THE CHOICELESS CARDINALS ARE INCONSISTENT

RUPERT M[c]CALLUM

1. Inconsistency of a super-Reinhardt cardinal

We present a proof that the following extension of ZF is inconsistent. Add a unary function symbol **j** to the language and add Separation and Replacement axioms for formulas involving **j**, and add the axiom that the proper class function j which is the referent of the symbol **j** is an elementary embedding $V \prec V$ and also DC_λ where λ is the least fixed point of j above the critical point of j. We wish to claim that this theory is inconsistent, and it follows from this that a super-Reinhardt cardinal is inconsistent with ZF.

Lemma 1.1. Work in the theory described above. Then it is provable on the stated assumptions that V_κ is a model for the existence of a cardinal κ' with the following property. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ where $\kappa_0 = \kappa'$, such that if we let $\lambda := \sup\{\kappa_i : i \in \omega\}$, then for any $X \in V_{\lambda+2}$ there exists an $\alpha < \kappa_0$ and an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X) \prec (V_{\lambda+1}, X')$ with critical point α such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$.

Proof. If we let j be the elementary embedding originally assumed to exist for κ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness the stated reflection property for κ, and the existence of the elementary embedding j means that we can find a $\kappa' < \kappa$ such that the stated reflection property holds when we let $\kappa_0 := \kappa'$ and $\kappa_n := j^{n-1}(\kappa)$ for $n > 0$, κ_0 being reflected to κ' and each κ_n for $n > 0$ being reflected to κ_{n-1}. Let j_0 be an elementary embedding with domain $V_{\lambda+2}$ witnessing this instance of reflection. Then after the first reflection, κ_0 can be reflected to some ordinal between κ' and κ_0, and so on. At stage n for $n > 0$, let j_n be an elementary embedding with domain $V_{\lambda+2}$ witnessing the reflection, that is, an elementary embedding with the property that $j_n(\kappa'_i) = \kappa'_{i+1}$ for $0 \le i < n$, and $j_n(\kappa'_n) = \kappa_0$, and $j_n(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$. Thus for each $n > 0$ we can find a sequence $\langle \kappa'_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa'$ and $\kappa'_i < \kappa$ for integers i such that $0 \le i \le n$ and $\kappa'_i = \kappa_{i-n-1}$ for integers $i > n$, and the stated reflection property holds for $\langle \kappa'_i : i \in \omega \rangle$ and λ as before.

Then we can reflect the entire sequence $\langle\kappa_n : n \in \omega\rangle$ to a sequence $\langle\kappa'_n : n \in \omega\rangle$, with $\kappa'_n < \kappa$ for all n, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for κ'_0. First of all let us describe how we use the restrictions of j_n to V_λ to construct an embedding $j'_X : V_{\lambda'+1} \to V_{\lambda'+1}$ with critical point α and $j'_X(\alpha) = \kappa'_0$ and $j'_X(\kappa'_i) = \kappa'_{i+1}$ for each $i \geq 0$. We denote by j_X an embedding witnessing the reflection property for the parameter X and assume that its critical point α is less than κ'_0. (This can be achieved by choosing the family of embeddings $\langle j_n : n \in \omega\rangle$ in such a way that $\kappa'_0 > \alpha$). The embedding j_{n+1} has critical point κ'_0 and we denote by U_n the range of the embedding j_{n+1} and define $A^{U_n} := A \cap U_n$ for $A \in V_{\kappa'_0+1}$ and $A^{U_n} := \{B^{U_n} : B \in A\}$ for $A \notin V_{\kappa'_0+1}$. We obtain the embedding j'_X by gluing together $(j_X \mid V_{\kappa_n})^{U_n}$ for all $n \in \omega$, where j_X is an embedding witnessing the stated reflection property for the parameter X for the sequence $\langle\kappa_n : n \in \omega\rangle$. We need to establish that this gluing is indeed possible.

So, recall that j_n is an elementary embedding that sends κ'_0 to κ'_1, κ'_1 to $\kappa'_2, \ldots \kappa'_{n-1}$ to κ'_n and κ'_n to κ_0, then κ_0 to κ_1, and so on. We would like to show that we can choose j_{n+1} in such a way that it agrees with j_n on $V_{\kappa'_n}$ (but disagrees at the ordinal κ'_n). This will be sufficient to show that the desired gluing is possible.

Recall how the j_n and j_{n+1} are constructed. Once we have constructed j_n, then we use the elementary embedding j to obtain a reflection so that κ_0 gets reflected to κ'_{n+1} and all κ_k for $k > 0$ are reflected to κ_{k-1}. Denote by U the range of the embedding k_{n+1} with critical point κ'_{n+1} witnessing the reflection, and for $A \in V_{\kappa'_{n+1}+1}$ define $A^U := A \cap U$, and for $A \notin V_{\kappa'_{n+1}+1}$ define $A^U := \{B^U : B \in A\}$. In this way we can speak of to which object a particular object A is reflected.

We wish to claim that it can be arranged so that $j_n \mid V_{\kappa_0}$ is reflected to an embedding $j_{n+1} \mid V_{\kappa'_{n+1}}$ which agrees with j_n on $V_{\kappa'_n}$ but disagrees at the ordinal κ'_n, and for each positive integer k, $j_n \mid V_{\kappa_k}$ gets reflected to an embedding $j_{n+1} \mid V_{\kappa_{k-1}}$. Recall that we are obtaining this reflection from an elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ with critical point κ_0, and the desired constraint on the reflection, agreement of j_{n+1} with j_n on $V_{\kappa'_n}$, can be achieved by making $j_n \mid V_{\kappa'_n}$ a parameter of the formula being reflected. In this way we can construct the embedding j_{n+1} with the necessary agreement condition satisfied and this shows that the gluing is possible.

So the desired gluing is possible as claimed, and the mapping j'_X is made by gluing together $(j_X \mid V_{\kappa_n})^{U_n}$, where U_n is the range of the embedding j_{n+1}, for all $n \in \omega$. This completes the description of the mapping j'_X, and we have made the claim that it is an elementary embedding from $V_{\lambda'+1} \to V_{\lambda'+1}$, let us argue this point. In order to provide the argument we need to describe an elementary embedding $e : V_{\lambda'+1} \to V_{\lambda+1}$.

Suppose we have a $Y' \subseteq V_{\lambda'}$. To obtain $e(Y)$, glue together $j_0(Y' \cap V_{\kappa'_0})$, $j_1^2(Y' \cap V_{\kappa'_1})$, and so on. Again the previous coherence condition established between the j_n for different $n \in \omega$ shows that this gluing is indeed possible. Suppose that we have some formula ϕ in the first-order language of set theory such that $\phi^{V_{\lambda'+1}}(Y'_1, Y'_2, \ldots Y'_k)$ holds. In the case where ϕ is Σ_1 or Π_1 we clearly have elementarity of the mapping e in the sense that $\phi^{V_{\lambda+1}}(e(Y'_1), e(Y'_2), \ldots e(Y'_k))$ holds. To show the case of a Π_1 formula note that elementarity for formulas with quantifiers relativised to V_λ is easy, and then to get the case of a Π_1 formula of the form $\forall Y \psi(Y, Y_1, \ldots Y_k)$, note that the formula holds if and only if it holds with Y relativised to the range of e. Then an induction argument, with similar reasoning for the induction step, generalises this to the case of a Σ_k or Π_k formula for all positive integers k. In this way we establish that e is indeed an elementary embedding.

Now, having shown that e is an elementary embedding and realising that j'_X is just $e^{-1} \circ j_X \circ e$, we obtain the result that j'_X is an elementary embedding $V_{\lambda'+1} \to V_{\lambda'+1}$ as claimed.

Defining $X' \in V_{\lambda'+2}$ to be X^U where U is the range of the embedding e, and noting that any $X' \in V_{\lambda'+2}$ can be obtained in this way, we wish to find an $X'' \in V_{\lambda'+2}$ for which $j'_X : (V_{\lambda'+1}, X') \prec (V_{\lambda'+1}, X'')$ will hold. Change the use of the notation X, denote by X the set $\{e(Y') : Y' \in X'\}$. Now we have $X \in V_{\lambda+2}$, then let $X'' := (j(X))^U$ where $U := \{e(Y') : Y' \in V_{\lambda'+1}\}$ and $A^U := A \cap U$ for $A \in V_{\kappa_0+1}$ and $A^U := \{B^U : B \in A\}$ for $A \in V_{\lambda+1} \setminus V_{\kappa_0+1}$. This is the desired X'' with the property which we seek. To see this note that $e(X'')$ agrees with $j(X)$ for its intersection with $\{e(Y') : Y' \in V_{\lambda'+1}\}$, and this is sufficient to show that e is elementary from $(V_{\lambda'+1}, X'')$ into $(V_{\lambda+1}, j(X))$. And of course j is elementary from $(V_{\lambda+1}, X)$ into $(V_{\lambda+1}, j(X))$, and e is elementary from $(V_{\lambda'+1}, X')$ into $(V_{\lambda+1}, X)$. So, combining all these claims, we obtain the desired result that j' is elementary from $(V_{\lambda'+1}, X')$ into $(V_{\lambda'+1}, X'')$, as claimed.

Thus V_κ is a model for the existence of a cardinal κ' satisfying the stated reflection principle. □

Lemma 1.2. The existence of a cardinal with the stated reflection property is still inconsistent with choice.

Proof. The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume ZFC and let $g \in V_{\lambda+2}$ be an ω-Jonsson function over λ. Then let g' be such that there is an elementary embedding $j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g')$ with critical point $\alpha < \kappa_0$ and $j(\alpha) = \kappa_0$ and $j(\kappa_n) = \kappa_{n+1}$. We have $g'(x) = \alpha$ for some $x \in [j\text{"}\lambda]^\omega$. Let $\beta := g(j^{-1}(x))$. We get $j(\beta) = \alpha$ which contradicts α being the critical point of j. □

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent.

2. Getting the Kunen inconsistency in ZF

Building on the results of the previous section we now show that it is inconsistent with ZF that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

Theorem 2.1. *The theory ZF together with the assumption that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ is inconsistent.*

Proof. Work in ZF and assume that there is a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$, and assume without loss of generality that $\lambda = \sup\{j^n(\mathrm{crit}(j)) : n \in \omega\}$. Let Θ be the least ordinal such that there is no surjection from $V_{\lambda+1}$ onto Θ and let $N := (HOD(j \mid V_\lambda))^{H_\Theta}$. Let $\kappa := \mathrm{crit}(j)$. Since $(V^N_{\lambda+2}$ is closed under j, the model N satisfies the assertion that for all $X \in V_{\lambda+2}$ there exists an $X' \in V_{\lambda+2}$ such that $j : (V_{\lambda+1}, X) \prec ((V^{j(N)}_{\lambda+1}, X')$, and this property of κ in N is rank-reflected as before by Lemma 1.1. Furthermore, $(j \circ j) \mid (V_\lambda)^{j(N)}) = j(j \mid V^N_\lambda)$. We also have $(V_\lambda)^N$ is a model of AC and $j(j\text{"}\lambda) \in (V_{\lambda+1})^{j(N)}$. So V_κ is a model for AC together with the assertion that there exists a κ' and an elementary embedding j' with critical point κ' such that for all $X \in V_{\lambda'+2}$ there exists an $X' \in V_{\lambda'+2}$ such that $j' : (V_{\lambda'+1}, X) \prec (M_{\lambda'+1}, X')$ with $j'((j')\text{"}\lambda') \in M_{\lambda'+1}$ and $(j \circ j) \mid M_{\lambda'} = j(j \mid V_{\lambda'})$, where $\lambda' = \sup\{(j')^n(\kappa') : n \in \omega\}$. Let g be an ω-Jonsson function $[\lambda']^\omega \to \lambda'$. Then we have g' is an ω-Jonsson function from $([\lambda']^\omega)^{M_{\lambda'+1}} \to \lambda'$. We can choose $\langle \kappa_n : n \in \omega \rangle$ such that $g'(\langle j'(j' \mid V_{\lambda'})(\kappa_n) : n \in \omega \rangle) = \kappa'$, and $j'(j' \mid V_{\lambda'})(\kappa_n) = j(j(\kappa_n))$. Then we have $g(\langle j'(\kappa_n) : n \in \omega \rangle) = \alpha$ where $j'(\alpha) = \kappa'$, but κ' is the critical point of j', contradiction.

□

References

[1] Akihiro Kanamori. The Higher Infinite: Large cardinals in set theory from their beginnings, 2nd edition. Springer Monographs in Mathematics, 2003.

[2] M. Victoria Marshall R. Higher order reflection principles, *Journal of Symbolic Logic,* vol. 54, no. 2, 1989, pp. 474–489.