THE CHOICELESS CARDINALS ARE INCONSISTENT

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1. The Kunen inconsistency is provable in ZF

The Kunen inconsistency is the result that the existence of a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ for some ordinal λ is not consistent with ZFC. We wish to show that it is not consistent with ZF either. Using Woodin's iterated collapse forcing, we can show that the theory ZF together with the existence of a non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$ is equiconstent with the same theory together with the assumption that V_λ can be well-ordered. We present a proof that this extension of ZF is inconsistent.

Definition 1.1. We say that a cardinal κ has reflection property R if the following holds. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa$, and if we define $\lambda := \sup\{\kappa_i : i \in \omega\}$, then there exists an $\alpha < \kappa$ such that, for any $X \in V_{\lambda+2}$ there exists an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X) \prec (V_{\lambda+1}, X')$ with critical point α, such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$.

Lemma 1.2. Work in the theory described in the first paragraph, before the statement of Definition 1.1, and denote by κ the critical point of the elementary embedding j. Then it is provable on the stated assumptions that V_κ is a model for the existence of a cardinal κ' with reflection property R.

Proof. If we let j be the elementary embedding originally assumed to exist for κ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness that κ has reflection property R. For each $X \in V_{\lambda+2}$, let j_X be an embedding witnessing reflection property R for that particular X with critical point α, α being independent of X, and $j_X(\alpha) = \kappa_0 := \kappa$ and $j_X(\kappa_n) = \kappa_{n+1}$ for all $n \in \omega$ where $\kappa_n := j^n(\kappa)$. The existence of the elementary embedding j means that we can find a κ' such that $\alpha < \kappa' < \kappa$ such that the stated reflection property holds for the sequence $\langle \kappa', \kappa_0, \kappa_1, \ldots \rangle$, κ_0 being reflected to κ' and each κ_n for $n > 0$ being reflected to κ_{n-1}. Let j_0 be an elementary embedding with domain $V_{\lambda+1}$ witnessing this instance of reflection, that is j_0 has critical point $\kappa'_0 := \kappa'$ and $j(\kappa'_0) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$. Also j_0 can be chosen to be an elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$. This can be achieved by reflecting a formula with a truth set for $V_{\lambda+1}$ as parameter which states all the Tarski axioms.

Suppose we have built up a sequence of embeddings $\langle j_i : 0 \leq i \leq n \rangle$, so that each j_i is an embedding $V_{\lambda+1} \to V_{\lambda+1}$ with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_i, \kappa_0, \kappa_1, \ldots \rangle$, and for each i such that $0 \leq i < n$ we have that j_{i+1} agrees with j_i on $V_{\kappa'_i}$. Define $A_n := j_n \mid V_{\kappa'_n}$. Then reflect the formula "There is an embedding $k : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ'_0 and critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_n, \kappa_0, \kappa_1, \ldots \rangle$ such that $k \mid V_{\kappa'_n}$ agrees with A_n", in such a way that the critical sequence $\langle \kappa_n : n \in \omega \rangle$ is reflected to $\langle \kappa'_{n+1}, \kappa_0, \kappa_1, \ldots \rangle$ via an elementary embedding $k_{n+1} : V_{\lambda+1} \to V_{\lambda+1}$ with critical point κ'_{n+1} such that $k_{n+1}(\kappa'_{n+1}) = \kappa_0$ and $k_{n+1}(\kappa_i) = \kappa_{i+1}$ for all $i \in \omega$. To ensure that k_{n+1} can indeed be chosen to be an elementary embedding as claimed, we need to reflect not just the previously stated formula but also a formula with a truth set for $V_{\lambda+1}$ as parameter stating all the Tarski axioms. In this way we obtain a new embedding $j_{n+1} : V_{\lambda+1} \to V_{\lambda+1}$ with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_{n+1}, \kappa_0, \kappa_1, \ldots \rangle$ which agrees with j_n on $V_{\kappa'_n}$. Note that $k_{n+1}(j_{n+1} \mid V_\lambda) = j_n \mid V_\lambda$.

Thus for each $n > 0$ we can find an elementary embedding with critical sequence $\langle \kappa'_0, \kappa'_1, \ldots \kappa'_n, \kappa_0, \kappa_1, \ldots \rangle$. Then we can reflect the entire sequence $\langle \kappa_n : n \in \omega \rangle$ to a sequence $\langle \kappa'_n : n \in \omega \rangle$, with $\kappa'_n < \kappa$ for all n, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for κ'_0. First of all let us describe how we use the sequence of embeddings $\langle j_n : n \in \omega \rangle$ and the embedding j_X to construct a mapping $j'_X : V_{\lambda'+1} \to V_{\lambda'+1}$ with critical point α and $j'_X(\alpha) = \kappa'_0$ and $j'_X(\kappa'_i) = \kappa'_{i+1}$ for each $i \in \omega$, the plan being to show that this mapping is also an elementary embedding.

Let us denote by U_n the range of the embedding $(k_0 \circ k_1 \circ k_2 \circ \ldots \circ k_n) \mid V_{\kappa'_n}$. Then, for $A \in V_{\kappa_{n+1}}$, we define $A^{U_n} := A \cap U_n$ for $A \in V_{\kappa_0+1}$ and $A^{U_n} := \{B^{U_n} : B \in A\}$ for $A \notin V_{\kappa_0+1}$. The map from A to A^{U_n} is a kind of transitive collapsing map. It can be thought of as indicating to which object A is reflected. V_{κ_i} will be reflected to $V_{\kappa'_i}$ for all integers i such that $0 \leq i \leq n$. For $A \in U_n$ we will have $A^{U_n} = (k_0 \circ k_1 \circ k_2 \circ \ldots \circ k_n)^{-1}(A)$. We obtain the embedding j'_X by gluing together $(j_X \mid V_{\kappa_{n-1}})^{U_n}$ for all $n \in \omega \setminus \{0\}$. Note that for each $n > 0$, $(j_X \mid V_{\kappa_{n-1}})^{U_n}$ is a mapping with domain $V_{\kappa'_{n-1}}$, and from the fact that $U_n \subseteq U_{n+1}$ for all $n \in \omega$ it follows that it is possible to glue all these maps together to obtain a mapping with domain $V_{\lambda'}$ where

$\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We shall eventually show that this map is an elementary embedding $V_{\lambda'} \to V_{\lambda'}$ with the usual unique extension to a mapping $V_{\lambda'+1} \to V_{\lambda'+1}$ which we will show is also an elementary embedding.

So the mapping j'_X is made by gluing together $(j_X \mid V_{\kappa_{n-1}})^{U_n}$, for all $n \in \omega \setminus \{0\}$. In order to argue the point that the canonical extension of j'_X to $V_{\lambda'+1}$ is an elementary embedding, we need to describe an elementary embedding $e : V_{\lambda'+1} \to V_{\lambda+1}$.

Suppose we have a $Y' \subseteq V_{\lambda'}$. To obtain $e(Y)$, glue together $k_0(Y' \cap V_{\kappa'_0})$, $k_0(k_1(Y' \cap V_{\kappa'_1}))$, and so on. Suppose that we have some formula ϕ in the first-order language of set theory such that $\phi^{V_{\lambda'+1}}(Y'_1, Y'_2, \dots Y'_k)$ holds, witih $Y'_i \in V_{\lambda'+1}$ for all i such that $1 \leq i \leq k$. In the case where ϕ is Σ_1 we clearly have elementarity of the mapping e in the sense that $\phi^{V_{\lambda+1}}(e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds. We need to show that elementarity also holds in the case where ϕ is Π_1. If ϕ is of the form $\forall X \psi$ where ψ is Σ_0, then clearly $\psi(Y, e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds whenever Y is in the range of e. Now consider the case of an arbitrary Y. All of the k_n are elementary embeddings $V_{\lambda+1} \to V_{\lambda+1}$. If we let W_n be the range of the embedding $k_0 \circ k_1 \circ \dots \circ k_n$, and U be the range of e, then $U = \bigcap_{n\in\omega} W_n$. We have $\psi(Y, e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds if and only if $\psi((k_0 \circ k_1 \circ \dots \circ k_n)(Y^{W_n}), e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds for all n, so it follows that $\psi(Y, e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds if and only if $\psi(e(Y^U), e(Y'_1), e(Y'_2), \dots e(Y'_k))$ holds, where the notations A^{W_n} and A^U are defined by similar definitions to those given previously. Then an induction argument, with similar reasoning for the induction step, generalises this to the case of a Σ_k or Π_k formula for all positive integers k. In this way we establish that e is indeed an elementary embedding.

Now, having shown that e is an elementary embedding and realising that j'_X is just $e^{-1} \circ j_X \circ e$, we obtain the result that j'_X is an elementary embedding $V_{\lambda'+1} \to V_{\lambda'+1}$ as claimed.

Define $X'' \in V_{\lambda'+2}$ to be X^U where U is the range of the embedding e, and note that any $X'' \in V_{\lambda'+2}$ can arise in this way. We wish to find an $X''' \in V_{\lambda'+2}$ for which $j'_X : (V_{\lambda'+1}, X'') \prec (V_{\lambda'+1}, X''')$ will hold. Change the use of the notation X, denote by X the set $e(X'' \cap V_{\lambda'}) \cup \{e(Y') : Y' \in X'' \cap (V_{\lambda'+1} \setminus V_{\lambda'})\}$. Now we have $X \in V_{\lambda+2}$, then let $X''' := (X')^U$ where $U := \{e(Y') : Y' \in V_{\lambda'+1}\}$, and X' is such that j_X is elementary from $(V_{\lambda+1}, X)$ into $(V_{\lambda+1}, X')$. This is the desired X''' with the property which we seek. To see this note that e is elementary from $(V_{\lambda'+1}, X''')$ into $(V_{\lambda+1}, X')$. And of course j_X is elementary from

$(V_{\lambda+1}, X)$ into $(V_{\lambda+1}, X')$, and e is elementary from $(V_{\lambda'+1}, X'')$ into $(V_{\lambda+1}, X)$. So, combining all these claims, we obtain the desired result that j' is elementary from $(V_{\lambda'+1}, X'')$ into $(V_{\lambda'+1}, X''')$, as claimed.

Thus V_κ is a model for the existence of a cardinal κ' satisfying the stated reflection principle. □

Lemma 1.3. The existence of a cardinal with the stated reflection property is still inconsistent with choice.

Proof. The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume ZFC and let $g \in V_{\lambda+2}$ be an ω-Jonsson function over λ. Then let g' be such that there is an elementary embedding $j : (V_{\lambda+1}, g) \prec (V_{\lambda+1}, g')$ with critical point $\alpha < \kappa_0$ and $j(\alpha) = \kappa_0$ and $j(\kappa_n) = \kappa_{n+1}$. We have $g'(x) = \alpha$ for some $x \in [j"\lambda]^\omega$. Let $\beta := g(j^{-1}(x))$. We get $j(\beta) = \alpha$ which contradicts α being the critical point of j. □

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent, and from this we obtain the conclusion that the Kunen inconsistency is provable in ZF.

References

[1] Akihiro Kanamori. The Higher Infinite: Large cardinals in set theory from their beginnings, 2nd edition. Springer Monographs in Mathematics, 2003.

[2] M. Victoria Marshall R. Higher order reflection principles, *Journal of Symbolic Logic,* vol. 54, no. 2, 1989, pp. 474–489.