THE CHOICELESS CARDINALS ARE INCONSISTENT

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1. Inconsistency of a super-Reinhardt cardinal

We present a proof that the following extension of ZF is inconsistent. Add a constant symbol $j$ to the language and add Separation and Replacement axioms for formulas involving $j$, and add the axiom that $j$ is an elementary embedding $V \prec V$ and also $DC_\lambda$ where $\lambda$ is the least fixed point of $j$ above the critical point of $j$. We wish to claim that this theory is inconsistent, and it follows from this that a super-Reinhardt cardinal is inconsistent with ZF.

Lemma 1.1. Work in the theory described above. Then it is provable on the stated assumptions that $V_\kappa$ is a model for the existence of a cardinal $\kappa'$ with the following property. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ where $\kappa_0 = \kappa'$, such that if we let $\lambda := \sup\{\kappa_i : i \in \omega\}$, then for any $X \in V_{\lambda+2}$ there exists an $\alpha < \kappa_0$ and an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X) \prec (V_{\lambda+1}, X')$ with critical point $\alpha$ such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$.

Proof. If we let $j$ be the elementary embedding originally assumed to exist for $\kappa$ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness the stated reflection property for $\kappa$, and the existence of the elementary embedding $j$ means that we can find a $\kappa' < \kappa$ such that the stated reflection property holds when we let $\kappa_0 := \kappa'$ and $\kappa_n := j^{n-1}(\kappa)$ for $n > 0$, $\kappa_0$ being reflected to $\kappa'$ and each $\kappa_n$ for $n > 0$ being reflected to $\kappa_{n-1}$. Let $j_0$ be an elementary embedding with domain $V_{\lambda+2}$ witnessing this instance of reflection. Then after the first reflection, $\kappa_0$ can be reflected to some ordinal between $\kappa'$ and $\kappa_0$, and so on. At stage $n$ for $n > 0$, let $j_n$ be an elementary embedding with domain $V_{\lambda+2}$ witnessing the reflection. Thus for each $n > 0$ we can find a sequence $\langle \kappa'_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa'$ and $\kappa'_i < \kappa$ for integers $i$ such that $0 \leq i \leq n$ and $\kappa'_i = \kappa_{i-n-1}$ for integers $i > n$, and the stated reflection property holds for $\langle \kappa'_i : i \in \omega \rangle$ and $\lambda$ as before. Then we can reflect the entire sequence $\langle \kappa_n : n \in \omega \rangle$ to a sequence $\langle \kappa'_n : n \in \omega \rangle$, with $\kappa'_n < \kappa$ for all $n$, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for $\kappa'_0$. We can use the
restrictions of \( j_n \) to \( V\lambda \) to construct an embedding \( j': V\lambda+1 \to V\lambda+1 \). Consider an \( X' \in V\lambda+2 \), we wish to find an \( X'' \in V\lambda+2 \) for which \( j': (V\lambda+1, X') \prec (V\lambda+1, X'') \) will hold. Suppose \( Y' \in X' \subseteq V\lambda+1 \), and let \( Y'' := Y' \cap V\alpha\gamma \), \( Y_n := (j_n^{-1}(j | V\lambda))^{n+1}(Y_n'') \). We can glue the \( Y_n \) together to obtain a \( Y \in V\lambda+1 \) for each \( Y' \in X' \), denote by \( Y \) the set of all such \( Y \). We have just defined a mapping \( V\lambda+1 \to V\lambda+1 \) which we will denote by \( e \). Now we have \( X \in V\lambda+2 \), then let \( X'' := (j(X))^{U} \) where \( U := \{ e(Y') : Y' \in V\lambda+1 \} \) and \( A^U := A \cap U \) for \( A \in V_{\lambda+1} \) and \( A^U := \{ B^U : B \in A \} \) for \( A \in V_{\lambda+1} \setminus V_{\lambda+1} \). This is the desired \( X'' \) with the property which we seek. Thus \( V_\kappa \) is a model for the existence of a cardinal \( \kappa' \) satisfying the stated reflection principle. \qed

**Lemma 1.2.** The existence of a cardinal with the stated reflection property is still inconsistent with choice.

**Proof.** The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume \( \text{ZFC} \) and let \( g \in V_{\lambda+2} \) be an \( \omega \)-Jonsson function over \( \lambda \). Then let \( g' \) be such that there is an elementary embedding \( j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g') \) with critical point \( \alpha < \kappa_0 \) and \( j(\alpha) = \kappa_0 \) and \( j(\kappa_\gamma) = \kappa_{\gamma+1} \). We have \( g'(x) = \alpha \) for some \( x \in [j^n\lambda]^\omega \). Let \( \beta := g(j^{-1}(x)) \). We get \( j(\beta) = \alpha \) which contradicts \( \alpha \) being the critical point of \( j \). \qed

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent.

2. **Getting the Kunen inconsistency in ZF**

Building on the results of the previous section we now show that it is inconsistent with \( \text{ZF} \) that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \).

**Theorem 2.1.** The theory \( \text{ZF} \) together with the assumption that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \) is inconsistent.

**Proof.** Work in \( \text{ZF} \) and assume that there is a non-trivial elementary embedding \( j : V_{\lambda+2} \to V_{\lambda+2} \), and assume without loss of generality that \( \lambda = \sup \{ j^n(\text{crit}(j)) : n \in \omega \} \). Let \( \Theta \) be the least ordinal such that there is no surjection from \( V_{\lambda+1} \) onto \( \Theta \) and let \( N := (\text{HOD}(j | V_\lambda))^{H_\Theta} \). Let \( \kappa := \text{crit}(j) \). Since \( (V_{\lambda+2})^N \) is closed under \( j \), the model \( N \) satisfies the assertion that for all \( \lambda \in V_{\lambda+2} \) there exists an \( X' \in V_{\lambda+2} \) such that \( j : (V_{\lambda+1}, X) \prec ((V_{\lambda+1})^N, X') \), and this property of \( \kappa \) in \( N \) is rank-reflected as before by Lemma 1.1. Furthermore, \( (j \circ j) | (V_\lambda)^{j(N)} = j(j | V_\lambda^N) \). We also have \( (V_\lambda)^N \) is a model of \( \text{AC} \) and \( j(j^n\lambda) \in (V_{\lambda+1})^j(N) \). So \( V_\kappa \) is a model for \( \text{AC} \) together with the assertion that there exists a \( \kappa' \)
and an elementary embedding $j'$ with critical point $\kappa'$ such that for all $X \in V_{\lambda'+2}$ there exists an $X' \in V_{\lambda'+2}$ such that $j' : (V_{\lambda'+1}, X) \prec (M_{\lambda'+1}, X')$ with $j'((j')''\lambda') \in M_{\lambda'+1}$ and $(j \circ j) | M_{\lambda'} = j(j | V_{\lambda'})$, where $\lambda' = \sup \{j''m(\kappa') : n \in \omega\}$. Let $g$ be an $\omega$-Jonsson function $[\lambda']^\omega \rightarrow \lambda'$. Then we have $g'$ is an $\omega$-Jonsson function from $([\lambda']^\omega)^{M_{\lambda'+1}} \rightarrow \lambda'$. We can choose $\langle \kappa_n : n \in \omega \rangle$ such that $g'((j'(j' | V_{\lambda'})(\kappa_n)) : n \in \omega)) = \kappa'$, and $j'(j' | V_{\lambda'})(\kappa_n) = j(j(\kappa_n))$. Then we have $g((j'(\kappa_n) : n \in \omega)) = \alpha$ where $j'(\alpha) = \kappa'$, but $\kappa'$ is the critical point of $j'$, contradiction.

$\square$
REFERENCES
