Parallels in universality between the universal algorithm and the universal finite set

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Oxford Logic Seminar 2018
Warm thanks to

Rasmus Blanck, Ali Enayat, Victoria Gitman, Roman Kossak, Øystein Linnebo, Volodya Shavrukov, Albert Visser, Philip Welch, Kameryn J. Williams and W. Hugh Woodin

for insightful comments and helpful discussions.

This list includes several of my co-authors on papers relevant for this talk.
Introduction

A delightful theorem

The universal algorithm

and its generalization to set theory,

The universal finite set

with applications to set-theoretic potentialism.
Theorem (Woodin)

There is a Turing machine program $e$ such that:

1. Program $e$ enumerates a finite sequence only, and PA proves this.
Review of the universal algorithm

Theorem (Woodin)

There is a Turing machine program $e$ such that:

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2. Program $e$ enumerates the empty sequence in the standard model $\mathbb{N}$. 
Review of the universal algorithm

Theorem (Woodin)

There is a Turing machine program $e$ such that:

1. Program $e$ enumerates a finite sequence only, and PA proves this.

2. Program $e$ enumerates the empty sequence in the standard model $\mathbb{N}$.

3. If $M \models \text{PA}$ and $e$ enumerates $s$ in $M$, then for any larger $t$ in $M$ there is an end-extension $N \models \text{PA}$ in which $e$ enumerates $t$.

In particular, every finite sequence $s \in \mathbb{N}^\omega$ is enumerated by $e$ in some model $M \models \text{PA}$. 
History

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- Rasmus Blanck and Ali Enayat generalize it, remove restriction to countable models, 2017.
- Hamkins provides a simplified proof, 2017.
- Visser points out affinity with the classical ‘exile’ argument in proof theory.
- Weak forms go back to Mostowski and Kripke 1960s.
- Current generalizations to set theory by Hamkins and Woodin 2017, and Hamkins, Williams, Welch 2018.
My simplified proof

Let’s define the universal algorithm $e$. 

Proceed in stages, releasing the sequence in batches. Stage $n$ succeeds, if there is a proof from fragment $\text{PA}_k$ with $k_n$ smaller than all earlier $k_i$, of a statement of the form 

"it is not the case that $e$ has exactly $n$ stages and releases $s$ at stage $n,"$

where $s$ is an explicitly listed sequence of numbers. In this case, release $s$ at stage $n$. Proceed to next stage.
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Proof of universal algorithm

Succinctly:

The program $e$ enumerates $s$ at stage $n$, if it finds proof, in a strictly smaller fragment of PA each time, that it does not do so as its last stage.
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The program $e$ enumerates $s$ at stage $n$, if it finds proof, in a strictly smaller fragment of $\text{PA}$ each time, that it does not do so as its last stage.

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Upon finding a rule forbidding certain behavior, it immediately exhibits that behavior.
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Note

Use Kleene recursion theorem to find $e$, solving the circular definition.
Finiteness

Observation

The universal sequence is finite.
Finiteness

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Proof.

The fragments \( k_n \) must descend, and so there can be at most finitely many successful stages.

Thus, the sequence enumerated by \( e \) will be finite.

And PA can undertake this argument.
Empty in the standard model

Claim

If stage $n$ is successful, then $k_n$ is nonstandard.
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Proof.

Consider last stage $n$. The assertion

“$e$ enumerates $s$ with exactly $n$ successful stages”

has complexity $\Sigma^0_2$. For standard $k$, the Mostowski reflection theorem shows $\text{PA} \vdash \text{Con}(\text{Tr}_k)$.

So $M$ cannot have proof from $\text{PA}_k$ of something contrary to actual behavior. So $k_n$ must be nonstandard.
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In particular, $e$ enumerates empty sequence in $\mathbb{N}$. 

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Proving the extension property of $e$

Assume $e$ enumerates $s$ in $M$ and $s \subseteq t$. 

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**Key Observation**

Since stage $n$ was not successful, $M$ must think that

$$\text{PA}_k + \ \text{“}e \ \text{has exactly} \ n \ \text{stages and enumerates} \ t \text{”}$$

is consistent.
Proving the extension property of $e$

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So $M$ can build a Henkin model of this theory, $N$. 

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So $M$ can build a Henkin model of this theory, $N$.

So $N$ end-extends $M$ and thinks $e$ enumerates $t$. And $N \models \text{PA}$ since $k$ is nonstandard. This proves the theorem. $\square$
Classical consequences

Several classical results in the model theory of arithmetic can be seen as immediate consequences of the universal algorithm.

Let us explore a few examples.
Maximal $\Sigma_1$ diagrams

Corollary

No model of PA has a maximal $\Sigma_1$ diagram.
Maximal $\Sigma_1$ diagrams

**Corollary**

No model of $\text{PA}$ has a maximal $\Sigma_1$ diagram.

**Proof.**

If $M \models \text{PA}$, then there is an unsuccessful stage $n$, which becomes successful in an end-extension $N$.

So the assertion “stage $n$ is successful” is a new $\Sigma_1$ statement about $n$ true in $N$, false in $M$.

For example, there is a diophantine equation, with coefficients in $M$, having no solution in $M$, but it has a solution in $N$. 
Independent $\Pi^0_1$ sentences

**Corollary (Kripke, Mostowski)**

There are infinitely many independent $\Pi^0_1$ sentences

\[ \eta_0, \eta_1, \eta_2, \ldots \]

Any desired true/false pattern is consistent with $\text{PA}$.
Independent $\Pi^0_1$ sentences

Corollary (Kripke, Mostowski)

There are infinitely many independent $\Pi^0_1$ sentences

$\eta_0, \eta_1, \eta_2, \ldots$

Any desired true/false pattern is consistent with PA.

Proof.

Let $\eta_k = "k$ does not appear on the universal sequence."
Independent buttons

Corollary “independent buttons”

There are $\Sigma^0_1$ sentences

$$\rho_0, \rho_1, \rho_2, \ldots$$

all false in $\mathbb{N}$ and for any $M \models \text{PA}$, any pattern $I$ coded in $M$,
there is end-extension $N$ with

1. Every $\rho_k$ becomes true in $N$ for $k \in I$.
2. Truth of $\rho_k$ is not changed for $k \notin I$. 
Independent buttons

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1. Every $\rho_k$ becomes true in $N$ for $k \in I$.
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Proof.

Let $\rho_k = \text{“} k \text{ appears on the universal sequence.} \text{”}$
Independent Orey sentences

**Corollary “Independent switches”**

There is an infinite list of independent Orey sentences

\[ \sigma_0, \sigma_1, \sigma_2, \ldots \]

For any \( M \models \text{PA} \) any pattern \( I \) coded in \( M \), there is end-extension \( N \) with

1. \( \sigma_k \) is true in \( N \) for \( k \in I \).
2. \( \sigma_k \) is false in \( N \) for \( k \notin I \).

Proof. Let \( \sigma_k = "k \text{ is amongst the numbers added at the last stage.}" \)
Independent Orey sentences

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Flexible formulas

**Corollary (Kripke)**

For \( n \geq 2 \), there is a \( \Sigma^0_n \) formula \( \varphi(x) \) that can be made so as to agree with any desired \( \Sigma^0_n \) formula \( \phi(x) \) in an end-extension.

That is, for any \( M \models \text{PA} \) and any such \( \phi \), there is an end-extension \( N \) satisfying

\[
\forall x \varphi(x) \leftrightarrow \phi(x).
\]
Flexible formulas

Corollary (Kripke)

For $n \geq 2$, there is a $\Sigma_n^0$ formula $\varphi(x)$ that can be made so as to agree with any desired $\Sigma_n^0$ formula $\phi(x)$ in an end-extension.

That is, for any $M \models \text{PA}$ and any such $\phi$, there is an end-extension $N$ satisfying

$$\forall x \varphi(x) \leftrightarrow \phi(x).$$

Proof.

Let $\varphi(x) = \Phi(k, x)$ where $k$ is the last element of the universal sequence and $\Phi$ is a universal $\Sigma_n^0$ formula.
Flexible formula, uniform version

**Theorem**

There is a computable sequence $\sigma_n(x)$ of $\Sigma_n$ formulas, for $n \geq 2$, such that for any $M \models \text{PA}$ and any sequence of $\Sigma_n$ formulas $\phi_n$ coded in $M$, there is end-extension $N$ satisfying

$$\forall x \quad \sigma_n(x) \leftrightarrow \phi_n(x).$$
Set-theoretic analogue

What is the set-theoretic analogue of the universal algorithm?

One wants a version of the theorem for models of set theory.
Arithmetic vs. set theory

Arithmetic

The computably enumerable sets are gradually revealed as time proceeds. Elements are confirmed at some stage of time.
### Arithmetic vs. set theory

#### Arithmetic

The computably enumerable sets are gradually revealed as time proceeds. Elements are confirmed at some stage of time.

#### Set theory

The locally verifiable sets have members confirmed in some $V_\alpha$, as the set-theoretic universe grows.

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\{ x \mid \varphi(x) \}, \quad \text{where} \quad \varphi(x) \leftrightarrow \exists\alpha \ V_\alpha \models \psi(x).
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Arithmetic vs. set theory

**Arithmetic**

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The locally verifiable sets have members confirmed in some $V_\alpha$, as the set-theoretic universe grows.

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**Elementary Fact**

Locally verifiable sets $= \Sigma_2$ definable.
Question

So the set-theoretic analogue of c.e. is $\Sigma_2$ definable.
So the set-theoretic analogue of c.e. is $\Sigma_2$ definable.

**Question (Hamkins)**

Is there a $\Sigma_2$ definable set $\{ x \mid \varphi(x) \}$ with the following?

- $\text{ZFC}$ proves $\{ x \mid \varphi(x) \}$ is a set.
- For every countable model $M \models \text{ZFC}$, if

$$M \models \{ x \mid \varphi(x) \} = y \subseteq z,$$

then there is top-extension $N$ with

$$N \models \{ x \mid \varphi(x) \} = z.$$
Partial progress

I had initially made partial progress.

- I could do it with $\Pi_3$ definitions.
- I could do it with $\Sigma_2$, restricting to models of certain theories, such as eventual GCH or $V \neq \text{HOD}$.
- I could do it with $\Sigma_2$, if we allow $\{ x \mid \varphi(x) \}$ sometimes to be a proper class.

Woodin and I began to work together on the problem in September 2017.
Universal finite set

Ultimately, we were able to answer the question.

Theorem (Hamkins + Woodin)

There is a $\Sigma_2$ definition $\varphi$ such that

1. ZFC proves $\{ x \mid \varphi(x) \}$ is finite.
Universal finite set

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Theorem (Hamkins + Woodin)

*There is a $\Sigma_2$ definition $\varphi$ such that*

1. *ZFC proves* $\{ x \mid \varphi(x) \}$ *is finite.*

2. *If $M$ is transitive, then* $M \models \{ x \mid \varphi(x) \} = \emptyset.$
Universal finite set

Ultimately, we were able to answer the question.

**Theorem (Hamkins + Woodin)**

There is a $\Sigma_2$ definition $\varphi$ such that

1. ZFC proves $\{ x \mid \varphi(x) \}$ is finite.
2. If $M$ is transitive, then $M \models \{ x \mid \varphi(x) \} = \emptyset$.
3. If $M$ is a countable model of ZFC with $M \models \{ x \mid \varphi(x) \} = y \subseteq z$, where $z$ is finite in $M$, then there is top-extension $N$ with $N \models \{ x \mid \varphi(x) \} = z$. 

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Universal countable set, universal set

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The union of the countable members of the universal finite set is an arbitrary countable set.

And so on for other cardinals or other kinds of universal sets.

Also, there is a sequence version of the theorem.
Proof sketch

We break the proof into two pieces.

- Process A will work with $\omega$-nonstandard models.
- Process B will work with $\omega$-standard models.

In the end, we unify the two processes into a single $\Sigma_2$ definition.
Process A

We define $\varphi_A(x)$ as follows.
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Stage $n$ succeeds, if there is a beth-fixed point $\beta_n$, above all previous $\beta_i$, and number $k_n$, below all previous $k_i$, such that

$$\langle V_{\beta_n}, \in \rangle \text{ has no topped extension to } N \models \text{ZFC}_{k_n} \text{ with } N \models \text{“process A ends at stage n with } \beta_n \text{ and this is preserved by } P(\beta_n)\text{-preserving forcing.”}$$

In this case, let $y_n$ be the finite set coded into the GCH pattern starting at $\beta_n^+$. Declare $\varphi_A(x)$ for all $x \in y_n$.

Use the Gödel-Carnap fixed-point lemma to find $\varphi_A(x)$ solving this circular definition.
Process A

Since the $k_n$ are descending, we have only finitely many successful stages. So the set is finite.
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Claim

If $n$ is successful, then $k_n$ is nonstandard.
The universal algorithm
Classical consequences
Universal finite set
Applications
Further generalizations

Process A

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Claim

If $n$ is successful, then $k_n$ is nonstandard.

Proof.

If $M \models \{ x \mid \varphi_A(x) \} = y$ (and maximal successful stages in any forcing extension), then for standard $k$ find many $\theta$ with $V_\theta^M \models \text{ZFC}_k + \{ x \mid \varphi_A(x) \} = y$. So $V_{\beta_{kn}}^M$ has many top-extensions in $M$ after all, but it wasn’t supposed to since $n$ was successful.
Process A extension property

Suppose $M \models \{ x \mid \varphi_A(x) \} = y \subseteq z$. Let $n$ be the first unsuccessful stage. Let $k$ be any nonstandard number below previous $k_i$. 

Let $\beta = \mathcal{V}_\beta^{M+}$. Stage $n$ still not successful in $M+$. In particular, not successful with $\beta$. So there is a topped extension $N \models \text{ZFC}$ of $\langle V_\beta, \in \rangle^{M+}$ with exactly stage $n$ successful at $\beta$, and preserved by forcing. $N$ top-extends $M$. 

Build $N[G]$ to code $z$ at $\beta+$. So $N[G] \models \{ x \mid \varphi_A(x) \} = z$, as desired.
Process A extension property

Suppose $M \models \{ x \mid \varphi_A(x) \} = y \subseteq z$. Let $n$ be the first unsuccessful stage. Let $k$ be any nonstandard number below previous $k_i$.

Go to an elementary top-extension $M \prec M^+$, and let $\beta = \exists \beta > M$ in $M^+$.
Process A extension property

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Stage $n$ still not successful in $M^+$. In particular, not successful with $\beta$.

So there is a topped extension $N \models \text{ZFC}_k$ of $\langle V_\beta, \in \rangle^{M^+}$ with exactly stage $n$ successful at $\beta$, and preserved by forcing. $N$ top-extends $M$. 
Process A extension property

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Stage $n$ still not successful in $M^+$. In particular, not successful with $\beta$.

So there is a topped extension $N \models \text{ZFC}_k$ of $\langle V_\beta, \in \rangle^{M^+}$ with exactly stage $n$ successful at $\beta$, and preserved by forcing. $N$ top-extends $M$.

Build $N[G]$ to code $z$ at $\beta^+$. So $N[G] \models \{ x \mid \varphi_A(x) \} = z$, as desired. □
Process B

Turn now to process B.

This will work with $\omega$-standard models.

In order to ensure finiteness, we shall need to find a different way to descend.
Process B definition

Define $\varphi_B(x)$. 
Process B definition

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Stage $n$ succeeds, if there is $\exists$-fixed point $\gamma_n$, larger than previous $\gamma_i$, and ordinal $\lambda_n$, smaller than previous $\lambda_i$, such that after collapse forcing $\text{Coll}(\omega, \gamma_n)$, the now-countable structure $\langle V_{\gamma_n}, \in \rangle$ has no topped extension to a model $N \models \text{ZFC}$ satisfying “process B has exactly $n$ stages, uses $\gamma_n$ at stage $n$, and this is preserved by $P(\gamma_n)$-preserving forcing,” and such that, furthermore, this true $\Pi^1_1$ assertion in $V[g]$ is witnessed by the well-foundedness of a tree of rank $\lambda_n$. 
Process B definition

Define $\varphi_B(x)$.

Stage $n$ succeeds, if there is $\mathbb{P}$-fixed point $\gamma_n$, larger than previous $\gamma_i$, and ordinal $\lambda_n$, smaller than previous $\lambda_i$, such that after collapse forcing $\text{Coll}(\omega, \gamma_n)$, the now-countable structure $\langle V_{\gamma_n}, \in \rangle$ has no topped extension to a model $N \models \text{ZFC}$ satisfying “process B has exactly $n$ stages, uses $\gamma_n$ at stage $n$, and this is preserved by $P(\gamma_n)$-preserving forcing,” and such that, furthermore, this true $\Pi^1_1$ assertion in $V[g]$ is witnessed by the well-foundedness of a tree of rank $\lambda_n$.

In this case, let $y_n$ be the finite set coded into the GCH pattern at $\gamma_n^+$ and declare $\varphi_B(x)$ for all $x \in y_n$. 
Process B

Since the $\lambda_n$ are descending, there are at most finitely many successful stages.
Process B

Since the $\lambda_n$ are descending, there are at most finitely many successful stages.

Claim

If stage $n$ is successful, then $\lambda_n$ is a nonstandard ordinal.

Proof.

The tree can’t actually be well-founded, since the statement isn’t true in the ambient universe.

The extension property follows as with property A.
Unify A and B

Finally, we unify the two processes into one $\Sigma_2$ definition $\varphi(x)$. Run the processes concurrently, but only accept the results of process B if no stages in A have yet been successful.

Process A can only succeed in an $\omega$-nonstandard model, so once that happens, we get extension by the process A analysis. Can verify the process A extension property without having new instances of process B in the extension.

So can unify into one definition $\{ x \mid \varphi(x) \}$. This proves the universal finite set theorem. $\Box$. 
Applications of the universal finite set

Let us similarly discuss some applications of the universal finite set theorem.
No model of set theory has maximal $\Sigma_2$ theory

**Theorem**

No model of set theory $M$ has a maximal $\Sigma_2$ diagram. Indeed, there is a $\Sigma_2$ assertion $\sigma(n)$ with some natural-number parameter $n \in \omega^M$, which is not true in $M$ but is consistent with the $\Sigma_2$ diagram of $M$. 
No model of set theory has maximal $\Sigma_2$ theory

**Theorem**

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**Proof.**

Let $\sigma(n)$ be the assertion that stage $n$ is successful. Some stage is not successful in $M$, but could become successful in a top-extension. So this is a $\Sigma_2$ assertion about $n$ that is not yet true, but consistent with ZFC plus the $\Sigma_2$ diagram of $M$. 

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Maximal $\Sigma_2$ extensions of ZFC

It is easy to find maximal consistent $\Sigma_2$ theories $T$ extending ZFC.
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If $M \models T$, then $M$ must have stage $n$ successful for every standard $n$. 
Maximal $\Sigma_2$ extensions of ZFC

It is easy to find maximal consistent $\Sigma_2$ theories $T$ extending ZFC.

If $M \models T$, then $M$ must have stage $n$ successful for every standard $n$.

**Corollary**

No $\omega$-standard model of ZFC has a maximal $\Sigma_2$ theory.

In particular, no transitive model of ZFC has a maximal $\Sigma_2$ theory.

This raises issues for certain arguments in the philosophy of set theory.
Theorem

In any countable model of set theory $M$, every element becomes $\Sigma_2$ definable from a natural number parameter in some top-extension of $M$.

Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension $N$. 

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Theorem

In any countable model of set theory $M$, every element becomes $\Sigma_2$ definable from a natural number parameter in some top-extension of $M$.

Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension $N$.

Proof.

The definition is, “the unique set added at stage $n$,” where $n$ is the first unsuccessful stage in $M$. 

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Parameter-free $\Sigma_2$ definability

**Corollary**

For any countable $\omega$-standard model of set theory $M$, every $a \in M$ becomes $\Sigma_2$ definable without parameters in some top-extension $N$ of $M$.

Since $N$ is also $\omega$-standard, the result can be iterated.
The tree of top-extensions

This picture leads into the topic of potentialism, with which I’ve been recently preoccupied with several papers and talks.
The universal algorithm

Classical consequences

Universal finite set

Applications

Further generalizations

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<tbody>
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<td><strong>In the potentialist system consisting of the countable models of ZFC under top-extensions, considered as a Kripke model:</strong></td>
</tr>
<tr>
<td>1. <em>Exactly S4 is valid with respect to assertions in $L_{\in}$ with parameters.</em></td>
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<tr>
<td>2. <em>For any countable $M \models ZFC$, there is parameter $n \in \omega^M$ such that exactly S4 is valid with respect to assertions in $L_{\in}(n)$.</em></td>
</tr>
<tr>
<td>3. <em>For sentences, the validities are between S4 and S5.</em></td>
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<tr>
<td>4. <em>These bounds are sharp; both endpoints are realized.</em></td>
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</table>

The modal analysis raises certain issues with regard to the philosophy of potentialism versus actualism in the philosophy of mathematics.
Universal $\Sigma_1$-definable finite sequence

In recent work with myself, Kameryn Williams and Philip Welch, we have found a version of the theorem using $\Sigma_1$ definitions in set theory, working with end-extensions of models of any theory extending $\text{ZF}$. 

October 2018 Oxford
Theorem

Every c.e. extension $\overline{ZF} \supseteq ZF$ defines a $\Sigma_1$ sequence $a_0, a_1, \ldots, a_n$

with the following properties:

1. $ZF$ proves that the sequence is finite.
2. In any transitive model $M$ of $\overline{ZF}$, the sequence is empty.
3. If sequence is $s$ in countable $M \models \overline{ZF}$, then for every extension $s \subseteq t$ there is covering end-extension $M \subseteq N \models \overline{ZF}$ in which sequence is $t$.
4. It suffices merely that $M \models ZF$, provided there is an inner model $W \subseteq M$ with $W \models \overline{ZF}$. 
Infinitary-logic-free proof of Barwise theorem

The proof method of the universal algorithm provides a new proof of the Barwise extension theorem, without any use of Barwise compactness or indeed infinitary logic.

Corollary

Every countable model of ZF has an end-extension to a model of ZFC + $V = L$. 
Further work and questions

- Find universal finite sequences for other extension concepts.
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- View further classical results in the model theory of arithmetic and set theory as consequences of the universal finite set.
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- View further classical results in the model theory of arithmetic and set theory as consequences of the universal finite set.

- Remove the countability requirement from the set-theoretic case of universal finite set.

- Further applications to potentialism.

- View the universal finite sequence as a generalized Laver sequence or $\Diamond$ sequence; use this perspective to find new applications in forcing and set-theoretic combinatorics.
References


Thank you.


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