# THE CHOICELESS CARDINALS ARE INCONSISTENT

RUPERT M^c^CALLUM

## 1. The Kunen inconsistency is provable in ZF

The Kunen inconsistency is the result that the existence of a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ for some ordinal $\lambda$ is not consistent with $\mathsf{ZFC}$. We wish to show that it is not consistent with $\mathsf{ZF}$ either. By an unpublished result of Gabriel Goldberg, using Woodin's iterated collapse forcing, we can show that the theory $\mathsf{ZF}$ together with the existence of a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ is equiconstent with the same theory together with the assumption that $V_\lambda$ can be well-ordered. We present a proof that this extension of $\mathsf{ZF}$ is inconsistent.

**Definition 1.1.** We say that a cardinal $\kappa$ has reflection property R if the following holds. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa$, and if we define $\lambda := \sup\{\kappa_i : i \in \omega\}$, then there exists an $\alpha < \kappa$ such that, for any $X \in V_{\lambda+2}$ there exists an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X) \prec (V_{\lambda+1}, X')$ with critical point $\alpha$, such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$.

**Lemma 1.2.** Assume $\mathsf{ZF}$, together with the assertion that there exists a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, where $\lambda$ is the first fixed point of $j$ above the critical point of $j$, and assume also that $V_\lambda$ can be well-ordered. Denote by $\kappa$ the critical point of the elementary embedding $j$. Then it is provable on the stated assumptions that $V_\kappa$ is a model for the existence of a cardinal $\kappa'$ with reflection property R.

*Proof.* If we let $j$ be the elementary embedding originally assumed to exist for $\kappa$ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness that $\kappa$ has reflection property R. For each $X \in V_{\lambda+2}$, let $j'$ be an embedding witnessing reflection property R for that particular $X$ with critical point $\alpha$, chosen in such a way that $j'$ and $\alpha$ are independent of $X$, and $j'(\alpha) = \kappa_0 := \kappa$ and $j'(\kappa_n) = \kappa_{n+1}$ for all $n \in \omega$ where $\kappa_n := j^n(\kappa)$. The existence of the elementary embedding $j$ means that we can find a $\kappa'$ such that $\alpha < \kappa' < \kappa$ such that the stated reflection property holds for the sequence $\langle \kappa', \kappa_0, \kappa_1, \ldots \rangle$, $\kappa_0$ being reflected to $\kappa'$ and each $\kappa_n$ for $n > 0$ being reflected to $\kappa_{n-1}$. Let $j_0$ be an elementary embedding with domain $V_{\lambda+1}$ witnessing this instance of reflection, that is $j_0$ has critical point $\kappa'_0 := \kappa'$ and $j(\kappa'_0) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$. Also $j_0$ can be chosen to be an elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$. This can be achieved by reflecting a formula with a truth set for $V_{\lambda+1}$ as parameter which states all the Tarski axioms. In addition, reflect the formula "For all $X \in V_{\lambda+2}$, $j' \mid V_\lambda$ induces on $V_{\lambda+1}$ an elementary embedding $(V_{\lambda+1}, X) \to (V_{\lambda+1}, X')$ with $j'(\alpha) = \kappa_0$ and $j'(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$" with parameter $j' \mid V_\lambda$ and a definable subclass of $V_{\lambda+2}$ which codes for a truth set for the structures $(V_{\lambda+1}, X)$ and $(V_{\lambda+1}, X')$ for each $X$. This guarantees that $j' \mid V_{\kappa_i}$ is in the range of the embedding $j_0$ for all $i \in \omega$. It can be shown that the value of $\kappa'$ can be chosen to be independent of $X$.

Suppose we have built up a sequence of embeddings $\langle j_i : 0 \leq i \leq n \rangle$, so that each $j_i$ is an embedding $V_{\lambda+1} \to V_{\lambda+1}$ with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_i, \kappa_0, \kappa_1, \ldots \rangle$, and for each $i$ such that $0 \leq i < n$ we have that $j_{i+1}$ agrees with $j_i$ on $V_{\kappa'_i}$. Define $A_n := j_n \mid V_{\kappa'_n}$. Then reflect the formula "There is an embedding $k : V_{\lambda+1} \to V_{\lambda+1}$ with critical point $\kappa'_0$ and critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_n, \kappa_0, \kappa_1, \ldots \rangle$ such that $k \mid V_{\kappa'_n}$ agrees with $A_n$", in such a way that the critical sequence $\langle \kappa_n : n \in \omega \rangle$ is reflected to $\langle \kappa'_{n+1}, \kappa_0, \kappa_1, \ldots \rangle$ via an elementary embedding $k_{n+1} : V_{\lambda+1} \to V_{\lambda+1}$ with critical point $\kappa'_{n+1}$ such that $k_{n+1}(\kappa'_{n+1}) = \kappa_0$ and $k_{n+1}(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$. In addition, ensure in the same way as before that $k_{n+1}(j' \mid V_{\kappa'_{n+1}}) = j' \mid V_{\kappa_0}$ and $k_{n+1}(j' \mid V_{\kappa_i}) = j' \mid V_{\kappa_{i+1}}$ for all $i \in \omega$. To ensure that $k_{n+1}$ can indeed be chosen to be an elementary embedding as claimed, we need to reflect not just the previously stated formulas but also a formula with a truth set for $V_{\lambda+1}$ as parameter stating all the Tarski axioms. In this way we obtain a new embedding $j_{n+1} : V_{\lambda+1} \to V_{\lambda+1}$ with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_{n+1}, \kappa_0, \kappa_1, \ldots \rangle$ which agrees with $j_n$ on $V_{\kappa'_n}$. Note that $k_{n+1}(j_{n+1} \mid V_\lambda) = j_n \mid V_\lambda$. Note that the claim of the existence of the sequences $\langle j_n : n \in \omega \rangle$ and $\langle k_n : n \in \omega \rangle$ is justified by the hypothesis that $V_\lambda$ is well-orderable. And again, it can be shown that each $\kappa'_i$ can be made independent of $X$.

Thus for each $n > 0$ we can find an elementary embedding with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_n, \kappa_0, \kappa_1, \ldots \rangle$. Then we can reflect the entire sequence $\langle \kappa_n : n \in \omega \rangle$ to a sequence $\langle \kappa'_n : n \in \omega \rangle$, with $\kappa'_n < \kappa$ for all $n$, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for $\kappa'_0$. First of all let us describe how we use the sequence of embeddings $\langle j_n : n \in \omega \rangle$ and the embedding $j'$ to construct a mapping $j'' : V_{\lambda'+1} \to V_{\lambda'+1}$ with critical point $\alpha$ and

$j''(\alpha) = \kappa'_0$ and $j''(\kappa'_i) = \kappa'_{i+1}$ for each $i \in \omega$, the plan being to show that this mapping is also an elementary embedding.

Let us denote by $U_n$ the range of the embedding $(k_0 \circ k_1 \circ k_2 \circ \ldots \circ k_n) \mid V_{\kappa'_n}$. Then, for $A \in V_{\kappa_n}$, we define $A^{U_n} := A \cap U_n$ for $A \in V_{\kappa_0+1}$ and $A^{U_n} := \{B^{U_n} : B \in A\}$ for $A \notin V_{\kappa_0+1}$. The map from $A$ to $A^{U_n}$ is a kind of transitive collapsing map. It can be thought of as indicating to which object $A$ is reflected. $V_{\kappa_i}$ will be reflected to $V_{\kappa'_i}$ for all integers $i$ such that $0 \leq i < n$. In the case where $A \in U_n$ we have $A^{U_n} = (k'_n)^{-1}(A)$ where $k'_n := k_0 \circ k_1 \circ \ldots \circ k_n$. We obtain the embedding $j''$ by gluing together $(j' \mid V_{\kappa_{n-1}})^{U_n}$ for all $n \in \omega \setminus \{0\}$. Note that for each $n > 0$, $(j' \mid V_{\kappa_{n-1}})^{U_n}$ is a mapping with domain $V_{\kappa'_{n-1}}$, and from the fact that $U_n \subseteq U_{n+1}$ for all $n \in \omega$ it follows that it is possible to glue all these maps together to obtain a mapping with domain $V_{\lambda'}$ where $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We shall eventually show that this map is an elementary embedding $V_{\lambda'} \to V_{\lambda'}$ with the usual unique extension to a mapping $V_{\lambda'+1} \to V_{\lambda'+1}$ which we will show is also an elementary embedding.

So the mapping $j''$ is made by gluing together $(j' \mid V_{\kappa_{n-1}})^{U_n}$, for all $n \in \omega \setminus \{0\}$. In order to argue the point that the canonical extension of $j''$ to $V_{\lambda'+1}$ is an elementary embedding, we need to describe an elementary embedding $e : V_{\lambda'+1} \to V_{\lambda+1}$.

Suppose we have a $Y' \subseteq V_{\lambda'}$. To obtain $e(Y')$, glue together $k_0(Y' \cap V_{\kappa'_0})$, $k_0(k_1(Y' \cap V_{\kappa'_1}))$, and so on. Suppose that we have some formula $\phi$ in the first-order language of set theory such that $\phi^{V_{\lambda'+1}}(Y'_1, Y'_2, \ldots Y'_k)$ holds, witih $Y'_i \in V_{\lambda'+1}$ for all $i$ such that $1 \leq i \leq k$. In the case where $\phi$ is $\Sigma_1$ we clearly have elementarity of the mapping $e$ in the sense that $\phi^{V_{\lambda+1}}(e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds. We need to show that elementarity also holds in the case where $\phi$ is $\Pi_1$. If $\phi$ is of the form $\forall X \psi$ where $\psi$ is $\Sigma_0$, then clearly $\psi(Y, e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds whenever $Y$ is in the range of $e$. Now consider the case of an arbitrary $Y$. All of the $k_n$ are elementary embeddings $V_{\lambda+1} \to V_{\lambda+1}$. If we let $W_n$ be the range of the embedding $k_0 \circ k_1 \circ \ldots \circ k_n$, and $U$ be the range of $e$, then $U = \bigcap_{n \in \omega} W_n$, and each element of $U$ can be expressed as a limit of a sequence whose $n$-th term is an element of $W_n$ in a natural way. We have $\forall Y \psi(Y, e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds if and only if $\forall Y \in W_n \psi(Y, e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds for all $n$, so it follows that $\psi(Y, e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds if and only if $\forall Y \in U \psi(Y, e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds. Then an induction argument, with similar reasoning for the induction step, generalises this to the case of a $\Sigma_k$ or $\Pi_k$ formula for all positive integers $k$. In this way we establish that $e$ is indeed an elementary embedding.

Further, given the way that the embeddings $k_i$ for $i \in \omega$ were constructed in such a way that $j' \mid V_{\kappa_n}$ is always in the range of the embedding $k_0 \circ k_1 \circ \ldots \circ k_n$, it follows that $j'$ maps the range of $e$ into the range of $e$ and the complement of the range of $e$ into the complement of the range of $e$. It also follows from this that if $j'$ is elementary from $(V_{\lambda+1}, X)$ into $(V_{\lambda+1}, X')$, then $j'$ is elementary from $(V_{\lambda+1}, e"(X^U))$ into $(V_{\lambda+1}, (j')"(e"(X^U))) = (V_{\lambda+1}, e"((X')^U))$.

It can also be shown that $e$ is an elementary embedding $(V_{\lambda'+1}, X^U) \to (V_{\lambda+1}, e"(X^U))$, and that a similar result is also true for the set $X'$. This can be proved using the constraints that were given on the construction of the embeddings $k_i$ together, by means of the same argument that was used to prove that $e$ is an elementary embedding $V_{\lambda'+1} \to V_{\lambda+1}$.

Now, having shown that $e$ is an elementary embedding and realising that $j''$ is just $e^{-1} \circ j' \circ e$, we obtain the result that $j''$ is an elementary embedding $V_{\lambda'+1} \to V_{\lambda'+1}$ as claimed.

Define $X'' \in V_{\lambda'+2}$ to be $X^U$ where $U$ is the range of the embedding $e$, and $X^U$ is defined as before, and note that any $X'' \in V_{\lambda'+2}$ can arise in this way. We wish to find an $X''' \in V_{\lambda'+2}$ for which $j'' : (V_{\lambda'+1}, X'') \prec (V_{\lambda'+1}, X''')$ will hold. Now we have $X \in V_{\lambda+2}$, then let $X''' := (X')^U$ where $X'$ is such that $j'$ is elementary from $(V_{\lambda+1}, X)$ into $(V_{\lambda+1}, X')$. This is the desired $X'''$ with the property which we seek. To see this note that $j'$ is elementary from $(V_{\lambda+1}, e"(X^U))$ into $(V_{\lambda+1}, e"((X')^U))$, as follows from the construction of the embeddings $k_n$ from which the embedding $e$ was constructed, and it follows from our construction of the embedding $e$ that the mapping $e$ acting from $(V_{\lambda'+1}, X^U)$ into $(V_{\lambda+1}, e"(X^U))$, and from $(V_{\lambda'+1}, (X')^U)$ into $(V_{\lambda+1}, e"((X')^U)$, is elementary in both cases. So, combining all these claims, we obtain the desired result that $j'$ is elementary from $(V_{\lambda'+1}, X'')$ into $(V_{\lambda'+1}, X''')$, as claimed.

Thus $V_\kappa$ is a model for the existence of a cardinal $\kappa'$ satisfying the stated reflection principle. $\square$

**Lemma 1.3.** The existence of a cardinal with the stated reflection property is still inconsistent with choice.

*Proof.* The proof of this claim essentially the same as the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience.

Assume ZFC and let $g \in V_{\lambda+2}$ be an $\omega$-Jonsson function over $\lambda$. Then let $g'$ be such that there is an elementary embedding $j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g')$ with critical point $\alpha < \kappa_0$ and $j(\alpha) = \kappa_0$ and $j(\kappa_n) = \kappa_{n+1}$. We have $g'(x) = \alpha$ for some $x \in [j``\lambda]^\omega$. Let $\beta := g(j^{-1}(x))$. We get $j(\beta) = \alpha$ which contradicts $\alpha$ being the critical point of $j$. $\square$

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent, and from this we obtain the conclusion that the Kunen inconsistency is provable in ZF.

## References

[1] Akihiro Kanamori. The Higher Infinite: Large cardinals in set theory from their beginnings, 2nd edition. Springer Monographs in Mathematics, 2003.

[2] M. Victoria Marshall R. Higher order reflection principles, *Journal of Symbolic Logic,* vol. 54, no. 2, 1989, pp. 474–489.