Open and clopen determinacy for proper class games

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Games in set theory

The past half-century of set theory has revealed a robust connection between infinitary game theory and fundamental set-theoretic principles.

Let me exhibit another such connection, in context of proper class games, such as games played on the ordinals.

It will turn out that clopen determinacy for class games is related to the principle of elementary transfinite recursion ETR, which transcends ZFC in consistency strength.

Proper class games

Consider two-player games of perfect information on a class X, such as $X = \mathrm{Ord}$.

Player I	α_0		α_2		α_4		$\cdots$
Player II		α_1		α_3		α_5		$\cdots$

Player I wins if the resulting play $\vec{\alpha}$ is in fixed payoff set $A \subseteq X^\omega$.

The usual notions of open game, strategy, winning strategy are all expressible for class games in Gödel-Bernays GBC set theory.

Open games

A game is open for a player, if every winning play is won at some finite stage.

Thus, open games generalize the finite games.

Clopen games are open for each player—all plays of the game end in finitely many moves.

Theorem (Gale-Stewart 1953)

Every open game is determined.

Question

Does open determinacy hold for class games?

Two classical proofs of open determinacy

Let's review two proofs that every open game is determined.

First proof

Let W be the set of positions from which the open player has a winning strategy. If this includes the initial position, then open has a winning strategy. Otherwise, the closed player can play so as to remain outside W. This is a winning strategy, since play will never reach a position where open has already won. Therefore, the game is determined. □

A second proof of open determinacy

Second proof

Ordinal game values. Already-won positions have value 0. Position p has value $\alpha + 1$, if the open player can play to a position with value α, minimal. For opposing player, take supremums. Fundamental observation: from any valued position, open player can decrease value, but opponent cannot increase it; from any unvalued position, closed player can preserve this. So the game is determined. □

Generalizing to proper class games

Question

Do the open determinacy proofs extend to proper class games?

Unfortunately, those arguments cannot be readily carried out in ZFC or even GBC for class games.

The won-position proof suffers because the class W of won-positions for open is not first-order definable, but Σ^1_1-definable. We do not have second-order separation in ZFC or GBC.

The game values proof founders when the values exceed Ord.

Open determinacy for class games

Question

Can we prove open determinacy for class games?

Does every definable open class game in ZFC admit a definable winning strategy for one of the players? In GBC, must every open class game have a winning strategy?

We shall prove that the answers to both questions is no.

Main Theorems

1. In ZFC, there is a first-order definable clopen proper-class game with no definable winning strategy for either player.
2. In GBC, the game in (1) is determined iff $\exists$ satisfaction class, a truth predicate for first-order truth.
3. Consequently, clopen determinacy for class games implies $\mathrm{Con}(\mathrm{ZFC})$ and iterated $\mathrm{Con}^\alpha(\mathrm{ZFC})$ and more.
4. Clopen determinacy for class games is equivalent over GBC to the principle ETR of elementary transfinite recursion, strictly weaker than $\mathrm{GBC} + \Pi^1_1$-comprehension, strictly weaker than Kelley-Morse KM set theory.
5. Open determinacy for class games is provable in $\mathrm{GBC} + \Pi^1_1$-comprehension.

Natural analogue with second-order arithmetic

The main result (clopen determinacy iff ETR) is a second-order set theory analogue of Steel's 1977 result in second-order arithmetic: clopen determinacy for games on $\omega \leftrightarrow \text{ATR}_0$.

But one should not expect tight analogy between Σ^0_n-determinacy in arithmetic and Σ^0_n-determinacy for classes. For example, ZF refutes Δ^0_2-determinacy in Lévy hierarchy.

Instead, connection is with Borel hierarchy of Ord^ω. Important disanalogies:

- Ord^ω not separable, but Baire space ω^ω is separable;
- well-foundedness for class relations is first-order, but Π^1_1-complete in arithmetic;
- game plays on Ord are first-order objects, so payoff is a class; but in arithmetic, payoff is third-order.

Failure of open determinacy for definable classes

Theorem

In ZFC, there is a definable clopen proper-class game with no definable winning strategy for either player.

This is a uniform counterexample to clopen determinacy for definable class games and strategies.

This is a theorem scheme of ZFC: no particular definition and parameter defines a winning strategy in the game.

Clopen determinacy and truth

A stronger result:

Theorem

There is a definable clopen proper-class game, whose determinacy is equivalent in GBC *to the existence of a satisfaction class for first-order set-theoretic truth.*

Consequently, in GBC the principle of clopen determinacy for class games implies $\mathrm{Con}(\mathrm{ZFC})$, as well as iterated consistency assertions $\mathrm{Con}^{\alpha}(\mathrm{ZFC})$ and much more.

The game is: the truth-telling game.

The truth-telling game

Two players, in a court of law

- The *truth-teller*, in the witness box, answering questions
- The *interrogator*, posing the tricky questions

On each turn, interrogator asks: $\varphi(\vec{a})$?

Truth-teller answers: *true* or *false*.

Existential proviso: if $\exists x\, \varphi(x, \vec{a})$ is declared true, then truth-teller must also provide witness $\varphi(b, \vec{a})$.

A play of the game consists of a sequence of inquiries and truth pronouncements.

Winning conditions for the truth-telling game

The truth-teller need not necessarily answer truthfully to win!

Rather, the truth-teller wins, if she does not violate the recursive Tarskian truth conditions.

- Atomic truth assertions must be truthful
- Truth assertions must respect Boolean connectives
- Truth assertions must respect quantifiers.

This is an open game for the interrogator, since any violation will occur at a finite stage.

Satisfaction classes

Definition

A *satisfaction class* or *truth predicate* for first-order truth is a class Tr of pairs $\langle\varphi, \vec{a}\rangle$, which obeys the Tarskian recursive definition of truth.

Theorem (Tarski)

In any sufficient theory, no definable class is a truth predicate for first-order truth.

Meanwhile, Kelley-Morse set theory KM proves that there is a truth predicate for first-order truth.

Indeed, this is provable in GBC + ETR, since the Tarskian recursion is itself an elementary recursion of height ω, defining the truth of $\varphi(\vec{a})$ in terms of $\psi(\vec{b})$ for simpler formulas ψ.

Truth-teller wins $\leftrightarrow$ truth predicate

Lemma

The truth-teller has a winning strategy in the truth-telling game if and only if there is a satisfaction class for first-order truth.

Proof.

($\leftarrow$) If there is a satisfaction class for first-order truth, then truth-teller can win by playing in accordance with it. Use global well-order (GBC) to pick witnesses.

($\rightarrow$) Suppose that the truth-teller has a winning strategy τ in the truth-telling game. I claim that the truth pronouncements made by τ are independent of the play in which they occur. Prove by induction on formulas. This provides a truth predicate. □

By Tarski's non-definability of truth, there is no definable winning strategy for the truth-teller.

Interrogator cannot win

Lemma

The interrogator has no winning strategy in the truth-telling game.

Proof.

Consider any strategy σ for interrogator, directing him to issue certain challenges $\varphi(\vec{a})$. By reflection, there is θ with V_θ closed under σ: if all challenges and witnesses come from V_θ, then σ replies in V_θ. Let truth-teller answer with theory of $\langle V_\theta, \in \rangle$. This will survive against σ, and so σ is not winning for interrogator. □

Open determinacy $\to$ satisfaction class

Conclusion

If open determinacy holds for class games, then there is a satisfaction class for first-order truth.

If open determinacy holds, then the truth-telling game is determined. Interrogator cannot have a winning strategy. So truth-teller has a winning strategy, and from this we can build a satisfaction class.

But: what about *clopen* determinacy?

The counting-down truth-telling game

Modify the truth-telling game: interrogator counts down in ordinals during play. Truth-teller wins when count hits zero.

This is now a clopen game.

Interrogator still cannot win, since the game got harder.

If truth-teller has winning strategy, then the truth assertions made by it are independent of the play, provided that the remaining count is high enough. Proved by induction on formulas.

Conclusion

Clopen determinacy for class games implies there is a satisfaction class.

Satisfaction class is strong

In GBC, the existence of a satisfaction class Tr has nontrivial consistency strength.

By reflection theorem, find a class club of θ with $\langle V_\theta, \in, \mathrm{Tr} \restriction V_\theta \rangle \prec_{\Sigma_1} \langle V, \in, \mathrm{Tr} \rangle$.

This implies $V_\theta \prec V$ and get an elementary chain

$$V_{\theta_0} \prec V_{\theta_1} \prec \cdots \prec V_\lambda \prec \cdots \prec V.$$

This implies $\mathrm{Con}(\mathrm{ZFC})$ and $\mathrm{Con}^\alpha(\mathrm{ZFC})$ and more (but subtle point).

Conclusion

The consistency strength of GBC + clopen determinacy is strictly stronger than GBC.

Class parameters

One may easily relativize the truth-telling games to allow any fixed class parameter.

In particular, we may get iterated truth predicates *truth-about-truth* and *truth-about-truth-about-truth* and so on.

One may iterate this much further, to get iterated truth predicates Tr_β for any ordinal β, concerning truth in the structure $\langle V, \in, \mathrm{Tr}_\alpha \rangle_{\alpha<\beta}$.

More uniformly, can have a single binary predicate $\mathrm{Tr} \subseteq \mathrm{Ord} \times V$, whose slices Tr_α are truth predicates over $\mathrm{Tr} \restriction \alpha \times V$.

One might hope to iterate further than Ord.

Clopen determinacy → global choice

Theorem (Folklore)

In Gödel-Bernays set theory GB*, the principle of clopen determinacy implies the global axiom of choice.*

Proof.

Consider the game where player I plays a nonempty set b an player II plays a set a, with player II winning if $a \in b$. This is a clopen game, since it is over after one move for each player. Clearly, player I can have no winning strategy, since if b is nonempty, then player II can win by playing any element $a \in b$. But a winning strategy for player II amounts exactly to a global choice function, selecting uniformly from each nonempty set an element. □

Amusing fact: ZF refutes universal axiom of determinacy

Theorem

There is a definable non-determined game of complexity Δ^0_2 in the Lévy hierarchy.

Proof.

Player I plays a nonempty set $A \subseteq \omega^\omega$; player II plays an element $a \in A$; after this, player I plays a non-determined set $B \subseteq \omega^\omega$, and then play proceeds as in that game.

In ZF, neither player has a winning strategy. Player I cannot win the game determined by B. If player II had a winning strategy, then AC holds for sets of reals, so there is non-determined B, and then player II cannot win that game. □

ZFC does not prove open class determinacy

Since ZFC does not prove global choice, it follows that there is a model of ZFC with a definable clopen game having no definable winning strategy.

Meanwhile, the truth-telling game shows a stronger uniform result: every model of ZFC, including those with global choice, have a definable clopen game with no definable winning strategy.

Furthermore, the definition of the game is uniform.

ETR: Elementary transfinite recursion

I'd like next to prove a stronger result, which explains why clopen determinacy for class games is not provable in ZFC.

Specifically, clopen determinacy is equivalent over GBC to the principle of elementary transfinite recursion ETR.

Definition

The principle of *elementary transfinite recursion* ETR is the assertion that every first-order recursive definition along any well-founded binary class relation has a solution.

Class recursion

Consider a well-founded binary class relation $\lhd$ on a class I. (Note: well-foundedness is first-order.)

Any formula $\varphi(x, b, F, Z)$ determines a recursion along $\lhd$.

Namely, a *solution* of the recursion is class $F \subseteq I \times V$ such that for every $b \in I$, the b^{th} slice of the solution $F_b = \{ x \mid \varphi(x, b, F \upharpoonright b) \}$ is defined by the recursive rule φ.

Each slice F_b is determined via φ by the earlier slices F_c for $c \lhd b$.

ETR asserts: every such recursion has a solution.

It doesn't matter whether you use well-orders, well-founded relations, trees, etc.

ETR implies truth predicate

Central example: definition of satisfaction

We define the truth of $\varphi(\vec{a})$ recursively in terms of simpler formulas, and the recursion is well-founded with height ω. The solution of the recursion is a satisfaction class.

Thus, ETR strictly exceeds ZFC and GBC in consistency strength.

Meanwhile, it is easy to prove ETR in Kelley-Morse set theory.

ETR ↔ iterated truth

In fact, one can get very long iterated truth predicates.

If $\langle I, \lhd\rangle$ is a class well-order, then T is an *iterated truth predicate* over $\langle I, \lhd\rangle$, if every slice T_i is a truth predicate for the structure $\langle V, \in, Z, T \upharpoonright i\rangle$.

Theorem

The principle ETR of elementary transfinite recursion is equivalent over GBC *to the assertion that for every class parameter Z and every class well-ordering $\langle I, \lhd\rangle$ there is an iterated truth predicate T along $\langle I, \lhd\rangle$ over the parameter Z.*

ETR $\leftrightarrow$ iterated truth

($\rightarrow$) Easy, because the truth predicate itself is defined by recursion.

($\leftarrow$) Conversely, from an iterated truth predicate, one can extract a solution of any fixed recursion φ. (It is a little subtler than one might expect, uses Carnap fixed-point lemma.) $\square$

Clopen determinacy ↔ ETR ↔ iterated truth

Theorem

In Gödel-Bernays set theory GBC, *the following are equivalent.*

1. *Clopen determinacy for class games.*
2. *The principle ETR: every well-founded class recursion has a solution.*
3. *Every class well-order* $\langle I, \lhd \rangle$ *admits an iterated truth predicate.*

We've already proved (2) $\leftrightarrow$ (3).

ETR implies clopen determinacy

Assume ETR and consider the game tree of any clopen game. This is well-founded, because the game is clopen. Consider the back-propagation labeling of positions in the game tree, recursively labeled with the winner from that position. By ETR, there is such a labeling.

Whichever player gets their label on the initial position has a winning strategy: stay on positions with their label.

Clopen determinacy implies ETR

Assume clopen determinacy for class games.

To prove ETR, suppose we are faced with a recursion $\varphi(x, b, F)$ along a well-founded class partial-order $\lhd$ on I.

To find a solution of the recursion, we shall use the *recursion game*. A winning strategy for that game will give us a solution of the recursion.

The recursion game

Like the truth-telling game, with the interrogator and the truth-teller, but we now have a predicate symbol for the solution of the recursion.

The truth-teller answers questions about truth in $\langle V, \in, \lhd, F\rangle$.

Truth-teller must obey the Tarski recursion for truth, must provide explicit witnesses for existentials, and must agree that F obeys the desired recursion.

The game is open for the interrogator.

The interrogator cannot win

Lemma

The interrogator has no winning strategy in the recursion game.

Proof.

Suppose σ is a strategy for the interrogator. By reflection, find V_θ closed under σ, and we may assume $\langle V_\theta, \in, \lhd \cap V_\theta, \sigma \cap V_\theta \rangle \prec \langle V, \in, \lhd, \sigma \rangle$. Consider $\lhd \cap V_\theta$, still well-founded. This set-sized recursion has (unique) solution $f \subseteq (I \cap V_\theta) \times V_\theta$, where i^{th} slice $f_i = \{ x \in V_\theta \mid \langle V_\theta, \in, \lhd \cap V_\theta, f \rangle \models \varphi(x, i, f \restriction i) \}$ obeys φ. If truth-teller plays in accordance with $\langle V_\theta, \in, \lhd \cap V_\theta, f \rangle$, then she will defeat σ. □

Truth-teller wins $\leftrightarrow$ recursion solution

Lemma

The truth-teller has a winning strategy in the recursion game if and only if there is a solution of the recursion.

($\leftarrow$) If F is a solution of the recursion, then the truth-teller can play according to truth in $\langle V, \in, F\rangle$. (Use clopen determinacy to get a truth predicate for that structure.)

($\rightarrow$) Suppose that τ is a winning strategy for the truth-teller in the recursion game. The truth pronouncements made by τ about $\langle V, \in, \lhd\rangle$ do not depend on the play, and provide a truth predicate. By induction on i and formulas, the truth pronouncements made by τ about truth in $\langle V, \in, \lhd, F \upharpoonright i\rangle$ are independent of the play. They provide a truth predicate for this class and a solution of the recursion. $\square$

The counting down recursion game

The recursion game was open, but we assumed only clopen determinacy.

Modify the game to require the interrogator to count down in the natural numbers. Truth-teller wins when he gets to zero.

This is a clopen game. Interrogator still cannot win, and a win for the truth-teller yields a solution to the recursion.

So clopen determinacy $\leftrightarrow$ ETR. $\square$

Proving open determinacy

In second-order arithmetic, Steel proved that open determinacy and clopen determinacy are equivalent to each other, and to ATR_0.

But in second-order set theory, the proof of clopen determinacy from ETR does not generalize to open determinacy.

Meanwhile, open determinacy for class games is provable in stronger theories.

Theorem

Open determinacy for class games is provable in $\mathrm{GBC} + \Pi^1_1$*-comprehension.*

Follow a classical proof of open determinacy: let W be the positions from which open has a winning strategy. Use Π^1_1-comprehension to form this class. If open has no winning strategy, then the closed player can play so as to remain outside W.

A subtle issue in the proof: uses a class-choice principle to unify strategies. But then, argument shows how to eliminate this use by building the contructible universe beyond Ord and finding canonical strategies.

Separating open from clopen determinacy

Sherwood Hachtman has answered several of our questions.

Theorem (Hachtman)

If there is a transitive model of $\mathrm{ZF}^- + \kappa$ is inaccessible, then there is a model of GBC, in which clopen determinacy for class games holds, but open determinacy fails.

The Borel class games obtained by σ-algebra generated by open classes $B \subseteq \mathrm{Ord}^\omega$. (Note: not the same as Δ^1_1.)

Theorem (Hachtman)

Kelley-Morse set theory, if consistent, does not prove the determinacy of Borel class games.

New work on class forcing

Current joint work with JDH, Victoria Gitman, Kameryn Williams, Philipp Schlicht and Peter Holy.

Theorem

The forcing theorem for all class forcing notions is equivalent over GBC *to* $\mathrm{ETR}_{\mathrm{Ord}}$.

The forcing theorem for $\mathbb{P}$ asserts that there is a forcing relation $\Vdash_{\mathbb{P}}$, such that forced statements are true in the extension and all true statements there are forced.

Thank you.

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