Class forcing and second-order arithmetic

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Content

The Continuum Hypothesis

One of the most prominent open problems in mathematics in the beginning of the 20th century was the

Continuum Hypothesis (CH)

There is no set whose cardinality is strictly between the cardinality of the natural numbers and that of the real numbers. In other words, $2^{\aleph_0} = \aleph_1$.

Kurt Gödel proved its consistency in 1940 and Paul Cohen proved its independence of the axioms of ZFC in 1963.

Independence results in set theory

Definition

A property Ψ is **independent** of ZFC, if neither Ψ nor its negation is provable from ZFC.

The way to prove independence is to construct models M, M' of ZFC such that

- $M \models \Psi$ and
- $M' \models \neg\Psi$.

A method to construct such models is **forcing**.

The idea of forcing

We extend a given model M of ZFC to a new model $M[G]$ by adding a generic object G.

Every element of $M[G]$ has **name** σ in M which is evaluated as σ^G, i.e.

$$M[G] = \{\sigma^G \mid \sigma \text{ is a name}\}.$$

This is similar to the case of field extensions in algebra:

Consider $\mathbb{Q}$ and an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$.

- In $\mathbb{Q}$, the polynomial $X^3 - 2$ **names** the roots $\sqrt[3]{2}, \zeta\sqrt[3]{2}, \zeta^2\sqrt[3]{2} \in \bar{\mathbb{Q}}$ ($\zeta = e^{\frac{2\pi i}{3}}$).
- If we extend $\mathbb{Q}$ to $\mathbb{Q}[\sqrt[3]{2}]$, then $\sqrt[3]{2}$ is the **evaluation** of $X^3 - 2$.
- $\mathbb{Q}[\sqrt[3]{2}]$ is a **minimal** field extension of $\mathbb{Q}$ with $\sqrt[3]{2} \in \mathbb{Q}$.

Forcing

Let M denote a countable transitive model of (some fragment of) ZFC.

Forcing uses a **partial order** $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ to approximate the generic object. Elements of P are called **conditions**.

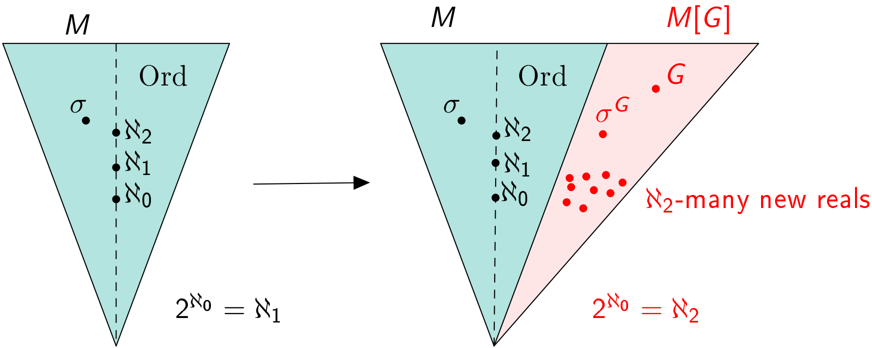
Definition

1. A subset $D \subseteq P$ is said to be **dense**, if for every $p \in P$ there is some $q \leq_{\mathbb{P}} p$ with $q \in D$.
2. A subset $G \subseteq P$ is said to be a $\mathbb{P}$-**generic filter**, if it has the following properties:
   - If $p \leq_{\mathbb{P}} q$ and $p \in G$, then $q \in G$.
   - If $p, q \in G$ then there is $r \in G$ such that $r \leq_{\mathbb{P}} p, q$.
   - If $D \subseteq P$ is a dense set which is in M, then $G \cap D \neq \emptyset$.

Such filters do not exist in M.

Cohen forcing

By adding $\aleph_2$-many reals instead of just one we obtain that $M[G] \models \neg\mathrm{CH}$.



$M[G] = \{\sigma^G \mid \sigma \text{is a } \mathbb{P}\text{-name}\}$.

The forcing theorem

Let M be a countable transitive model of ZFC and $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ a partial order.

Theorem

If G is $\mathbb{P}$-generic over M then $M[G]$ is a transitive model of ZFC with $M \cup \{G\} \subseteq M[G]$ and $\mathrm{Ord}^{M[G]} = \mathrm{Ord}^M$.

Let $p \in P$, $\varphi(x)$ a formula and σ a $\mathbb{P}$-name. We say that p **forces** $\varphi(\sigma)$, denoted

$$p \Vdash_{\mathbb{P}} \varphi(\sigma),$$

if for every $\mathbb{P}$-generic filter G with $p \in G$, $M[G] \models \varphi(\sigma^G)$.

The proof of the theorem above relies on

Theorem (Forcing theorem)

1. *The forcing relation $p \Vdash_{\mathbb{P}}^M \varphi(\sigma)$ is definable over M* ***(Definability lemma)****.*
2. *If $M[G] \models \varphi(\sigma^G)$ then there is $p \in G$ such that $p \Vdash_{\mathbb{P}}^M \varphi(\sigma)$* ***(Truth lemma)****.*

Further properties of set forcing

1. Every partial order $\mathbb{P}$ has a Boolean completion, i.e. there is an injective dense embedding from $\mathbb{P}$ into some complete Boolean algebra.
2. If there is a dense embedding $\mathbb{P} \to \mathbb{Q}$, then $\mathbb{P}$-generic extensions and $\mathbb{Q}$-generic extensions coincide.
3. Every set of ordinals in a $\mathbb{P}$-generic extension has a nice name, i.e. a name of the form $\bigcup_{\alpha<\gamma}\{\check{\alpha}\} \times A_\alpha$, where each $A_\alpha \subseteq P$ is an antichain.

The generalized Continuum Hypothesis GCH

Consider the following strengthening of CH:

Generalized Continuum Hypothesis (GCH)

$2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α.

This motivates the following

Question

Is it possible to obtain the GCH *(or failures at every level) using forcing?*

Using a set-sized partial order $\mathbb{P}$ it is impossible to modify the continuum function above $2^{|\mathbb{P}|}$.

The solution is to use **class-sized** partial orders instead of set-sized ones.

A general setting for class forcing

We study class forcing in a **second-order** context.

Definition

Let ZF^- denote ZFC without the power set axiom and the axiom of choice. We denote by $\mathbf{GB}^-$ the theory in the two-sorted language with variables for sets and classes, with

- set axioms given by ZF^- with class parameters allowed in the schemata of Separation and Collection
- class axioms of extensionality, foundation and first-order class comprehension (i.e. involving only set quantifiers).

Examples are $\langle M, \mathrm{Def}(M)\rangle$, where M is a countable transitive model of ZF^-, and models of Kelley-Morse class theory KM.

Class forcing extensions

Let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ be a countable transitive model of GB^-. We work with partial orders $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ such that $\leq_{\mathbb{P}}, P \in \mathcal{C}$.

- $M^{\mathbb{P}}$ denotes the set of $\mathbb{P}$-names which are in M (**set names**).
- $\mathcal{C}^{\mathbb{P}}$ denotes the set of $\mathbb{P}$-names which are in $\mathcal{C}$ (**class names**).

A filter G is $\mathbb{P}$-**generic** over $\mathbb{M}$ if it meets all dense subclasses of M which are in $\mathcal{C}$. Evaluations of names are defined as usual.

We set $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G] \rangle$, where

- $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$
- $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$.

The forcing theorem

Let $\mathbb{P}$ be a class-sized partial order, $\sigma \in M^{\mathbb{P}}$ and $\Gamma \in \mathcal{C}^{\mathbb{P}}$. We write $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma, \Gamma)$ if for every $\mathbb{P}$-generic filter G with $p \in G$, $\mathbb{M}[G] \models \varphi(\sigma^G, \Gamma^G)$.

Definition

We say that $\mathbb{P}$ satisfies the **forcing theorem** over $\mathbb{M}$, if for every $\mathcal{L}_\in$-formula $\varphi(x, C)$ allowing class parameters and for every $\Gamma \in \mathcal{C}^{\mathbb{P}}$,

1. $\{\langle p, \sigma \rangle \in P \times M^{\mathbb{P}} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma, \Gamma)\} \in \mathcal{C}$ (**definability lemma**)
2. whenever G is $\mathbb{P}$-generic over $\mathbb{M}$, $\sigma \in M^{\mathbb{P}}$ and $\Gamma \in \mathcal{C}^{\mathbb{P}}$ such that $\mathbb{M}[G] \models \varphi(\sigma^G, \Gamma^G)$ then there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma, \Gamma)$ (**truth lemma**).

A failure of Replacement

In the following, we present some examples which illustrate that class forcing behaves differently from set forcing.

From now on, let $\mathbb{M} = \langle M, \mathcal{C} \rangle$ denote a countable transitive model of GB^-.

Observation

Let $\mathbb{P} = \mathsf{Col}(\omega, \mathrm{Ord})$ denote the partial order whose conditions are finite functions $p : \mathrm{dom}(p) \to \mathrm{Ord}^M$, $\mathrm{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion. Then $\mathbb{P}$ adds a surjective function $\omega \to \mathrm{Ord}^M$. In particular, Replacement fails in the generic extension.

Failures of the forcing theorem

... but it can get even worse:

Theorem (Holy, K., Lücke, Njegomir, Schlicht 2015)

Let M be a countable transitive model of ZF^-. There is a partial order $\mathbb{P} \subseteq M$ which is definable over M such that $\mathbb{P}$ does not satisfy the definability lemma over M.

... and even worse than that:

Theorem (Holy, K., Lücke, Njegomir, Schlicht 2015)

There are countable transitive models M of ZF^- for which there is a partial order $\mathbb{P}$ that is definable over M such that $\mathbb{P}$ does not satisfy the truth lemma over M.

Boolean completions

Definition

1. A Boolean algebra $\mathbb{B}$ is M-**complete** if the supremum $\sup_{\mathbb{B}} A$ of all elements in A exists in $\mathbb{B}$ for every $A \in M$ with $A \subseteq \mathbb{B}$.
2. We say that $\mathbb{P}$ **has a Boolean completion in** $\mathbb{M}$ if there is an M-complete Boolean algebra $\mathbb{B}$ such that its domain, all Boolean operations of $\mathbb{B}$ are in $\mathcal{C}$ and there is an injective dense embedding from $\mathbb{P}$ into $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ in $\mathcal{C}$.

In set forcing, every partial order has a Boolean completion.

Theorem (Holy, K., Lücke, Njegomir, Schlicht)

Suppose that $\mathcal{C}$ contains a global well-order. Then a partial order $\mathbb{P}$ satisfies the forcing theorem if and only if it has a Boolean completion.

In particular, there are class-sized partial orders which do not have a Boolean completion.

Failures of the extension maximality principle

Definition

A partial order $\mathbb{P}$ satisfies the **extension maximality principle (EMP)** over $\mathbb{M} \models \mathsf{GB}^-$ if for every partial order $\mathbb{Q}$ such that $\mathbb{P}$ is dense in $\mathbb{Q}$ and for every $\mathbb{Q}$-generic filter G over $\mathbb{M}$, $M[G] = M[G \cap \mathbb{P}]$.

Every set-sized partial order satisfies the EMP.

Observation

Let $\mathrm{Col}_*(\omega, \mathrm{Ord})$ denote the suborder of $\mathrm{Col}(\omega, \mathrm{Ord})$ of conditions p with $\mathrm{dom}(p) \in \omega$. Clearly, $\mathrm{Col}_*(\omega, \mathrm{Ord})$ is dense in $\mathrm{Col}(\omega, \mathrm{Ord})$. However, $\mathrm{Col}(\omega, \mathrm{Ord})$ collapses all M-cardinals but $\mathrm{Col}_*(\omega, \mathrm{Ord})$ does not add any new sets.

Non-existence of nice names

Let $\mathbb{P}$ be a partial order. A **nice name** is a $\mathbb{P}$-name of the form $\bigcup_{\alpha<\gamma}\{\check{\alpha}\} \times A_\alpha$, where $A_\alpha \subseteq \mathbb{P}$ is an antichain in M and $\gamma \in \mathrm{Ord}^M$.

Definition

$\mathbb{P}$ is **nice**, if for every $\gamma \in \mathrm{Ord}^M, \sigma \in M^{\mathbb{P}}$ and for every $\mathbb{P}$-generic filter G such that $\sigma^G \subseteq \gamma$ there is a nice name $\tau \in M^{\mathbb{P}}$ with $\sigma^G = \tau^G$.

Consider the forcing notion $\mathbb{P} = \mathrm{Col}(\omega, \mathrm{Ord})$ and $\sigma = \{\langle \check{n}, \{\langle n, 0\rangle\}\rangle \mid n \in \omega\}$. The set $\omega \setminus \sigma^G$ has a name, but no nice name:

Suppose that $\mu = \bigcup_{n\in\omega}\{\check{n}\} \times A_n$ and $p \Vdash_{\mathbb{P}} \mu = \check{\omega} \setminus \sigma$. Take $n \notin \mathrm{dom}(p)$ and $\alpha > \mathrm{rank}(A_n)$ and put $q = p \cup \{\langle n, \alpha\rangle\}$. Then $q \Vdash_{\mathbb{P}} \check{n} \in \mu$ so there must be $r \in A_n$ which is compatible with q. But then $n \in \mathrm{dom}(r)$ and so $r(n) = \alpha$, a contradiction.

Conclusion

$\mathrm{Col}(\omega, \mathrm{Ord})$ is not nice.

Motivation

We need to place some restrictions on the class-sized partial orders used for forcing.

Question

1. *Under what conditions does a class-sized partial order satisfy the forcing theorem and preserve the axioms of* GB^-*?*
2. *Is there a (minimal) property which ensures that all properties of set forcing can be transferred to class forcing?*

Pretameness

The following notion was introduced by Sy Friedman.

Definition

We say that a class-sized partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ is **pretame** for $\mathbb{M} = \langle M, \mathcal{C} \rangle$ if for every $p \in P$ and for every sequence of dense classes $\langle D_i \mid i \in I \rangle \in \mathcal{C}$ with $I \in M$ there is $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I \rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below q.

Pretameness allows us to pass from dense classes to predense sets by strengthening a given condition.

Pretameness

Theorem (S. Friedman)

Let $\mathbb{M}$ be a model of GB^- such that either $M \models$ the power set axiom, or $\mathcal{C}$ contains a set-like well-order. Then the following statements hold for every class-sized partial order $\mathbb{P}$:

1. *If $\mathbb{P}$ is pretame then $\mathbb{P}$ satisfies the forcing theorem.*
2. *If $\mathbb{P}$ is pretame iff $\mathbb{M}[G] \models \mathrm{GB}^-$ for every $\mathbb{P}$-generic filter G.*

Pretameness and the forcing theorem

Notation

Let Ψ be some property of partial orders. We say that a partial order $\mathbb{P}$ satisfies Ψ **densely**, if every partial order $\mathbb{Q}$ such that there is a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$ satisfies Ψ.

We have seen that the forcing theorem may fail for class forcing. On the other hand, there are non-pretame partial orders such as $\mathrm{Col}(\omega, \mathrm{Ord})$ which do satisfy the forcing theorem.

Theorem (Holy, K., Schlicht)

Suppose that $\mathbb{M} \models \mathsf{GB}^-$ and $\mathcal{C}$ contains a set-like well-order but no first-order truth predicate. Then a class-sized partial order $\mathbb{P}$ is pretame if and only if it densely satisfies the forcing theorem.

The main theorem

Pretameness of a class-sized partial order $\mathbb{P}$ is - under sufficient conditions on the ground model $\mathbb{M}$ - equivalent to each of the following properties:

- $\mathbb{P}$ preserves the axioms of GB^-.
- $\mathbb{P}$ preserves Separation/Replacement/Collection.
- $\mathbb{P}$ does not add a cofinal/surjective function from some ordinal κ into Ord^M.
- $\mathbb{P}$ satisfies the EMP.
- $\mathbb{P}$ densely satisfies the forcing theorem.
- $\mathbb{P}$ is densely nice.
- $\mathbb{P}$ densely has a Boolean completion.

All properties above always hold for set-sized partial orders; this suggests that pretame forcings are the "right" class of class forcings to consider.

Characterizations of the Ord-cc

A class-sized partial order $\mathbb{P}$ is said to satisfy the Ord-**cc**, if all its antichains are elements of M.

We can strenghten many previously considered properties and obtain characterizations of the Ord-cc. The following conditions are - under sufficient conditions on the ground model $\mathbb{M}$ - equivalent to $\mathbb{P}$ satisfying the Ord-cc:

- $\mathbb{P}$ satisfies the strong extension maximality principle.
- $\mathbb{P}$ satisfies the maximality principle.
- $\mathbb{P}$ is densely very nice.
- $\mathbb{P}$ has a unique Boolean completion.
- $\mathbb{P}$ has a Boolean completion $\mathbb{B}$ such that every subclass of $\mathbb{B}$ which is in $\mathcal{C}$ has a supremum in $\mathbb{B}$.

Other properties which imply the forcing theorem

We have found other properties than pretameness which imply the forcing theorem.

Definition

We say that a partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ has the **set decision property**, if for every $p \in P$ and every set $A \subseteq P$ in M, there is an extension $q \leq_{\mathbb{P}} p$ of p such that for every $a \in A$, either $q \leq_{\mathbb{P}} a$ or $q \perp_{\mathbb{P}} a$.

Theorem (Holy, K., Schlicht)

1. *$\mathbb{P}$ has the set decision property iff it doesn't add any new sets.*
2. *If $\mathbb{P}$ has the set decision property, then it satisfies the forcing theorem.*

Thank you for your attention!

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Second-order arithmetic and ZFC

Theorem

SOA *and* $\mathsf{ZFC}^- + \mathsf{V} = \mathsf{HC}$ *bi-interpret each other.*

By using the $\mathbf{\Pi}^1_1$-PSP to prove that $\mathsf{L} \models \mathsf{ZFC}$, and using a class version of the Lévy collapse one can prove

Theorem (Koepke-Moellerfeld)

$\mathsf{SOA} + \mathbf{\Pi}^1_1$-PSP *and* ZFC *are equiconsistent.*

Let $\mathsf{ZFC}^\#$ denote the theory ZFC+ "every set of ordinals has a sharp".

Theorem (K.)

$\mathsf{SOA} + \mathbf{\Pi}^1_1\text{-Det} + \mathbf{\Pi}^1_2$-PSP *and* $\mathsf{ZFC}^\#$ *are equiconsistent.*

Class forcing over SOA

Pretame class forcing over SOA corresponds to class forcing with pretame partial orders which satisfy the forcing theorem over models of ZFC^-.

All "common" tree forcings

- have a pre-Boolean completion
- satisfy the forcing theorem
- are pretame.

This additionally uses **dependent choice** DC.

In particular, the tree forcings preserve SOA.

Preservation of the perfect set property

Theorem (Castiblanco-Schlicht)

All "common" tree forcings such as Sacks forcing or Mathias forcing preserve the $\mathbf{\Pi}^1_1$-PSP *over models of* $\mathsf{SOA} + \mathsf{DC}$.

On the other hand, using almost disjoint coding and reshaping, we obtain the following:

Theorem (K.)

There is a forcing notion $\mathbb{P}$ *such that in every* $\mathbb{P}$*-generic extension* $M[G]$ *there is a real* x *with* $M[G] = L[x]$*. In particular, the* $\Pi^1_1[x]$-PSP *fails in* $M[G]$.