation "$x \in W_r$" is first order expressible in set theory.

This reduces second-order statements about grounds to first-order statements about parameters.

Let's give some examples.

The ground axiom

For example, the *ground axiom* (JDH,Reitz) is the assertion that the universe was not obtained by nontrivial forcing.

$$\forall r\, V = W_r.$$

In other words, there are no nontrivial grounds.

Although one might reasonably have hoped that the ground axiom implied regular structural properties, Jonas proved in his dissertation among other things that $\mathrm{ZFC} + \mathrm{GA}$ has no new Π_2-consequences. It cannot settle CH, or $\Diamond$ or any other locally verifiable assertion.

The mantle

The *mantle* is $M = \bigcap_r W_r$.

This inner model became the focus of set-theoretic geology.

Ancient Paradise

The analysis of the mantle engages with an interesting philosophical idea:

Ancient Paradise. This is a perspective in set-theoretic ontology that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If we could sweep the accumulated material away, we should find an ancient paradise.

The mantle, of course, wipes away an entire strata of forcing.

So the ancient paradise view suggests that the mantle may be highly regular.

Every model is a mantle

Unfortunately, our initial main theorem tends to refute this perspective:

Theorem (Fuchs, Hamkins, Reitz)

Every model of ZFC *is the mantle of another model of* ZFC*.*

By sweeping away the accumulated sands of forcing, what we find is not a highly regular ancient core, but rather: an arbitrary model of set theory.

Conclusion: we will not be able to prove any highly regular structural features of the mantle.

Downward directedness hypotheses

The ground-model enumeration theorem allows one to express directedness in the first-order language of set theory.

Definition

1. The Downward Directed Grounds Hypothesis DDG asserts that the grounds are downward directed.

   For every r and s there is t such that $W_t \subseteq W_r \cap W_s$.

2. The Strong DDG asserts that they are downward set-directed.

   For every set I there is t with $W_t \subseteq \bigcap_{r \in I} W_r$.

Thus, seemingly second-order assertions are brought into the realm of ZFC set theory.

Naive answer attempt: intersect the models

Before continuing, let's dispense with a naive solution attempt. The intersection of ZFC models need not model ZFC.

Example

A model of set theory with two grounds, whose intersection is not a model of ZFC.

Proof.

Start in V with GCH. Using Easton product forcing, let $V[G]$ force failure of GCH at every $\aleph_n$, and let c be a $V[G]$-generic Cohen real.

Consider $V[G][c]$, with grounds $V[G]$ and $V[c][G \restriction c]$.

Intersection has GCH failing at $\aleph_n$ iff $n \in c$, but it does not have c. This violates separation axiom. $\square$

Downward-directed Grounds hypothesis

Let's turn now to Usuba's theorem.

Theorem (Usuba)

The strong DDG *is true.*

In other words, for any set I, there is a ground W that is contained in every ground W_r with $r \in I$.

$$W \subseteq \bigcap_{r \in I} W_r$$

Let me present a detailed proof.

Review of some tree combinatorics

König's Lemma

Every finitely-branching infinite tree has a cofinal branch.

In other words, every tree of height ω with all levels finite has a branch.

How does this generalize to higher cardinals?

Aronszajn trees, of course, block one approach.

But consider a tree of height ω_1 with all levels finite; or a tree of height ω_2 with all levels countable.

More generally, consider a *tall narrow* tree...

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More generally, consider a *tall narrow* tree...

Every tall narrow tree has a branch, but not too many

Lemma (Kurepa)

If T is a tree, height λ, all levels size $< \delta$, with δ regular, $\delta < \mathrm{cof}(\lambda)$, then T has a cofinal branch, but fewer than δ many.

Proof.

For levels β with cofinality δ, find $\nu_\beta < \beta$ such that level β nodes separate already by level ν_β. By Fodor, there is stationary $S \subseteq \lambda$ with constant value ν. So all nodes on level $\beta \in S$ separate by level ν. By pigeon-hole, one node on level ν has successors on cofinally many levels, and these must cohere to form a branch.

Meanwhile, any two cofinal branches have separated by level ν, so there are fewer than δ many branches. $\square$

Tall narrow trees gain no new branches

Lemma

If $W \subseteq V$ is an inner model of ZFC and T is a tall narrow tree in W, then all cofinal branches of T are already in W.

Proof.

We know T has fewer than δ many branches in W. If b is a new branch through T in V, then there must be a level by which b is distinguished from those branches. So there is a node p on b, but not on any branch in W. But T_p is a tall narrow tree in W, and so has a branch in W, contradiction. □

Approximation and cover properties

Suppose that W is a transitive class and that δ is a cardinal.

Definition

- The extension $W \subseteq V$ satisfies the δ-*approximation property*, if whenever $A \subseteq W$, $A \in V$ and $A \cap a \in W$ for all $a \in W$ of size less than δ, then $A \in W$.
- The extension $W \subseteq V$ satisfies the δ-*cover property*, if for every $A \subseteq W$ with $A \in V$ and $|A| < \delta$, there is $B \in W$ with $A \subseteq B$ and $|B| < \delta$.

These concepts are central in set-theoretic geology, used in my proof of the ground-model definability theorem.

Uniform cover property

Consider a *uniform* version of covering: every sequence of small sets is uniformly covered.

The *uniform* δ-cover property for $W \subseteq V$

If $A_i \subseteq W$ with $|A_i| < \delta$ every $i \in I$, with $I \in W$, then there is a covering sequence in W:

- $\langle B_i \mid i \in I \rangle \in W$.
- $A_i \subseteq B_i$.
- $|B_i| < \delta$.

Using the axiom of choice, it suffices to consider $\langle A_\alpha \mid \alpha < \lambda \rangle$ for λ ordinal and $A_\alpha \subseteq \lambda$.

Uniform covering, alternative formulations

If $W \subseteq V$ and δ regular, $\delta \leq \lambda$, then the following are equivalent:

1. For every $\langle A_\alpha \mid \alpha < \lambda \rangle$ with $A_\alpha \subseteq \lambda$ size $< \delta$, there is $\langle B_\alpha \mid \alpha < \lambda \rangle \in W$ with $A_\alpha \subseteq B_\alpha$ and $|B_\alpha| < \delta$.
2. For every $A \subseteq \lambda \times \lambda$, with all vertical sections size $< \delta$, there is $B \in W$ with $A \subseteq B$ and all sections size $< \delta$.
3. For every function $f : \lambda \to \lambda$ there is $B \in W$ with $B \subseteq \lambda \times \lambda$, all vertical sections size $< \delta$ and $f \subseteq B$.

Let's call this: λ-*uniform* δ-*covering*.

Uniform covering implies approximation

Lemma

If $W \subseteq V$ has λ-uniform δ-cover property, with δ regular and λ strong limit, then it has δ^+-approximation property for subsets of λ.

Proof.

Assume $s \in 2^\lambda$ has all δ^+-small approximations in W; aim to show $s \in W$. Assume inductively that $s \restriction \alpha \in W$ all $\alpha < \eta$. By uniform covering, can find a tree $T \in W$ height η, levels size $< \delta$, such that $s \restriction \alpha \in T$ for $\alpha < \eta$. If $\delta < \mathrm{cof}(\eta)$, this is tall narrow tree, and so $s \in W$.

Otherwise, $\mathrm{cof}(\eta) \leq \delta$. By a simple closure argument, find cofinal $J \subseteq \eta$ size δ, such that distinct nodes p, q on level $\beta \in J$ have $p(\alpha) \neq q(\alpha)$ some $\alpha \in J$. By approximation assumption, $s \restriction J \in W$, and this determines $s \restriction \eta$. $\square$

Bukovskỳ ground model characterization

Theorem (Bukovskỳ, 1973)

Suppose that $W \subseteq V$ is an inner model of ZFC. Then the following are equivalent:

1. *W is a ground of V.*
2. *For some cardinal δ, the extension $W \subseteq V$ exhibits the uniform δ-cover property.*

The direction 1 $\to$ 2 is immediate, using the δ-chain condition.

The direction 2 $\to$ 1 is reminiscent of Vopěnka's proof that every set is HOD-generic, uses an infinitary logic.

Downward-directed Grounds hypothesis

Let us now put it together to prove the main result.

Theorem (Usuba)

The downward-directed grounds hypothesis is true.

Fix any set I and consider the grounds W_r for $r \in I$. We seek to find a ground W with $W \subseteq W_r$ for all $r \in I$.

For each ground W_r, we may realize V as a forcing extension $V = W_r[G_r]$, where $G_r \subseteq \mathbb{Q}_r \in W_r$ is W_r-generic.

Let κ be regular, larger than every $|\mathbb{Q}_r|^{W_r}$ and larger than $|I|$.

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Let κ be regular, larger than every $|\mathbb{Q}_r|^{W_r}$ and larger than $|I|$.

Consider large $\theta = \beth_\theta > \kappa$.

Lemma

There is $A \subseteq \theta$ with $L[A] \subseteq W_r$ all $r \in I$, such that $L[A] \subseteq V$ has $<\theta$-uniform κ^+-covering property.

Proof.

Let $h : \theta \to \theta$ be universal, in that every $t : \lambda \to \lambda$ for $\lambda < \theta$ occurs as a block in h. Since $W_r \subseteq V$ has uniform κ-covering, there is $H_{0,r} \subseteq \theta \times \theta$ with all vertical sections size $< \kappa$ and $h \subseteq H_{0,r} \in W_r$. Continuing, for $\eta < \kappa$ find $H_{\eta,r} \in W_r$ with all sections size $< \kappa$ and $H_{\nu,s} \subseteq H_{\eta,r}$ for $\nu < \eta$, $s \in I$. Let H be the union of all $H_{\eta,r}$, so $H \subseteq \theta \times \theta$, with all sections size $\leq \kappa$. Note that $H \in W_r$ all $r \in I$ by approximation property, since $H \cap a = H_{\eta,r} \cap a$ for κ-small $a \in W$. Let $A \subseteq \theta$ code $H \subseteq \theta \times \theta$. So $L[A] \subseteq W_r$ all $r \in I$. Also, $L[A] \subseteq V$ has $<\theta$-uniform κ^+-covering, since $h \subseteq H$. $\square$

So for each $\theta = \beth_\theta$ above κ, we have $L[A_\theta] \subseteq \bigcap_r W_r$ and $L[A_\theta] \subseteq V$ has $<\theta$-uniform κ^+-covering, hence κ^{++}-approximation for bounded subsets of θ.

It follows by the pseudo-ground-model definability theorem that $V_\theta^{L[A]}$ is definable in V_θ from parameter $p = (2^{<\kappa^{++}})^{L[A_\theta]}$.

Some such p must be used for unboundedly many θ, and the $V_\theta^{L[A_\theta]}$ cohere by the ground-model definability theorem. Let $W = \bigcup_\theta V_\theta^{L[A_\theta]}$ be the union of these.

Note that W is closed under Gödel operations, is almost universal and has well-orders. So $W \models \mathrm{ZFC}$.

Also, $W \subseteq W_r$ for all $r \in I$, since $W_\theta \subseteq L[A_\theta] \subseteq W_r$.

Finally, $W \subseteq V$ has uniform κ^{++}-cover property, since $V_\theta^{L[A_\theta]} \subseteq V_\theta$ had $<\theta$-uniform κ^{++}-covering. So by Bukovskỳ, W is a ground of V, contained in every W_r for $r \in I$, as desired. QED

Conclusion: in any model of ZFC, any set-indexed family of grounds have a common deeper ground.

$$W \subseteq \bigcap_r W_r \subseteq V$$

Let's now mention consequences of the DDG.

Consequences

Bedrock models are unique when they exist

This is immediate, since any two bedrocks would have a common ground below.

This was the question in Reitz's dissertation that originally motivated the DDG.

DDG → mantle is absolute

The mantle is absolute by forcing

For any forcing extension $V \subseteq V[G]$, we have the easy inclusion

$$\mathbb{M}^V \quad \mathbb{M}^{V[G]}$$

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For any forcing extension $V \subseteq V[G]$, we have the easy inclusion

$$\mathbb{M}^V \supseteq \mathbb{M}^{V[G]}$$

Every ground of V is also a ground of $V[G]$, and so $\mathbb{M}^{V[G]}$ arises from a larger intersection.

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The mantle is absolute by forcing

For any forcing extension $V \subseteq V[G]$, we have the easy inclusion

$$\mathbb{M}^V \supseteq \mathbb{M}^{V[G]}$$

Every ground of V is also a ground of $V[G]$, and so $\mathbb{M}^{V[G]}$ arises from a larger intersection.

Conversely, if $x \notin \mathbb{M}^{V[G]}$, then x is excluded by some ground $W \subseteq V[G]$. Find a common ground W_0 below V, W, and so $x \notin W_0$. So $\mathbb{M}^V = \mathbb{M}^{V[G]}$.

Mantle = generic Mantle

The mantle is the same as the generic mantle

The generic mantle is the intersection of the grounds of all the forcing extensions of V. Since the ground-model grounds are dense amongst those new grounds, the intersection is the same.

The generic mantle was originally introduced because we could prove more about it than we could about the mantle. But now that we know they are the same, there is no longer any need for the generic mantle concept.

The mantle is a ZFC model

The mantle is a model of ZFC

It is a transitive class containing all ordinals, closed under the Gödel operations, and almost-universal (this uses the DDG for absoluteness). So it has ZF.

For AC, use the DDG again: for every set in the mantle, if each well-order is excluded by some ground, then can find a common ground excluding all well-orders, contradiction.

The mantle is the largest forcing-invariant class

The mantle is the intersection of the generic multiverse

Since it is forcing invariant, it is contained in that intersection. But the mantle is the intersection of grounds, which are among the models of the generic multiverse.

The mantle is the largest forcing-invariant definable class

Any other such class would be contained in the intersection of the generic multiverse.

Thus, the mantle is a highly canonical definable inner model of ZFC.

The mantle is locally a ground

The mantle is, locally, a ground model

That is, for any ordinal γ, there is a ground W with

$$(V_\gamma)^W = (V_\gamma)^{\mathbb{M}}.$$

To see this, apply the strong DDG. Every $x \in V_\gamma - \mathbb{M}$ is omitted by some ground W_{r_x}. But there are only set-many x's, so we can find a single ground W that omits all such x.

This might suggest that every set is generic over the mantle, but that isn't necessarily true.

Diameter of the generic multiverse

One travels in the generic multiverse from one model to another in a zig-zag path of forcing extensions and grounds.

Question

How many zig-zags are needed?

This question was important in the investigation of "generic multiverse possibility."

$$\Diamond\varphi \quad \text{iff} \quad \varphi \text{ is true in some model in the generic multiverse.}$$

If the number of steps is uniformly bounded, then this modality is expressible in the language of set theory.

Generic multiverse diameter

Woodin had noted: for generic multiverse truth, 3 steps suffice.

Löwe and I reduced this to 2 steps: up and then down.

To reach the actual models, as opposed to realizing a given truth, we had no idea how many steps were required.

This is now settled by the DDG.

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Structure of the generic multiverse

The generic multiverse of M consists precisely of the ground-extensions of M, that is, the models of the form $W_r[G]$, the forcing extensions of the grounds of M.

Certainly these are all in the generic multiverse of M, and closed under forcing extensions.

Conversely, the DDG shows that these are closed under grounds. So they exhaust the generic multiverse.

Thus, the diameter of the generic multiverse is 2.

Inclusion in the generic multiverse

The *generic multiverse* of M consists of all models obtainable by successively passing to a forcing extension or a ground.

Question

If M, N are in the same generic multiverse and $M \subseteq N$, must M be a ground of N?

Theorem

Yes. The inclusion relation agrees with the ground-of relation in the generic multiverse.

The point is that the DDG implies that the generic multiverse of M consists of all the forcing extensions of grounds of M, since this collection is closed under forcing extensions and grounds.

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Modal logic of forcing

I had introduced the forcing modalities.

$\Box\varphi$ if φ holds in all forcing extensions.

$\Diamond\varphi$ if φ holds in some forcing extension.

This modal language can express diverse forcing principles.

Maximality principle (Stavi+Väänänen, indep. JDH) asserts

$$\Diamond\Box\varphi \to \varphi$$

Question (JDH)

What exactly are the ZFC-provable forcing validities?

Theorem (JDH,Löwe)

If ZFC is consistent, then the ZFC-provably valid principles of forcing are precisely those in the modal theory S4.2.

Up and down forcing modalities

In set-theoretic geology, we naturally have *two* pairs of forcing modalities:

$\boxed{}\,\varphi$ if φ holds in all forcing extensions.

$\Diamond\!\!\!\!\Diamond\,\varphi$ if φ holds in some forcing extension.

$\square\,\varphi$ if φ holds in all grounds.

$\Diamond\!\!\!\!\Diamond\,\varphi$ if φ holds in some grounds.

Question (JDH,Löwe)

What exactly are the mixed-modality principles of forcing?

For example, these temporal-logic-like principles are valid:

$$\varphi \to \boxed{}\,\Diamond\!\!\!\!\Diamond\,\varphi \qquad\qquad \varphi \to \square\,\Diamond\!\!\!\!\Diamond\,\varphi$$

Modal logic of grounds

The DDG implies (and is nearly equivalent to) the validity of axiom .2 for the downward-logic.

$$\Diamond\Box\varphi \to \Box\Diamond\varphi$$

Directedness implies the possibility operators commute.

$$\Diamond\Diamond\varphi \to \Diamond\Diamond\varphi$$

Corollary

If ZFC is consistent, then the ZFC-provably valid downward principles of forcing are exactly S4.2.

The point is that Benedikt Löwe and I had already proven S4.2 as an upper bound, but our previously best lower bound was S4. With the DDG, we get S4.2, so this theory hits it exactly.

Generic multiverse modalities

Generic multiverse possibility is expressible

$$\Diamond\varphi \quad \leftrightarrow \quad \Diamond\Diamond\varphi \quad \leftrightarrow \quad \Diamond\Diamond\varphi$$

Similarly for generic multiverse necessity

$$\Box\varphi \quad \leftrightarrow \quad \Box\Box\varphi \quad \leftrightarrow \quad \Box\Box\varphi$$

Let's now turn to another important development, revealing a connection between extremely large cardinals and the forcing-theoretic structure of the universe.

Large cardinal → few grounds

Theorem (Usuba)

If there is a hyper-huge cardinal, then the universe has a bedrock.

In other words, the mantle is a ground.

I find this to be an incredible connection between large cardinal existence and the structure of grounds!

A cardinal κ is *hyper-huge*, if for every ordinal λ there is $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$ and $M^{j(\lambda)} \subseteq M$.

supercompact $<$ super-huge $<$ hyper-huge $<$ super almost 2-huge

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In other words, the mantle is a ground.

I find this to be an incredible connection between large cardinal existence and the structure of grounds!

A cardinal κ is *hyper-huge*, if for every ordinal λ there is $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$ and $M^{j(\lambda)} \subseteq M$.

supercompact $<$ super-huge $<$ hyper-huge $<$ super almost 2-huge

Lemma

If κ is hyper-huge and W is a ground of V, then V is a forcing extension of W by forcing of size $< \kappa$.

Proof.

$V = W[G]$ for some $G \subseteq \mathbb{P} \in W$. Pick inaccessible $\lambda > |\mathbb{P}|$ and let $j : V \to M$ witness hyper-hugeness. Fix stationary partition $\langle S_\alpha \mid \alpha < j(\lambda)\rangle$ of $\mathrm{Cof}_\omega \cap j(\lambda)$ in W, and argue that $j(\vec{S})_\beta$ is stationary in $\sup j\text{ ``}j(\lambda)$ in $j(W)$ iff $\beta \in j\text{ ``}j(\lambda)$. Conclusion: $j\text{ ``}j(\lambda) \in j(W)$. It follows that

$$W_{j(\lambda)} \subseteq j(W)_{j(\lambda)} \subseteq M_{j(\lambda)} \subseteq V_{j(\lambda)} = W[G]_{j(\lambda)}.$$

So $M_{j(\lambda)}$ is extension of $j(W)_{j(\lambda)}$ by forcing size $< j(\kappa)$. By elementarity, V_λ is a forcing extension of W_λ by forcing of size $< \kappa$. □

Thus, V has a bedrock, by the strong DDG.

Hyperhuge → only small forcing

General conclusion

If there is a hyperhuge cardinal, then essentially,

The universe was not obtained by forcing,

except possibly for the trivial kind of small forcing allowed for in the Lévy-Solovay theorem.

The result showed that if κ is hyperhuge, then V is a forcing extension of the mantle, which is a bedrock model, by forcing of size less than κ.

$$V = \mathbb{M}[G]$$

All other grounds are therefore trapped between $\mathbb{M}$ and V, also by small forcing.

Automatic absoluteness of very large cardinals

This implies a kind of automatic absoluteness of very large cardinals to a highly canonical definable inner model.

Namely, if there is a hyperhuge cardinal κ, then the universe is a κ-small forcing extension of the mantle. Consequently, by the Lévy-Solovay phenomenon, all other large cardinals above κ are also absolute to the mantle.

The mantle is a canonical forcing-invariant definable inner model of ZFC, the largest forcing-invariant class.

So this is a case of automatic absoluteness for very large cardinals to a canonically defined inner model.

Chain condition of the common ground

Usuba's proof of the DDG shows that if every W_r is a κ-c.c. ground for $r \in I$, then there is a common ground $W \subseteq \bigcap_{r \in I} W_r$ that is κ^{++}-c.c.

So the chain condition of the common ground jumps twice.

Question

Can this be improved?

For example, can we always find a κ-c.c. common ground, or at least a κ^{+}-c.c. common ground?

Relativize to a class of forcing Γ

Question

For which classes Γ of forcing does the downward-directed grounds hypothesis hold?

We already know several instances where DDG_Γ can fail.

- $\mathrm{DDG}_{\mathrm{ccc}}$ can fail.
- $\mathrm{DDG}_{\mathrm{Cohen}}$ can fail.
- $\mathrm{DDG}_{\mathrm{proper}}$ can fail.
- $\mathrm{DDG}_{\sigma\text{-closed}}$ can fail.

Improving the large cardinal hypothesis

Usuba proved that if there is a hyperhuge cardinal, then there is a bedrock.

Question

Which large cardinals imply that the universe has a bedrock?

Supercompact is insufficient. But the cases of superhuge cardinal, a super-almost-huge cardinal and even an extendible cardinal are extremely interesting.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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