THE CHOICELESS CARDINALS ARE INCONSISTENT

RUPERT M[c]CALLUM

1. Inconsistency of a super-Reinhardt cardinal

We present a proof that the following extension of ZF is inconsistent. Add a constant symbol $\mathbf{j}$ to the language and add Separation and Replacement axioms for formulas involving $\mathbf{j}$, and add the axiom that j is an elementary embedding $V \prec V$ and also DC_λ where λ is the least fixed point of j above the critical point of j. We wish to claim that this theory is inconsistent, and it follows from this that a super-Reinhardt cardinal is inconsistent with ZF.

Lemma 1.1. Work in the theory described above. Then it is provable on the stated assumptions that V_κ is a model for the existence of a cardinal κ' with the following property. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ where $\kappa_0 = \kappa'$, such that if we let $\lambda := \sup\{\kappa_i : i \in \omega\}$, then for any $X \in V_{\lambda+2}$ there exists an $\alpha < \kappa_0$ and an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X) \prec (V_{\lambda+1}, X')$ with critical point α such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$.

Proof. If we let j be the elementary embedding originally assumed to exist for κ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness the stated reflection property for κ, and the existence of the elementary embedding j means that we can find a $\kappa' < \kappa$ such that the stated reflection property holds when we let $\kappa_0 := \kappa'$ and $\kappa_n := j^{n-1}(\kappa)$ for $n > 0$, κ_0 being reflected to κ' and each κ_n for $n > 0$ being reflected to κ_{n-1}. Let j_0 be an elementary embedding with domain $V_{\lambda+2}$ witnessing this instance of reflection. Then after the first reflection, κ_0 can be reflected to some ordinal between κ' and κ_0, and so on. At stage n for $n > 0$, let j_n be an elementary embedding with domain $V_{\lambda+2}$ witnessing the reflection. Thus for each $n > 0$ we can find a sequence $\langle \kappa'_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa'$ and $\kappa'_i < \kappa$ for integers i such that $0 \leq i \leq n$ and $\kappa'_i = \kappa_{i-n-1}$ for integers $i > n$, and the stated reflection property holds for $\langle \kappa'_i : i \in \omega \rangle$ and λ as before. Then we can reflect the entire sequence $\langle \kappa_n : n \in \omega \rangle$ to a sequence $\langle \kappa'_n : n \in \omega \rangle$, with $\kappa'_n < \kappa$ for all n, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for κ'_0. We can use the restrictions of j_n to V_λ to construct an embedding $j' : V_{\lambda'+1} \to V_{\lambda'+1}$. Consider an $X' \in V_{\lambda'+2}$, we wish to find an $X'' \in V_{\lambda'+2}$ for which $j' : (V_{\lambda'+1}, X') \prec (V_{\lambda'+1}, X'')$ will hold. Suppose $Y' \in X' \subseteq V_{\lambda'+1}$, and let $Y'_n := Y' \cap V_{\kappa'_n}$, $Y_n := (j_n^{-1}(j \mid V_\lambda))^{n+1}(Y'_n)$. We can glue the Y_n together to obtain a $Y \in V_{\lambda+1}$ for each $Y' \in X'$, denote by X the set of all such Y. We have just defined a mapping $V_{\lambda'+1} \to V_{\lambda+1}$ which we will denote by e. Now we have $X \in V_{\lambda+2}$, then let $X'' := (j(X))^U$ where $U := \{e(Y') : Y' \in V_{\lambda'+1}\}$ and $A^U := A \cap U$ for $A \in V_{\kappa_0+1}$ and $A^U := \{B^U : B \in A\}$ for $A \in V_{\lambda+1} \setminus V_{\kappa_0+1}$. This is the desired X'' with the property which we seek. Thus V_κ is a model for the existence of a cardinal κ' satisfying the stated reflection principle. □

Lemma 1.2. The existence of a cardinal with the stated reflection property is still inconsistent with choice.

Proof. The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume ZFC and let $g \in V_{\lambda+2}$ be an ω-Jonsson function over λ. Then let g' be such that there is an elementary embedding $j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g')$ with critical point $\alpha < \kappa_0$ and $j(\alpha) = \kappa_0$ and $j(\kappa_n) = \kappa_{n+1}$. We have $g'(x) = \alpha$ for some $x \in [j``\lambda]^\omega$. Let $\beta := g(j^{-1}(x))$. We get $j(\beta) = \alpha$ which contradicts α being the critical point of j. □

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent.

2. Getting the Kunen inconsistency in ZF

Building on the results of the previous section we now show that it is inconsistent with ZF that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

Theorem 2.1. *The theory ZF together with the assumption that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ is inconsistent.*

Proof. Work in ZF and assume that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, and assume without loss of generality that $\lambda = \sup\{j^n(\mathrm{crit}(j)) : n \in \omega\}$. Let Θ be the least ordinal such that there is no surjection from $V_{\lambda+1}$ onto Θ and let $N := (HOD(j \mid V_\lambda))^{H_\Theta}$. Let $\kappa := \mathrm{crit}(j)$. Since $(V_{\lambda+2}^N$ is closed under j, the model N satisfies the assertion that for all $X \in V_{\lambda+2}$ there exists an $X' \in V_{\lambda+2}$ such that $j : (V_{\lambda+1}, X) \prec ((V_{\lambda+1}^{j(N)}, X')$, and this property of κ in N is rank-reflected as before by Lemma 1.1. Furthermore, $(j \circ j) \mid (V_\lambda)^{j(N)}) = j(j \mid V_\lambda^N)$. We also have $(V_\lambda)^N$ is a model of AC and $j(j``\lambda) \in (V_{\lambda+1})^{j(N)}$. So V_κ is a model for AC together with the assertion that there exists a κ'

and an elementary embedding j' with critical point κ' such that for all $X \in V_{\lambda'+2}$ there exists an $X' \in V_{\lambda'+2}$ such that $j' : (V_{\lambda'+1}, X) \prec (M_{\lambda'+1}, X')$ with $j'((j')"\lambda') \in M_{\lambda'+1}$ and $(j \circ j) \mid M_{\lambda'} = j(j \mid V_{\lambda'})$, where $\lambda' = \sup\{j'^n(\kappa') : n \in \omega\}$. Let g be an ω-Jonsson function $[\lambda']^\omega \to \lambda'$. Then we have g' is an ω-Jonsson function from $([\lambda']^\omega)^{M_{\lambda'+1}} \to \lambda'$. We can choose $\langle \kappa_n : n \in \omega \rangle$ such that $g'(\langle j'(j' \mid V_{\lambda'})(\kappa_n) : n \in \omega \rangle) = \kappa'$, and $j'(j' \mid V_{\lambda'})(\kappa_n) = j(j(\kappa_n))$. Then we have $g(\langle j'(\kappa_n) : n \in \omega \rangle) = \alpha$ where $j'(\alpha) = \kappa'$, but κ' is the critical point of j', contradiction.

$\square$

References

[1] Akihiro Kanamori. The Higher Infinite: Large cardinals in set theory from their beginnings, 2nd edition. Springer Monographs in Mathematics, 2003.

[2] M. Victoria Marshall R. Higher order reflection principles, *Journal of Symbolic Logic,* vol. 54, no. 2, 1989, pp. 474–489.