1. Inconsistency of a super-Reinhardt cardinal

We present a proof that the following extension of \( ZF \) is inconsistent. Add a constant symbol \( j \) to the language and add Separation and Replacement axioms for formulas involving \( j \), and add the axiom that \( j \) is an elementary embedding \( V \prec V \) and also \( DC_\lambda \) where \( \lambda \) is the least fixed point of \( j \) above the critical point of \( j \). We wish to claim that this theory is inconsistent, and it follows from this that a super-Reinhardt cardinal is inconsistent with \( ZF \).

**Lemma 1.1.** Work in the theory described above. Then it is provable on the stated assumptions that \( V_\kappa \) is a model for the existence of a cardinal \( \kappa' \) with the following property. There exists a sequence \( \langle \kappa_i : i \in \omega \rangle \) where \( \kappa_0 = \kappa' \), such that if we let \( \lambda := \sup \{ \kappa_i : i \in \omega \} \), then for any \( X \in V_{\lambda+2} \) there exists an \( \alpha < \kappa_0 \) and an \( X' \in V_{\lambda+2} \) such that there is an elementary embedding \( j : (V_{\lambda+1},X) \prec (V_{\lambda+1},X') \) with critical point \( \alpha \) such that \( j(\alpha) = \kappa_0 \) and \( j(\kappa_i) = \kappa_{i+1} \).

**Proof.** If we let \( j \) be the elementary embedding originally assumed to exist for \( \kappa \) itself then \( \langle j^n(\kappa) : n \in \omega \rangle \) and \( \lambda := \sup \{ j^n(\kappa) : n \in \omega \} \) witness the stated reflection property for \( \kappa \), and the existence of the elementary embedding \( j \) means that we can find a \( \kappa' < \kappa \) such that the stated reflection property holds when we let \( \kappa_0 := \kappa' \) and \( \kappa_n := j^{n-1}(\kappa) \) for \( n > 0 \), \( \kappa_0 \) being reflected to \( \kappa' \) and each \( \kappa_n \) for \( n > 0 \) being reflected to \( \kappa_{n-1} \). Let \( j_0 \) be an elementary embedding with domain \( V_{\lambda+2} \) witnessing this instance of reflection. Then after the first reflection, \( \kappa_0 \) can be reflected to some ordinal between \( \kappa' \) and \( \kappa_0 \), and so on. At stage \( n \) for \( n > 0 \), let \( j_n \) be an elementary embedding with domain \( V_{\lambda+2} \) witnessing the reflection. Thus for each \( n > 0 \) we can find a sequence \( \langle \kappa'_i : i \in \omega \rangle \) such that \( \kappa_0 = \kappa' \) and \( \kappa'_i < \kappa \) for integers \( i \) such that \( 0 \leq i \leq n \) and \( \kappa'_i = \kappa_{i-n-1} \) for integers \( i > n \), and the stated reflection property holds for \( \langle \kappa'_i : i \in \omega \rangle \) and \( \lambda \) as before. Then we can reflect the entire sequence \( \langle \kappa_n : n \in \omega \rangle \) to a sequence \( \langle \kappa'_n : n \in \omega \rangle \), with \( \kappa'_n < \kappa \) for all \( n \), and take \( \lambda' := \sup \{ \kappa'_n : n \in \omega \} \). We want to claim that these data witness our reflection principle for \( \kappa'_0 \). We can use the
restrictions of \( j_n \) to construct an embedding \( j' : V_{\lambda+1} \to V_{\lambda+1} \). Consider an \( X' \in V_{\lambda+2} \), we wish to find an \( X'' \in V_{\lambda+2} \) for which \( j' : (V_{\lambda+1}, X') \prec (V_{\lambda+1}, X'') \) will hold. Suppose \( Y' \in X' \subseteq V_{\lambda+1} \), and let \( Y'' := Y' \cap V_{\kappa_0}, Y_n := (j_{n+1}(j | V_\lambda))^{n+1}(Y'') \). We can glue the \( Y_n \) together to obtain a \( Y \in V_{\lambda+1} \) for each \( Y' \in X' \), denote by \( X \) the set of all such \( Y \). We have just defined a mapping \( V_{\lambda+1} \to V_{\lambda+1} \) which we will denote by \( e \). Now we have \( X \in V_{\lambda+2} \), then let \( X'' := (j(X))^U \) where \( U := \{ e(Y') : Y' \in V_{\lambda+1} \} \) and \( A^U := A \cap U \) for \( A \in V_{\eta_0+1} \) and \( A^U := \{ B^U : B \in A \} \) for \( A \in V_{\lambda+1} \setminus V_{\eta_0+1} \). This is the desired \( X'' \) with the property which we seek. Thus \( V_\kappa \) is a model for the existence of a cardinal \( \kappa' \) satisfying the stated reflection principle. \( \square \)

**Lemma 1.2.** The existence of a cardinal with the stated reflection property is still inconsistent with choice.

*Proof.* The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume ZFC and let \( g \in V_{\lambda+2} \) be an \( \omega \)-Jonsson function over \( \lambda \). Then let \( g' \) be such that there is an elementary embedding \( j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g') \) with critical point \( \alpha < \kappa_0 \) and \( j(\alpha) = \kappa_0 \) and \( j(\kappa_n) = \kappa_{n+1} \). We have \( g'(x) = \alpha \) for some \( x \in [\alpha]^{\omega} \). Let \( \beta := g(j^{-1}(x)) \). We get \( j(\beta) = \alpha \) which contradicts \( \alpha \) being the critical point of \( j \).

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent.

2. **Getting the Kunen inconsistency in ZF**

Building on the results of the previous section we now show that it is inconsistent with ZF that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \).

**Theorem 2.1.** The theory ZF together with the assumption that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \) is inconsistent.

*Proof.* Work in ZF and assume that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \), and assume without loss of generality that \( \lambda = \sup\{ j^n(\text{crit}(j)) : n \in \omega \} \). Let \( \Theta \) be the least ordinal such that there is no surjection from \( V_{\lambda+1} \) onto \( \Theta \) and let \( N := (HOD(j | (V_{\lambda+2})^{H_\Theta}))^{H_\Theta} \). Let \( \kappa := \text{crit}(j) \). The model \( N \) satisfies the assertion that for all \( X \in V_{\lambda+2} \) there exists an \( X' \in V_{\lambda+2} \) such that \( j : (V_{\lambda+1}, X) \prec ((V_{\lambda+1})^{(N)}, X') \). Furthermore, \( (V_{\lambda})^{j(N)} \prec (V_{\lambda})^N \). We also have \( (V_{\lambda})^N \) is a model of AC and \( j(j(V_\lambda)) \in (V_{\lambda+1})^{j(N)} \). So \( V_\kappa \) is a model for AC together with the assertion that there exists a \( \kappa' \) and an elementary embedding \( j' \) with critical point \( \kappa' \) such that for all \( X \in V_{\lambda+2} \).
there exists an $X' \in V_{\lambda' + 2}$ such that $j' : (V_{\lambda' + 1}, X) \prec (M_{\lambda' + 1}, X')$ with $j'((j')^\omega \lambda') \in M_{\lambda' + 1}$ and $M_{\lambda'} \prec V_{\lambda}$, where $\lambda' = \sup\{j^n(\kappa') : n \in \omega\}$. Let $g$ be an $\omega$-Jonsson function $[\lambda']^\omega \rightarrow \lambda'$. Then we have $g'$ is an $\omega$-Jonsson function from $([\lambda']^\omega)_{M_{\lambda' + 1}} \rightarrow \lambda'$. We can choose $\langle \kappa_n : n \in \omega \rangle$ such that $g'(\langle j'(\kappa_n) : n \in \omega \rangle) = \kappa'$, but $j' | V_{\lambda'} : V_{\lambda'} \prec V_{\lambda}$, so $j'(\kappa_n) = j(j(\kappa_n))$. Then we have $g(\langle j'(\kappa_n) : n \in \omega \rangle) = \alpha$ where $j'(\alpha) = \kappa'$, but $\kappa'$ is the critical point of $j'$, contradiction. □
REFERENCES
