Bi-interpretation in set theory

Joel David Hamkins
Professor of Logic
Sir Peter Strawson Fellow

University of Oxford
University College, Oxford

University of Bristol
February 25, 2020
Joint work with:

Alfredo Roque Freire  
Professor of Logic  
University of Brasília, Brasília  
alfrfreire@gmail.com  
https://www.alfredoroquefreire.com/

Introduction

I should like to discuss the interpretation phenomenon in set theory.
Introduction

I should like to discuss the interpretation phenomenon in set theory.

Let’s begin by reviewing what it means to interpret one model in another or one theory in another.

This is a very general model-theoretic concept, which makes sense with any kind of model or theory.
Familiar examples of interpretation
Familiar examples of interpretation

The complex field $\mathbb{C}$ is interpretable in the real field $\mathbb{R}$

Represent complex number $a + bi$ with the pair $(a, b) \in \mathbb{R}^2$. 
Familiar examples of interpretation

<table>
<thead>
<tr>
<th>The complex field $\mathbb{C}$ is interpretable in the real field $\mathbb{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Represent complex number $a + bi$ with the pair $(a, b) \in \mathbb{R}^2$.</td>
</tr>
<tr>
<td>The complex field operations are definable:</td>
</tr>
<tr>
<td>$(a, b) + (c, d) = (a + c, b + d)$</td>
</tr>
<tr>
<td>$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$</td>
</tr>
<tr>
<td>Thus, one defines a copy of the complex field $\mathbb{C}$ inside $\mathbb{R}$.</td>
</tr>
</tbody>
</table>
Familiar examples of interpretation

The complex field \( \mathbb{C} \) is interpretable in the real field \( \mathbb{R} \)

Represent complex number \( a + bi \) with the pair \( (a, b) \in \mathbb{R}^2 \).
The complex field operations are definable:

\[
(a, b) + (c, d) = (a + c, b + d) \\
(a, b) \cdot (c, d) = (ac - bd, ad + bc)
\]

Thus, one defines a copy of the complex field \( \mathbb{C} \) inside \( \mathbb{R} \).

Conversely, \( \mathbb{R} \) is not actually interpretable in \( \mathbb{C} \), as fields.
Familiar examples of interpretation

The complex field \( \mathbb{C} \) is interpretable in the real field \( \mathbb{R} \)

Represent complex number \( a + bi \) with the pair \( (a, b) \in \mathbb{R}^2 \).

The complex field operations are definable:

\[
(a, b) + (c, d) = (a + c, b + d) \\
(a, b) \cdot (c, d) = (ac - bd, ad + bc)
\]

Thus, one defines a copy of the complex field \( \mathbb{C} \) inside \( \mathbb{R} \).

Conversely, \( \mathbb{R} \) is not actually interpretable in \( \mathbb{C} \), as fields.

But \( \mathbb{R} \) is interpretable in \( \langle \mathbb{C}, +, \cdot, \bar{z} \rangle \), with conjugation \( z \mapsto \bar{z} \), or in the complex plane \( \langle \mathbb{C}, +, \cdot, \text{Re}, \text{Im} \rangle \), which is bi-interpretable with \( \langle \mathbb{R}, +, \cdot \rangle \).
Interpretation of models and theories

**Integer ring** $\langle \mathbb{Z}, +, \cdot \rangle$ is interpretable in natural numbers $\langle \mathbb{N}, +, \cdot \rangle$

Every integer is the difference of two natural numbers.
```markdown
<table>
<thead>
<tr>
<th>Interpretation</th>
<th>Interpretation in set theory</th>
<th>Interpretation in ZF</th>
<th>Interpretation in ZFC</th>
<th>Interpretation in Z</th>
</tr>
</thead>
</table>

**Interpretation of models and theories**

### Integer ring \( \langle \mathbb{Z}, +, \cdot \rangle \) is interpretable in natural numbers \( \langle \mathbb{N}, +, \cdot \rangle \)

Every integer is the difference of two natural numbers. Interpret integers as \((n, m) \in \mathbb{N}^2\) under *same-difference* relation.

\[(n, m) \equiv (s, t) \iff n - m = s - t \iff n + t = s + m.\]

Integer addition and multiplication are well-defined.
```
Integer ring $\langle \mathbb{Z}, +, \cdot \rangle$ is interpretable in natural numbers $\langle \mathbb{N}, +, \cdot \rangle$

Every integer is the difference of two natural numbers. Interpret integers as $(n, m) \in \mathbb{N}^2$ under *same-difference* relation.

$$(n, m) \equiv (s, t) \iff n - m = s - t \iff n + t = s + m.$$  

Integer addition and multiplication are well-defined.

Rational field $\langle \mathbb{Q}, +, \cdot \rangle$ is interpretable in integer ring $\langle \mathbb{Z}, +, \cdot \rangle$

Rational number are represented as fractions $p/q$, essentially integer pairs $(p, q)$, with $q \neq 0$ under the *same ratio* relation.

$$\frac{p}{q} \sim \frac{r}{s} \iff ps = rq.$$  

The familiar fractional arithmetic is well-defined.
Finite set theory

The structure of hereditarily finite sets $\langle \text{HF}, \in \rangle$ is interpretable in arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. Use the Ackermann relation $n \in m$ $\iff$ the $n$th binary digit of $m$ is 1. This relation is definable in arithmetic and it is easily verified that $\langle \text{HF}, \in \rangle \models \langle \mathbb{N}, \in \rangle$. 
Finite set theory

The structure of hereditarily finite sets $\langle HF, \in \rangle$ is interpretable in arithmetic $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Use the Ackermann relation

$$n E m \iff n^{th} \text{ binary digit of } m \text{ is } 1.$$

This relation is definable in arithmetic and it is easily verified that $\langle HF, \in \rangle \cong \langle \mathbb{N}, E \rangle$. 
General definition

One structure $N = \langle N, R, f, c, \ldots \rangle$ is *interpreted* in another structure $M$ if there is a definable copy of $N$ inside $M$. 
General definition

One structure $N = \langle N, R, f, c, \ldots \rangle$ is interpreted in another structure $M$ if there is a definable copy of $N$ inside $M$.

More specifically, $\langle N, R, f, c, \ldots \rangle \cong \langle N^*, R^{N*}, f^{N*}, c^{N*}, \ldots \rangle / \sim$

- where $N^* \subseteq M^k$ is a definable set of $k$-tuples in $M$;
- $R^{N*}, f^{N*}, c^{N*}$ are $M$-definable relations/functions;
- $\sim$ is an $M$-definable equivalence relation, a congruence.
Some simplifications

In certain theories, some issues simplify.
Some simplifications

In certain theories, some issues simplify.

- In *sequential* theories, such as arithmetic and set theory, can eliminate need for \(k\)-tuples by internal coding.
Some simplifications

In certain theories, some issues simplify.

- In *sequential* theories, such as arithmetic and set theory, can eliminate need for $k$-tuples by internal coding.
- In models of arithmetic or set theory with global choice, can eliminate need for the equivalence relation $\simeq$ by picking least members.
Some simplifications

In certain theories, some issues simplify.

- In *sequential* theories, such as arithmetic and set theory, can eliminate need for $k$-tuples by internal coding.

- In models of arithmetic or set theory with global choice, can eliminate need for the equivalence relation $\simeq$ by picking least members.

- In ZF, even without global choice, can eliminate need for $\simeq$ via Scott’s trick with minimal rank representatives.
Some simplifications

In certain theories, some issues simplify.

- In *sequential* theories, such as arithmetic and set theory, can eliminate need for $k$-tuples by internal coding.

- In models of arithmetic or set theory with global choice, can eliminate need for the equivalence relation $\equiv$ by picking least members.

- In ZF, even without global choice, can eliminate need for $\equiv$ via Scott’s trick with minimal rank representatives.

- (foreshadowing: can’t generally eliminate $\equiv$ in ZFC$^-$)
Mutual interpretation of models

Models $M$ and $N$ are *mutually interpretable*, if each of them is interpreted in the other.
Mutual interpretation of models

Models $M$ and $N$ are *mutually interpretable*, if each of them is interpreted in the other.
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.

Each model is isomorphic to its iterated interpreted copy.

\[ M \sim N = M_{ij} \sim N_{ij} \]
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.

\[ M \sim = M^{j} : N^{*} \]
\[ N^{*} \sim = N \]

Bi-interpretation in set theory, Bristol 2020

Joel David Hamkins
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.
Mutual interpretations are naturally iterated

One finds copies within copies of the original models.

Each model is isomorphic to its iterated interpreted copy

\[ ji : M \cong \overline{M} \quad \text{and} \quad ij : N \cong \overline{N}. \]
Bi-interpretation

Models $M$ and $N$ are *bi-interpretable*, if they are mutually interpretable in such a way that the isomorphisms $ji : M \cong \overline{M}$ and $ij : N \cong \overline{N}$ arising in the interpretation are each definable in the original models.
A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model \( N \) with its interpreted copy inside \( M \).

This picture works with either mutual interpretation or bi-interpretation.
Cleaner picture

A cleaner picture emerges when we identify the model $N$ with its interpreted copy inside $M$.

This picture works with either mutual interpretation or bi-interpretation.
Synonymy

Models $M$ and $N$ are \textit{bi-interpretation synonymous}, also known as \textit{definitionally equivalent}, if there is a bi-interpretation for which the domains of the interpreted structures are in each case the whole structure and the equivalence relation is equality.
Synonymy

Models $M$ and $N$ are *bi-interpretable synonymous*, also known as *definitionally equivalent*, if there is a bi-interpretation for which the domains of the interpreted structures are in each case the whole structure and the equivalence relation is equality.

Every instance of bi-interpretation between models of ZF can be transformed to an instance of bi-interpretable synonymy.

- Don’t need $k$-tuples, since can encode sequences internally.
- Don’t need equivalence relations, by Scott’s trick.
- Can use whole domain, by Cantor-Schröder–Bernstein theorem for classes.
Interpretation of theories

It is traditional to consider interpretations of theories, rather than of models.
Interpretation of theories

It is traditional to consider interpretations of theories, rather than of models.

One theory $T_1$ is interpreted in another theory $T_2$, if one can uniformly define a model of $T_1$ inside any model of $T_2$. 
Interpretation of theories

It is traditional to consider interpretations of theories, rather than of models.

One theory $T_1$ is interpreted in another theory $T_2$, if one can uniformly define a model of $T_1$ inside any model of $T_2$.

There should be $\mathcal{L}_2$-formulas defining a domain of $k$-tuples, defining interpretations of the $\mathcal{L}_1$ structure and defining an equivalence relation, which provide recursively a translation of the $\mathcal{L}_1$ assertions into the language of $\mathcal{L}_2$,

$$\varphi \mapsto \varphi^*$$

in such a way that

$$T_1 \vdash \varphi \implies T_2 \vdash \varphi^*.$$ 

So theory $T_2$ proves that the interpretation is a model of $T_1$. 

Bi-interpretation in set theory, Bristol 2020
Joel David Hamkins
Mutual interpretation and bi-interpretation of theories

Theories $T_1$ and $T_2$ are *mutually interpretable*, if each of them is interpretable in the other.
Mutual interpretation and bi-interpretation of theories

Theories $T_1$ and $T_2$ are \textit{mutually interpretable}, if each of them is interpretable in the other.

Theories $T_1$ and $T_2$ are \textit{bi-interpretable}, if they are mutually interpretable in such a way that each model is provably definably isomorphic to its iterated interpreted copy.
Mutual interpretation and bi-interpretation of theories

Theories $T_1$ and $T_2$ are *mutually interpretable*, if each of them is interpretable in the other.

Theories $T_1$ and $T_2$ are *bi-interpretable*, if they are mutually interpretable in such a way that each model is provably definably isomorphic to its iterated interpreted copy.

For bi-interpretation, the theory $T_1$ proves that the universe is isomorphic, by a definable isomorphism map, to the model resulting by first interpreting to the defined model of $T_2$ and then interpreting to the model of $T_1$ inside that model; and similarly $T_2$ proves that its universe is definably isomorphic to the iterated interpreted model.
Interpretation in ZF set theory

There is an extremely robust mutual interpretability phenomenon in set theory.
The following theories are pairwise mutually interpretable.

1. ZF
2. ZFC
3. ZFC + GCH
4. ZFC + $V = L$
5. ZF + ¬AC
6. ZFC + ¬CH
7. ZFC + MA + ¬CH
8. ZFC + $b < d$
9. etc. etc. etc.

And many corresponding theorems for theories of higher consistency strength.
The following theories are pairwise mutually interpretable.

1. ZF
2. ZFC
3. ZFC + GCH
4. ZFC + V = L
5. ZF + ¬AC
6. ZFC + ¬CH
7. ZFC + MA + ¬CH
8. ZFC + b < ω
9. etc. etc. etc.

And many corresponding theorems for theories of higher consistency strength.
Inner models

The easy case occurs when one can define an inner model of the desired theory.

- $\text{ZFC}$ is interpretable in $\text{ZF}$.
Inner models

The easy case occurs when one can define an inner model of the desired theory.

- \( \text{ZFC} \) is interpretable in \( \text{ZF} \).
- \( \text{ZFC} + \text{CH} \) is interpretable in \( \text{ZF} \).
Inner models

The easy case occurs when one can define an inner model of the desired theory.

- ZFC is interpretable in ZF.
- ZFC + CH is interpretable in ZF.
- ZFC + $V = L_{\mu}$ is interpretable in ZFC + $\exists$ measurable cardinal.

In each case, we can go to a definable inner model where the interpreted theory holds.
Forcing

Meanwhile, forcing also provides an interpretation method.
Forcing

Meanwhile, forcing also provides an interpretation method.

To be sure, forcing is usually conceived as a way to define outer models, rather than inner models.
Forcing

Meanwhile, forcing also provides an interpretation method.

To be sure, forcing is usually conceived as a way to define outer models, rather than inner models.

Nevertheless, one can use forcing to define interpreted models by means of the Boolean ultrapower.
Interpretation via forcing

Suppose that $\mathbb{B}$ is a forcing notion in model $M$. 
Interpretation via forcing

Suppose that $\mathbb{B}$ is a forcing notion in model $M$. Let $U \subseteq \mathbb{B}$ ultrafilter in $M$. No need for genericity.
### Interpretation via forcing

Suppose that $\mathbb{B}$ is a forcing notion in model $M$.

Let $U \subseteq \mathbb{B}$ ultrafilter in $M$. No need for genericity.

Define Boolean ultrapower model $M^\mathbb{B}/U$, using

\[
\sigma =_U \tau \iff \left[ \sigma = \tau \right] \in U;
\]

\[
\sigma \in_U \tau \iff \left[ \sigma \in \tau \right] \in U.
\]
Interpretation via forcing

Suppose that $\mathbb{B}$ is a forcing notion in model $M$.

Let $U \subseteq \mathbb{B}$ ultrafilter in $M$. No need for genericity.

Define Boolean ultrapower model $M^\mathbb{B}/U$, using

$$
\sigma =_U \tau \iff \left[ [\sigma = \tau] \right] \in U;
\sigma \in_U \tau \iff \left[ [\sigma \in \tau] \right] \in U.
$$

The Łoś theorem shows

$$
M^\mathbb{B}/U \models \varphi \iff \left[ [\varphi] \right] \in U.
$$

So this is a model of everything forced by $\mathbb{B}$.
ZFC mutually interpretable with ZFC + ¬CH

To illustrate, let us interpret ZFC + ¬CH in ZFC.
ZFC mutually interpretable with \( \text{ZFC} + \neg \text{CH} \)

To illustrate, let us interpret \( \text{ZFC} + \neg \text{CH} \) in \( \text{ZFC} \).

To avoid parameters, we can define the Boolean ultrapower over a definable inner model, using a definable forcing and definable ultrafilter.
ZFC mutually interpretable with ZFC + ¬CH

To illustrate, let us interpret ZFC + ¬CH in ZFC.

To avoid parameters, we can define the Boolean ultrapower over a definable inner model, using a definable forcing and definable ultrafilter.

For example, in any model of ZFC, can define $L$ and the forcing $\text{Add}(\omega, \omega_2)^L$ and the $L$-least ultrafilter $U$ on Boolean completion $\mathcal{B}$. 
ZFC mutually interpretable with ZFC + ¬CH

To illustrate, let us interpret ZFC + ¬CH in ZFC.

To avoid parameters, we can define the Boolean ultrapower over a definable inner model, using a definable forcing and definable ultrafilter.

For example, in any model of ZFC, can define L and the forcing Add(ω, ω₂)⁰ and the L-least ultrafilter U on Boolean completion B.

Therefore, can define L^B / U, which is a model of ZFC + ¬CH.
From mutual interpretation to bi-interpretation?

Set theory supports a rich mutual interpretability phenomenon.
From mutual interpretation to bi-interpretation?

Set theory supports a rich mutual interpretability phenomenon. One interprets back and forth between models with AC and without, with CH and without, with certain features or others.
From mutual interpretation to bi-interpretation?

Set theory supports a rich mutual interpretability phenomenon. One interprets back and forth between models with AC and without, with CH and without, with certain features or others.

Question

Do these instances of mutual interpretation rise to the level of bi-interpretation?
From mutual interpretation to bi-interpretation?

Set theory supports a rich mutual interpretability phenomenon.

One interprets back and forth between models with AC and without, with CH and without, with certain features or others.

Question

Do these instances of mutual interpretation rise to the level of bi-interpretation?

In particular, can one get back home to the original model, rather than merely back to some model of the original theory?
From mutual interpretation to bi-interpretation?

Set theory supports a rich mutual interpretability phenomenon. One interprets back and forth between models with AC and without, with CH and without, with certain features or others.

**Question**

Do these instances of mutual interpretation rise to the level of bi-interpretation?

In particular, can one get back home to the original *model*, rather than merely back to some model of the original *theory*?

If not, does following an interpretation in set theory necessarily involve the loss of information?
Automatic bi-interpretability

Theorem

If a well-founded model $M$ of $\text{ZF}^-$ is interpreted in itself via $i: M \rightarrow \overline{M}/\simeq$, then $i$ is unique and definable.
Theorem

If a well-founded model $M$ of $\mathsf{ZF}^-$ is interpreted in itself via $i : M \to \overline{M}/\simeq$, then $i$ is unique and definable.

Proof.

Assume $\langle M, \in \rangle \models \mathsf{ZF}^-$ is interpreted in itself $i : \langle M, \in \rangle \cong \langle \overline{M}, \in \rangle / \simeq$. 
Theorem

If a well-founded model $M$ of $\text{ZF}^-$ is interpreted in itself via $i : M \rightarrow \overline{M}/\simeq$, then $i$ is unique and definable.

Proof.

Assume $\langle M, \in \rangle \models \text{ZF}^-$ is interpreted in itself $i : \langle M, \in \rangle \cong \langle \overline{M}, \bar{\in} \rangle/\simeq$. The relation $\bar{\in}$ is well-founded and extensional (modulo $\simeq$).
**Theorem**

If a well-founded model $M$ of $\text{ZF}^-$ is interpreted in itself via $i : M \to \overline{M}/\simeq$, then $i$ is unique and definable.

**Proof.**

Assume $\langle M, \in \rangle \models \text{ZF}^-$ is interpreted in itself $i : \langle M, \in \rangle \cong \langle \overline{M}, \in \rangle/\simeq$. The relation $\in$ is well-founded and extensional (modulo $\simeq$).

Furthermore, one can prove it is sufficiently set-like.
Theorem

If a well-founded model $M$ of $\text{ZF}^-$ is interpreted in itself via $i : M \rightarrow M/\sim$, then $i$ is unique and definable.

Proof.

Assume $\langle M, \in \rangle \models \text{ZF}^-$ is interpreted in itself $i : \langle M, \in \rangle \cong \langle M, \bar{\in} \rangle/\sim$. The relation $\bar{\in}$ is well-founded and extensional (modulo $\sim$).

Furthermore, one can prove it is sufficiently set-like.

Necessarily, $i$ is the inverse of the Mostowski collapse.
Theorem

If a well-founded model $M$ of $\text{ZF}^-$ is interpreted in itself via $i : M \to M/\simeq$, then $i$ is unique and definable.

Proof.

Assume $\langle M, \in \rangle \models \text{ZF}^-$ is interpreted in itself $i : \langle M, \in \rangle \cong \langle \overline{M}, \in \rangle/\simeq$.

The relation $\in$ is well-founded and extensional (modulo $\simeq$).

Furthermore, one can prove it is sufficiently set-like.

Necessarily, $i$ is the inverse of the Mostowski collapse.

So the map is definable.
Automatic bi-interpretability

Corollary

Every instance of mutual interpretation amongst well-founded models of $\text{ZF}^-$ is a bi-interpretation. Indeed, if $M$ is a well-founded model of $\text{ZF}^-$ and mutually interpreted with any structure $N$ of any theory, as in the figure below, then the isomorphism $i : M \rightarrow \overline{M}$ is definable in $M$. 

![Diagram showing a bi-interpretation between $M$ and $N$]
B i-interpretation in $ZF$ set theory

We explained the robust mutual interpretation phenomenon in set theory.
B i-interpretation in ZF set theory

We explained the robust mutual interpretation phenomenon in set theory.

Meanwhile, there is actually no nontrivial bi-interpretation phenomenon to be found.
B i-interpretation in $\text{ZF}$ set theory

We explained the robust mutual interpretation phenomenon in set theory.

Meanwhile, there is actually no nontrivial bi-interpretation phenomenon to be found.

Theorem (Enayat [Ena16])

1. Distinct non-isomorphic models of $\text{ZF}$ are never bi-interpretable. $\text{ZF}$ is solid.
B i-interpretation in $\mathsf{ZF}$ set theory

We explained the robust mutual interpretation phenomenon in set theory.

Meanwhile, there is actually no nontrivial bi-interpretation phenomenon to be found.

Theorem (Enayat [Ena16])

1. Distinct non-isomorphic models of $\mathsf{ZF}$ are never bi-interpretable. $\mathsf{ZF}$ is solid.
2. Distinct theories extending $\mathsf{ZF}$ are never bi-interpretable. $\mathsf{ZF}$ is tight.
Theorem (Enayat [Ena16])

*Distinct non-isomorphic models of ZF are never bi-interpretable.*
Theorem (Enayat [Ena16])

*Distinct non-isomorphic models of ZF are never bi-interpretable.*

Proof. Assume $M$ and $N$ are bi-interpretable.
Theorem (Enayat [Ena16])

Distinct non-isomorphic models of ZF are never bi-interpretable.

Proof. Assume $M$ and $N$ are bi-interpretable.

$N$ must see $\in^M$ as well-founded. So $\text{Ord}^N$ and $\text{Ord}^M$ are comparable.
Theorem (Enayat [Ena16])

**Distinct non-isomorphic models of ZF are never bi-interpretable.**

Proof. Assume $M$ and $N$ are bi-interpretable.

$N$ must see $\in^M$ as well-founded. So $\text{Ord}^N$ and $\text{Ord}^M$ are comparable.

If $\alpha \in M$, $\bar{\alpha} \in \bar{M}$, $\alpha^* \in N$ isomorphic, then

$$\langle V_\alpha, \in \rangle^M \cong \langle V_{\bar{\alpha}}, \in \rangle^{\bar{M}} \cong \langle V_{\alpha^*}, \in \rangle^N.$$  

by induction.
### Theorem (Enayat [Ena16])

**Distinct non-isomorphic models of ZF are never bi-interpretable.**

Proof. Assume $M$ and $N$ are bi-interpretable.

$N$ must see $\in^M$ as well-founded. So $\text{Ord}^N$ and $\text{Ord}^M$ are comparable.

If $\alpha \in M$, $\bar{\alpha} \in \overline{M}$, $\alpha^* \in N$ isomorphic, then

$$\langle V_\alpha, \in \rangle^M \cong \langle V_{\bar{\alpha}}, \in \rangle^{\overline{M}} \cong \langle V_{\alpha^*}, \in \rangle^N.$$  

by induction.

The isomorphism is unique, because transitive sets are rigid.
Theorem (Enayat [Ena16])

**Distinct non-isomorphic models of ZF are never bi-interpretable.**

Proof. Assume $M$ and $N$ are bi-interpretable.

$N$ must see $\in^M$ as well-founded. So $\text{Ord}^N$ and $\text{Ord}^M$ are comparable.

If $\alpha \in M$, $\bar{\alpha} \in \overline{M}$, $\alpha^* \in N$ isomorphic, then

$$\langle V_\alpha, \in \rangle^M \cong \langle V_{\bar{\alpha}}, \in \rangle^{\overline{M}} \cong \langle V_{\alpha^*}, \in \rangle^N.$$  

by induction.

The isomorphism is unique, because transitive sets are rigid.

If $\text{Ord}^N < \text{Ord}^M$, then $M$ will see universe bijective with a set, contradiction.
Theorem (Enayat [Ena16])

**Distinct non-isomorphic models of ZF are never bi-interpretable.**

Proof. Assume $M$ and $N$ are bi-interpretable.

$N$ must see $E^M$ as well-founded. So $\text{Ord}^N$ and $\text{Ord}^M$ are comparable.

If $\alpha \in M$, $\bar{\alpha} \in \bar{M}$, $\alpha^* \in N$ isomorphic, then

$$\langle V_\alpha, E^M \rangle \simeq \langle V_{\bar{\alpha}}, E^{\bar{M}} \rangle \simeq \langle V_{\alpha^*}, E^N \rangle.$$  

by induction.

The isomorphism is unique, because transitive sets are rigid.

If $\text{Ord}^N < \text{Ord}^M$, then $M$ will see universe bijective with a set, contradiction.

And similarly if $\text{Ord}^M < \text{Ord}^N$. So $\langle M, E^M \rangle \simeq \langle N, E^N \rangle$, as desired. □
Corollary

**ZF is tight.** That is, distinct theories extending ZF are never bi-interpretable.

Proof.

Every solid theory is tight.
Corollary

*ZF is tight.* That is, distinct theories extending ZF are never bi-interpretable.

Proof.

Every solid theory is tight.

In particular, ZF is not bi-interpretable with ZFC, nor with ZFC+CH, nor ZFC + ¬CH and so on.
Corollary

ZF is tight. That is, distinct theories extending ZF are never bi-interpretable.

Proof.

Every solid theory is tight.

In particular, ZF is not bi-interpretable with ZFC, nor with ZFC+CH, nor ZFC + ¬CH and so on.

ZFC+ large cardinals are not bi-interpretable with determinacy axioms or with canonical-inner-model hypotheses.
<table>
<thead>
<tr>
<th>Interpretation in set theory</th>
<th>Interpretation in ZF</th>
<th>Interpretation in ZFC</th>
<th>Interpretation in Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interpretable on all models</td>
<td>Interpretable on all models</td>
<td>No bi-interpretation in ZF</td>
<td>No bi-interpretation in ZF</td>
</tr>
</tbody>
</table>

**Corollary**

ZF is tight. That is, distinct theories extending ZF are never bi-interpretable.

**Proof.**

Every solid theory is tight.

In particular, ZF is not bi-interpretable with ZFC, nor with ZFC+CH, nor ZFC + ¬CH and so on.

ZFC+ large cardinals are not bi-interpretable with determinacy axioms or with canonical-inner-model hypotheses.

There is no nontrivial bi-interpretation phenomenon in set theory amongst the models or theories strengthening ZF.
History

- Albert Visser [Vis04] proved corresponding result for PA.
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
- Enayat proved the general theorem [Ena16] for all extensions of ZF, also for KM.
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
- Enayat proved the general theorem [Ena16] for all extensions of ZF, also for KM.
- Observed independently by H. Friedman and Visser.
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
- Enayat proved the general theorem [Ena16] for all extensions of ZF, also for KM.
- Observed independently by H. Friedman and Visser.
- Observed independently by Fedor Pakhomov.
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
- Enayat proved the general theorem [Ena16] for all extensions of ZF, also for KM.
- Observed independently by H. Friedman and Visser.
- Observed independently by Fedor Pakhomov.
- Observed independently by Freire and myself [Ham18].
History

- Albert Visser [Vis04] proved corresponding result for PA.
- Enayat proved ZF and ZFC not bi-interpretable, using involutions in automorphism groups.
- Enayat proved the general theorem [Ena16] for all extensions of ZF, also for KM.
- Observed independently by H. Friedman and Visser.
- Observed independently by Fedor Pakhomov.
- Observed independently by Freire and myself [Ham18].
- Result also follows from internal categoricity result of Väänänen [Vä19].
Well-founded models lack even mutual interpretability

We had seen a robust mutual interpretability amongst diverse theories extending of $\text{ZF}$. 
Well-founded models lack even mutual interpretability

We had seen a robust mutual interpretability amongst diverse theories extending of ZF.

And yet:

Theorem

Nonisomorphic well-founded models of ZF are never mutually interpretable.
Well-founded models lack even mutual interpretability

We had seen a robust mutual interpretability amongst diverse theories extending of $\mathsf{ZF}$.

And yet:

**Theorem**

*Nonisomorphic well-founded models of $\mathsf{ZF}$ are never mutually interpretable.*

**Proof.**

Every instance of mutual interpretation amongst the well-founded models of $\mathsf{ZF}$ is a bi-interpretation, but bi-interpretation occurs only between isomorphic models.
Interpretation: necessary loss of information

Well-established mutual interpretation between theories

- ZFC plus large cardinals
- ZF plus AD determinacy hypotheses
- Large cardinal canonical inner model hypotheses
Interpretation: necessary loss of information

Well-established mutual interpretation between theories

- ZFC plus large cardinals
- ZF plus AD determinacy hypotheses
- Large cardinal canonical inner model hypotheses

And yet, one cannot stay with well-founded models when following these mutual interpretations, because mutually interpretable well-founded models are isomorphic.
Interpretation: necessary loss of information

Well-established mutual interpretation between theories

- ZFC plus large cardinals
- ZF plus AD determinacy hypotheses
- Large cardinal canonical inner model hypotheses

And yet, one cannot stay with well-founded models when following these mutual interpretations, because mutually interpretable well-founded models are isomorphic.

One cannot get by interpretation back to the original model, even if one gets back to a model of the original theory.
Internal categoricity

Theorem (Väänänen [Vä19])

If \( \langle V, \in, \bar{\in} \rangle \) is a model of ZF(\( \in, \bar{\in} \)), then

\[ \langle V, \in \rangle \cong \langle V, \bar{\in} \rangle. \]

Furthermore, there is a unique definable isomorphism in
\( \langle V, \in, \bar{\in} \rangle \).

The hypothesis asserts, more precisely:

- ZF\(_\in(\bar{\in})\), using \( \in \) as membership and \( \bar{\in} \) as predicate; and
- ZF\(_{\bar{\in}}(\in)\), using \( \bar{\in} \) as membership and \( \in \) as predicate.
<table>
<thead>
<tr>
<th>Interpretation</th>
<th>Interpretation in set theory</th>
<th>Interpretation in $ZF$</th>
<th>Interpretation in $ZFC^{-}$</th>
<th>Interpretation in $Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No bi-interpretation in $ZF$ set theory</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Internal categoricity theorem (Väänänen)

If $\langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in})$, then $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$. 

**Proof.**

Assume $\langle V, \in, \bar{\in} \rangle$ satisfies $ZF(\in, \bar{\in})$, in common language using either $\in$ or $\bar{\in}$ as membership. Both $\in$ and $\bar{\in}$ are seen as well-founded by the other. So $\text{Ord} \langle V, \in \rangle$ and $\text{Ord} \langle V, \bar{\in} \rangle$ are comparable. Assume $\text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle$.

So every $\in$-ordinal $\alpha$ corresponds to $\bar{\in}$-ordinal $\alpha$.

Prove inductively that $\langle V_{\alpha}, \in \rangle \cong \langle V_{\alpha}, \bar{\in} \rangle$.

If $\text{Ord} \langle V, \in \rangle < \text{Ord} \langle V, \bar{\in} \rangle$, then $\langle V, \in \rangle$ is isomorphic to some $\langle V_{\gamma}, \in \rangle$.

So $\bar{\in}$-universe $V$ is bijective with a set, contradiction.

So $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$.
Internal categoricity theorem (Väänänen)

If $\langle V, \in, \in \rangle \models ZF(\in, \in)$, then $\langle V, \in \rangle \cong \langle V, \in \rangle$.

Proof.

Assume $\langle V, \in, \in \rangle$ satisfies $ZF(\in, \in)$, in common language using either $\in$ or $\in$ as membership.
Internal categoricity theorem (Väänänen)

If \( \langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in}) \), then \( \langle V, \in \rangle \cong \langle V, \bar{\in} \rangle \).

Proof.

Assume \( \langle V, \in, \bar{\in} \rangle \) satisfies \( ZF(\in, \bar{\in}) \), in common language using either \( \in \) or \( \bar{\in} \) as membership.

Both \( \in \) and \( \bar{\in} \) are seen as well-founded by the other.
Interpretation in set theory
Interpretation in ZF
Interpretation in ZFC
Interpretation in Z

No bi-interpretation in ZF set theory

Internal categoricity theorem (Väänänen)

If $\langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in})$, then $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$.

Proof.

Assume $\langle V, \in, \bar{\in} \rangle$ satisfies $ZF(\in, \bar{\in})$, in common language using either $\in$ or $\bar{\in}$ as membership.

Both $\in$ and $\bar{\in}$ are seen as well-founded by the other.

So $\text{Ord} \langle V, \in \rangle$ and $\text{Ord} \langle V, \bar{\in} \rangle$ are comparable. Assume $\text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle$. 

Bi-interpretation in set theory, Bristol 2020 Joel David Hamkins
**Internal categoricity theorem (Väänänen)**

If $\langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in})$, then $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$.

**Proof.**

Assume $\langle V, \in, \bar{\in} \rangle$ satisfies $ZF(\in, \bar{\in})$, in common language using either $\in$ or $\bar{\in}$ as membership.

Both $\in$ and $\bar{\in}$ are seen as well-founded by the other.

So $\text{Ord} \langle V, \in \rangle$ and $\text{Ord} \langle V, \bar{\in} \rangle$ are comparable. Assume $\text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle$.

So every $\in$-ordinal $\alpha$ corresponds to $\bar{\in}$-ordinal $\bar{\alpha}$. 

---

No bi-interpretation in ZF set theory
<table>
<thead>
<tr>
<th>Interpretation in set theory</th>
<th>Interpretation in ZF</th>
<th>Interpretation in ZFC^-</th>
<th>Interpretation in Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>No bi-interpretation in ZF set theory</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Internal categoricity theorem (Väänänen)**

If \( \langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in}) \), then \( \langle V, \in \rangle \cong \langle V, \bar{\in} \rangle \).

**Proof.**

Assume \( \langle V, \in, \bar{\in} \rangle \) satisfies \( ZF(\in, \bar{\in}) \), in common language using either \( \in \) or \( \bar{\in} \) as membership.

Both \( \in \) and \( \bar{\in} \) are seen as well-founded by the other.

So \( \text{Ord} \langle V, \in \rangle \) and \( \text{Ord} \langle V, \bar{\in} \rangle \) are comparable. Assume \( \text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle \).

So every \( \in \)-ordinal \( \alpha \) corresponds to \( \bar{\in} \)-ordinal \( \bar{\alpha} \).

Prove inductively that \( \langle V_\alpha, \in \rangle \langle V, \in \rangle \cong \langle V_{\bar{\alpha}}, \bar{\in} \rangle \langle V, \bar{\in} \rangle \).
Interpretation in set theory

No bi-interpretation in ZF set theory

Internal categoricity theorem (Väänänen)

If $\langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in})$, then $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$.

Proof.

Assume $\langle V, \in, \bar{\in} \rangle$ satisfies $ZF(\in, \bar{\in})$, in common language using either $\in$ or $\bar{\in}$ as membership.

Both $\in$ and $\bar{\in}$ are seen as well-founded by the other.

So $\text{Ord} \langle V, \in \rangle$ and $\text{Ord} \langle V, \bar{\in} \rangle$ are comparable. Assume $\text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle$.

So every $\in$-ordinal $\alpha$ corresponds to $\bar{\in}$-ordinal $\bar{\alpha}$.

Prove inductively that $\langle V_\alpha, \in \rangle \langle V, \in \rangle \cong \langle V_{\bar{\alpha}}, \bar{\in} \rangle \langle V, \bar{\in} \rangle$.

If $\text{Ord} \langle V, \in \rangle < \text{Ord} \langle V, \bar{\in} \rangle$, then $\langle V, \in \rangle$ is isomorphic to some $\langle V_\gamma, \bar{\in} \rangle$. 
Internal categoricity theorem (Väänänen)

If $\langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in})$, then $\langle V, \in \rangle \cong \langle V, \bar{\in} \rangle$.

Proof.

Assume $\langle V, \in, \bar{\in} \rangle$ satisfies $ZF(\in, \bar{\in})$, in common language using either $\in$ or $\bar{\in}$ as membership.

Both $\in$ and $\bar{\in}$ are seen as well-founded by the other.

So $\text{Ord} \langle V, \in \rangle$ and $\text{Ord} \langle V, \bar{\in} \rangle$ are comparable. Assume $\text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle$.

So every $\in$-ordinal $\alpha$ corresponds to $\bar{\in}$-ordinal $\bar{\alpha}$.

Prove inductively that $\langle V_\alpha, \in \rangle \langle V, \in \rangle \cong \langle V_{\bar{\alpha}}, \bar{\in} \rangle \langle V, \bar{\in} \rangle$.

If $\text{Ord} \langle V, \in \rangle < \text{Ord} \langle V, \bar{\in} \rangle$, then $\langle V, \in \rangle$ is isomorphic to some $\langle \bar{V}_\gamma, \bar{\in} \rangle$.

So $\bar{\in}$-universe $V$ is bijective with a set, contradiction.
**Internal categoricity theorem (Väänänen)**

If \( \langle V, \in, \bar{\in} \rangle \models ZF(\in, \bar{\in}) \), then \( \langle V, \in \rangle \cong \langle V, \bar{\in} \rangle \).

**Proof.**

Assume \( \langle V, \in, \bar{\in} \rangle \) satisfies \( ZF(\in, \bar{\in}) \), in common language using either \( \in \) or \( \bar{\in} \) as membership.

Both \( \in \) and \( \bar{\in} \) are seen as well-founded by the other.

So \( \text{Ord} \langle V, \in \rangle \) and \( \text{Ord} \langle V, \bar{\in} \rangle \) are comparable. Assume \( \text{Ord} \langle V, \in \rangle \leq \text{Ord} \langle V, \bar{\in} \rangle \).

So every \( \in \)-ordinal \( \alpha \) corresponds to \( \bar{\in} \)-ordinal \( \bar{\alpha} \).

Prove inductively that \( \langle V_{\alpha}, \in \rangle^{\langle V, \in \rangle} \cong \langle V_{\bar{\alpha}}, \bar{\in} \rangle^{\langle V, \bar{\in} \rangle} \).

If \( \text{Ord} \langle V, \in \rangle < \text{Ord} \langle V, \bar{\in} \rangle \), then \( \langle V, \in \rangle \) is isomorphic to some \( \langle \overline{V}_{\gamma}, \bar{\in} \rangle \).

So \( \bar{\in} \)-universe \( V \) is bijective with a set, contradiction.

So \( \langle V, \in \rangle \cong \langle V, \bar{\in} \rangle \).
Zermelo’s quasi-categoricity theorem

The internal categoricity argument is similar in important respects to Zermelo’s 1930 quasi-categoricity argument, showing that for any two models of $\text{ZF}_2$, one of them is isomorphic to a rank-initial segment of the other.
Tightness via internal categoricity

Let me explain how solidity and tightness for $\text{ZF}$ follows from the internal categoricity theorem.
Tightness via internal categoricity

Let me explain how solidity and tightness for $ZF$ follows from the internal categoricity theorem.

If two models of $ZF$ are bi-interpretable, then using Scott’s trick and class Cantor-Schröder-Bernstein, they can be placed in synonymy. And so we produce an instance $\langle V, \in, \bar{\in} \rangle$, where each relation is definable from the other. This gives $ZF(\in, \bar{\in})$. So by internal categoricity, they are isomorphic.
Tightness via internal categoricity

Let me explain how solidity and tightness for $\text{ZF}$ follows from the internal categoricity theorem.

If two models of $\text{ZF}$ are bi-interpretable, then using Scott’s trick and class Cantor-Schröder-Bernstein, they can be placed in synonymy. And so we produce an instance $\langle V, \in, \bar{\in} \rangle$, where each relation is definable from the other. This gives $\text{ZF}(\in, \bar{\in})$. So by internal categoricity, they are isomorphic.

For theories where the synonymy methods work, therefore, one can view internal categoricity as a strengthening of solidity/tightness, dropping the definability requirements.
Fundamental use of the $V_\alpha$ hierarchy

I had found it curious that in both cases, the proofs of solidity and tightness and the proof of internal categoricity make fundamental use of the $V_\alpha$ hierarchy.
Fundamental use of the $V_\alpha$ hierarchy

I had found it curious that in both cases, the proofs of solidity and tightness and the proof of internal categoricity make fundamental use of the $V_\alpha$ hierarchy.

I tried hard to prove the theorem via $\in$-recursion, rather than $V_\alpha$ recursion.
Fundamental use of the $V_\alpha$ hierarchy

I had found it curious that in both cases, the proofs of solidity and tightness and the proof of internal categoricity make fundamental use of the $V_\alpha$ hierarchy.

I tried hard to prove the theorem via $\in$-recursion, rather than $V_\alpha$ recursion.

But I found no such proof. This suggested the question:

Do the results hold for $\text{ZFC}_-$, without power set?
Fundamental use of the $V_\alpha$ hierarchy

I had found it curious that in both cases, the proofs of solidity and tightness and the proof of internal categoricity make fundamental use of the $V_\alpha$ hierarchy.

I tried hard to prove the theorem via $\in$-recursion, rather than $V_\alpha$ recursion.

But I found no such proof. This suggested the question:

**Question**

Do the results hold for $\text{ZFC}^-$, without power set?
Does solidity require full strength?

Enayat also had observed also that his proof seemed to require the full strength of ZF and of KM. He inquired whether this was necessary?
Does solidity require full strength?

Enayat also had observed also that his proof seemed to require the full strength of $\text{ZF}$ and of $\text{KM}$. He inquired whether this was necessary?

**Question**

Can one prove tightness and internal categoricity for weak set theories?
Internal categoricity fails for \( \text{ZFC}^- \)

For \( \text{ZFC}^- \), set theory without power set, the answer is no for internal categoricity.
Internal categoricity fails for $\text{ZFC}^-$

For $\text{ZFC}^-$, set theory without power set, the answer is no for internal categoricity.

Theorem (Freire, Hamkins)

*There is a transitive model $\langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in})$, where $\langle M, \in \rangle$ is not isomorphic to $\langle M, \bar{\in} \rangle$, both well-founded.*
Internal categoricity fails for $\text{ZFC}^-$

For $\text{ZFC}^-$, set theory without power set, the answer is no for internal categoricity.

Theorem (Freire, Hamkins)

There is a transitive model $\langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in})$, where $\langle M, \in \rangle$ is not isomorphic to $\langle M, \bar{\in} \rangle$, both well-founded.

We shall provide the counterexample model.
Theorem (Freire, Hamkins)

There is a transitive model \( \langle M, \in, \bar{\in} \rangle \models ZFC^- (\in, \bar{\in}) \), where \( \langle M, \in \rangle \) is not isomorphic to \( \langle M, \bar{\in} \rangle \), both well-founded.
Theorem (Freire, Hamkins)

There is a transitive model \( \langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in}), \) where \( \langle M, \in \rangle \) is not isomorphic to \( \langle M, \bar{\in} \rangle, \) both well-founded.

Proof.

To start, assume Luzin’s hypothesis, \( 2^\omega = 2^{\omega_1}. \)
**Theorem (Freire, Hamkins)**

*There is a transitive model* $\langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in})$, *where* $\langle M, \in \rangle$ *is not isomorphic to* $\langle M, \bar{\in} \rangle$, *both well-founded.*

**Proof.**

To start, assume Luzin’s hypothesis, $2^\omega = 2^{\omega_1}$.

So $H_{\omega_1}$ and $H_{\omega_2}$ are equinumerous.
Theorem (Freire, Hamkins)

*There is a transitive model $\langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in})$, where $\langle M, \in \rangle$ is not isomorphic to $\langle M, \bar{\in} \rangle$, both well-founded.*

Proof.

To start, assume Luzin’s hypothesis, $2^\omega = 2^{\omega_1}$.

So $H_{\omega_1}$ and $H_{\omega_2}$ are equinumerous. Fix bijection $\pi : H_{\omega_1} \to H_{\omega_2}$. 
Theorem (Freire, Hamkins)

There is a transitive model \( \langle M, \in, \in \rangle \models \text{ZFC}^-(\in, \in) \), where \( \langle M, \in \rangle \) is not isomorphic to \( \langle M, \in \rangle \), both well-founded.

Proof.

To start, assume Luzin’s hypothesis, \( 2^\omega = 2^{\omega_1} \).

So \( H_{\omega_1} \) and \( H_{\omega_2} \) are equinumerous. Fix bijection \( \pi : H_{\omega_1} \to H_{\omega_2} \).

Transfer the \( \in \) relations forward and back to form an isomorphism

\[
\pi : \langle H_{\omega_1}, \in, \in \rangle \cong \langle H_{\omega_2}, \tilde{\in}, \in \rangle.
\]
Theorem (Freire, Hamkins)

There is a transitive model \( \langle M, \in, \bar{\in} \rangle \models \text{ZFC}^- (\in, \bar{\in}) \), where \( \langle M, \in \rangle \) is not isomorphic to \( \langle M, \bar{\in} \rangle \), both well-founded.

Proof.

To start, assume Luzin’s hypothesis, \( 2^\omega = 2^{\omega_1} \).

So \( H_{\omega_1} \) and \( H_{\omega_2} \) are equinumerous. Fix bijection \( \pi : H_{\omega_1} \to H_{\omega_2} \).

Transfer the \( \in \) relations forward and back to form an isomorphism

\[
\pi : \langle H_{\omega_1}, \in, \bar{\in} \rangle \cong \langle H_{\omega_2}, \bar{\in}, \in \rangle.
\]

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \models \text{ZFC}^- (\bar{\in}) \), since one can add any predicate at all.
Theorem (Freire, Hamkins)

There is a transitive model \( \langle M, \in, \bar{\in} \rangle \models ZFC^{-}(\in, \bar{\in}) \), where \( \langle M, \in \rangle \) is not isomorphic to \( \langle M, \bar{\in} \rangle \), both well-founded.

Proof.

To start, assume Luzin’s hypothesis, \( 2^{\omega} = 2^{\omega_1} \).

So \( H_{\omega_1} \) and \( H_{\omega_2} \) are equinumerous. Fix bijection \( \pi : H_{\omega_1} \rightarrow H_{\omega_2} \).

Transfer the \( \in \) relations forward and back to form an isomorphism

\[ \pi : \langle H_{\omega_1}, \in, \bar{\in} \rangle \cong \langle H_{\omega_2}, \bar{\in}, \in \rangle. \]

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \models ZFC_{\bar{\in}}(\bar{\in}) \), since one can add any predicate at all.

Similarly, \( \langle H_{\omega_2}, \bar{\in}, \in \rangle \models ZFC_{\bar{\in}}(\bar{\in}) \).
Theorem (Freire, Hamkins)

*There is a transitive model* \( \langle M, \in, \bar{\in} \rangle \models ZFC^- (\in, \bar{\in}) \), *where* \( \langle M, \in \rangle \) *is not isomorphic to* \( \langle M, \bar{\in} \rangle \), *both well-founded.*

Proof.

To start, assume Luzin’s hypothesis, \( 2^\omega = 2^{\omega_1} \).

So \( H_{\omega_1} \) and \( H_{\omega_2} \) are equinumerous. Fix bijection \( \pi : H_{\omega_1} \to H_{\omega_2} \).

Transfer the \( \in \) relations forward and back to form an isomorphism

\[ \pi : \langle H_{\omega_1}, \in, \bar{\in} \rangle \cong \langle H_{\omega_2}, \tilde{\in}, \bar{\in} \rangle. \]

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \models ZFC^- (\bar{\in}) \), since one can add any predicate at all.

Similarly, \( \langle H_{\omega_2}, \tilde{\in}, \bar{\in} \rangle \models ZFC^- (\bar{\in}) \).

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \) satisfies \( ZFC^- (\in, \bar{\in}) \), violating internal categoricity.
Theorem (Freire, Hamkins)

*There is a transitive model* \( \langle M, \in, \bar{\in} \rangle \models \text{ZFC}^-(\in, \bar{\in}) \), *where* \( \langle M, \in \rangle \) *is not isomorphic to* \( \langle M, \bar{\in} \rangle \), *both well-founded.*

**Proof.**

To start, assume Luzin’s hypothesis, \( 2^\omega = 2^{\omega_1} \).

So \( H_{\omega_1} \) and \( H_{\omega_2} \) are equinumerous. Fix bijection \( \pi : H_{\omega_1} \to H_{\omega_2} \).

Transfer the \( \in \) relations forward and back to form an isomorphism

\[
\pi : \langle H_{\omega_1}, \in, \bar{\in} \rangle \cong \langle H_{\omega_2}, \bar{\in}, \in \rangle.
\]

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \models \text{ZFC}^-_{\bar{\in}} (\bar{\in}) \), since one can add any predicate at all.

Similarly, \( \langle H_{\omega_2}, \bar{\in}, \in \rangle \models \text{ZFC}^-_{\bar{\in}} (\bar{\in}) \).

So \( \langle H_{\omega_1}, \in, \bar{\in} \rangle \) satisfies \( \text{ZFC}^- (\in, \bar{\in}) \), violating internal categoricity.

For outright existence, omit Luzin via Shoenfield absoluteness.
Nonsolidity of $\text{ZFC}^-$

But to show $\text{ZFC}^-$ is not solid, we need such a model $\langle M, \in, \bar{\in} \rangle$ where the relations are not merely fulfilling $\text{ZFC}^-(\in, \bar{\in})$ but definable with respect to the other.
Nonsolidity of $\text{ZFC}^-$

But to show $\text{ZFC}^-$ is not solid, we need such a model $\langle M, \in, \bar{\in} \rangle$ where the relations are not merely fulfilling $\text{ZFC}^- (\in, \bar{\in})$ but definable with respect to the other.

Theorem (Freire, Hamkins)

*It is relatively consistent with ZFC that $\langle H_{\omega_1}, \in \rangle$ and $\langle H_{\omega_2}, \in \rangle$ are bi-interpretable.*
Nonsolidity of ZFC$^-$

But to show ZFC$^-$ is not solid, we need such a model $\langle M, \in, \inbar \rangle$ where the relations are not merely fulfilling $\text{ZFC}^-(\in, \inbar)$ but definable with respect to the other.

Theorem (Freire, Hamkins)

*It is relatively consistent with ZFC that* $\langle H_{\omega_1}, \in \rangle$ *and* $\langle H_{\omega_2}, \in \rangle$ *are bi-interpretable.*

Thus, there can be two well-founded models of ZFC$^-$ that are bi-interpretable, but not isomorphic.
Nonsolidity of $\mathsf{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing $\mathsf{MA}$.
Nonsolidity of $\text{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing $\text{MA}$.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$. 
Nonsolidity of $\text{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing MA.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$.

Every element $x \in H_{\omega_2}$ is coded by a set $A \subseteq \omega_1$. 
Nonsolidity of $\text{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing $\text{MA}$.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$.

Every element $x \in H_{\omega_2}$ is coded by a set $A \subseteq \omega_1$.

By $\text{MA}$ every $A \subseteq \omega_1$ is almost-disjoint encoded by some $a \subseteq \omega$. 
Nonsolidity of $\text{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing $\text{MA}$.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$.

Every element $x \in H_{\omega_2}$ is coded by a set $A \subseteq \omega_1$.

By $\text{MA}$ every $A \subseteq \omega_1$ is almost-disjoint encoded by some $a \subseteq \omega$.

In $H_{\omega_1}$ can define relations

\[
\begin{align*}
\bar{a} &\simeq b \iff \text{code the same set} \\
\bar{a} &\in b \iff \text{codes an element}
\end{align*}
\]
Nonsolidity of ZFC

We use the Solovay-Tennenbaum model $L[G]$ forcing MA.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$.

Every element $x \in H_{\omega_2}$ is coded by a set $A \subseteq \omega_1$.

By MA every $A \subseteq \omega_1$ is almost-disjoint encoded by some $a \subseteq \omega$.

In $H_{\omega_1}$ can define relations

\[ a \simeq b \iff \text{code the same set} \]

\[ a \bar{\in} b \iff \text{codes an element} \]

Inside $H_{\omega_1}$, define $\langle W, \bar{\in} \rangle / \simeq$, where $W$ are the codes.

Isomorphic to $\langle H_{\omega_2}, \in \rangle$. 
Nonsolidity of $\text{ZFC}^-$

We use the Solovay-Tennenbaum model $L[G]$ forcing $\text{MA}$.

There is a definable almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$ in $L$.

Every element $x \in H_{\omega_2}$ is coded by a set $A \subseteq \omega_1$.

By $\text{MA}$ every $A \subseteq \omega_1$ is almost-disjoint encoded by some $a \subseteq \omega$.

In $H_{\omega_1}$ can define relations

$$a \simeq b \iff \text{code the same set}$$

$$a \bar{\in} b \iff \text{codes an element}$$

Inside $H_{\omega_1}$, define $\langle W, \bar{\in} \rangle / \simeq$, where $W$ are the codes. Isomorphic to $\langle H_{\omega_2}, \in \rangle$.

Both $H_{\omega_1}$ and $H_{\omega_2}$ can see how the coding works, and from this one can show it is a bi-interpretation. □
Achieving synonymy for $H_{\omega_1}$ and $H_{\omega_2}$

**Theorem (Freire, Hamkins)**

*It is relatively consistent with ZFC that there is relation $\in$ definable in $\langle H_{\omega_1}, \in \rangle$ for which*

$$\langle H_{\omega_1}, \in \rangle \cong \langle H_{\omega_2}, \in \rangle,$$

*which makes $\langle H_{\omega_1}, \in \rangle$ and $\langle H_{\omega_2}, \in \rangle$ bi-interpretation synonymous.*
Achieving synonymy for $H_{\omega_1}$ and $H_{\omega_2}$

**Theorem (Freire, Hamkins)**

*It is relatively consistent with ZFC that there is relation $\in$ definable in $\langle H_{\omega_1}, \in \rangle$ for which*

$$\langle H_{\omega_1}, \in \rangle \cong \langle H_{\omega_2}, \in \rangle,$$

*which makes $\langle H_{\omega_1}, \in \rangle$ and $\langle H_{\omega_2}, \in \rangle$ bi-interpretation synonymous.*

Use Harrington [Har77], obtaining $\text{MA} + \neg \text{CH}$, with a projectively definable well-order of the reals. (Thanks to observation of Gabe Goldberg.)
Achieving synonymy for $H_{\omega_1}$ and $H_{\omega_2}$

**Theorem (Freire, Hamkins)**

*It is relatively consistent with ZFC that there is relation $\in$ definable in $\langle H_{\omega_1}, \in \rangle$ for which*

\[ \langle H_{\omega_1}, \in \rangle \cong \langle H_{\omega_2}, \in \rangle, \]

*which makes $\langle H_{\omega_1}, \in \rangle$ and $\langle H_{\omega_2}, \in \rangle$ bi-interpretation synonymous.*

Use Harrington [Har77], obtaining $\text{MA} + \neg \text{CH}$, with a projectively definable well-order of the reals. (Thanks to observation of Gabe Goldberg.)

This allows one to pick representatives, and avoid the quotient.
Meanwhile

In stronger large cardinal settings, however, we cannot expect to interpret $H_{\omega_2}$ inside $H_{\omega_1}$.

**Theorem**

*If there is no projectively definable $\omega_1$-sequence of distinct reals, then $\langle H_{\omega_2}, \in \rangle$ cannot be interpreted in $\langle H_{\omega_1}, \in \rangle$. In particular, in this case the structures are not bi-interpretable nor even mutually interpretable.*

The hypothesis is a consequence of sufficient large cardinals, since it is a consequence of $\text{AD}^L(\mathbb{R})$. 
ZFC$^-$ is not solid

Can appeal to absoluteness to get the outright result, instead of mere consistency.

Theorem (Freire, Hamkins)

*The theory ZFC$^-$ is not solid, not even for well-founded models. Indeed, there are transitive models $\langle M, \in \rangle, \langle N, \in \rangle$ of ZFC$^-$ that form a bi-interpretation synonymy, but are not isomorphic.*
ZFC$^-$ is not solid

Can appeal to absoluteness to get the outright result, instead of mere consistency.

**Theorem (Freire, Hamkins)**

The theory ZFC$^-$ is not solid, not even for well-founded models. Indeed, there are transitive models $\langle M, \in \rangle$, $\langle N, \in \rangle$ of ZFC$^-$ that form a bi-interpretation synonymy, but are not isomorphic.

**Proof.**

There are such transitive sets in $L[G]$. Can find countable such sets. Apply Shoenfield absoluteness to get them in $V$.  

---

*Bi-interpretation in set theory, Bristol 2020*  
Joel David Hamkins
ZFC\(^{-}\) is not tight

Theorem (Freire, Hamkins)

\[ \text{ZFC}^{-} \text{ is not tight.} \]
ZFC\(^{-}\) is not tight

Theorem (Freire, Hamkins)

ZFC\(^{-}\) is not tight.

Proof.

Let \(T_1\) and \(T_2\) be theories describing the situation of \(\langle H_\omega_1, \in \rangle\) and \(\langle H_\omega_2, \in \rangle\) in the previous theorem.
**ZFC^- is not tight**

**Theorem (Freire, Hamkins)**

**ZFC^- is not tight.**

**Proof.**

Let $T_1$ and $T_2$ be theories describing the situation of $\langle H_{\omega_1}, \in \rangle$ and $\langle H_{\omega_2}, \in \rangle$ in the previous theorem.

So $T_2$ asserts ZFC^- plus $\omega_1$ exists but not $\omega_2$, that $\omega_1 = \omega_1^L$, that $\omega_2 = \omega_2^L$, and that every subset of $\omega_1$ is coded by a real using the almost-disjoint coding with respect to the $L$-least almost-disjoint family $\langle a_\alpha \mid \alpha < \omega_1 \rangle$. 
**ZFC**⁻ is not tight

**Theorem (Freire, Hamkins)**

\[ \text{ZFC}^\neg \text{ is not tight.} \]

**Proof.**

Let \( T_1 \) and \( T_2 \) be theories describing the situation of \( \langle H_{\omega^1}, \in \rangle \) and \( \langle H_{\omega^2}, \in \rangle \) in the previous theorem.

So \( T_2 \) asserts \( \text{ZFC}^\neg \) plus \( \omega^1 \) exists but not \( \omega^2 \), that \( \omega^1 = \omega^L_1 \), that \( \omega^2 = \omega^L_2 \), and that every subset of \( \omega^1 \) is coded by a real using the almost-disjoint coding with respect to the \( L \)-least almost-disjoint family \( \langle a_\alpha \mid \alpha < \omega^1 \rangle \).

\( T_1 \) asserts \( \text{ZFC}^\neg \) plus every set is countable and that the interpretation of \( H_{\omega^2} \) in \( H_{\omega^1} \) used above defines a model of \( T_2 \).
**ZFC− is not tight**

**Theorem (Freire, Hamkins)**

\[ \text{ZFC}^\neg \text{ is not tight.} \]

**Proof.**

Let \( T_1 \) and \( T_2 \) be theories describing the situation of \( \langle H_{\omega_1}, \in \rangle \) and \( \langle H_{\omega_2}, \in \rangle \) in the previous theorem.

So \( T_2 \) asserts \( \text{ZFC}^\neg \) plus \( \omega_1 \) exists but not \( \omega_2 \), that \( \omega_1 = \omega_1^L \), that \( \omega_2 = \omega_2^L \), and that every subset of \( \omega_1 \) is coded by a real using the almost-disjoint coding with respect to the \( L \)-least almost-disjoint family \( \langle a_\alpha \mid \alpha < \omega_1 \rangle \).

\( T_1 \) asserts \( \text{ZFC}^\neg \) plus every set is countable and that the interpretation of \( H_{\omega_2} \) in \( H_{\omega_1} \) used above defines a model of \( T_2 \).

These two theories are bi-interpretable, but incompatible. \( \square \)
Zermelo set theory

Let’s now consider Zermelo set theory $\mathcal{Z}$. 
Zermelo set theory

Let’s now consider Zermelo set theory $\mathbb{Z}$.  

**Theorem (Freire, Hamkins)**

1. $\mathbb{Z}$ is not solid, not even for well-founded models. There are bi-interpretable well-founded models of Zermelo set theory that are not isomorphic.
Zermelo set theory

Let's now consider Zermelo set theory $\mathcal{Z}$.

**Theorem (Freire, Hamkins)**

1. **$\mathcal{Z}$ is not solid, not even for well-founded models.** *There are bi-interpretable well-founded models of Zermelo set theory that are not isomorphic.*

2. **$\mathcal{Z}$ is not tight.** *There are distinct bi-interpretable strengthenings of $\mathcal{Z}$.*
Zermelo set theory

Let’s now consider Zermelo set theory \( Z \).

**Theorem (Freire, Hamkins)**

1. \( Z \) is not solid, not even for well-founded models. There are bi-interpretable well-founded models of Zermelo set theory that are not isomorphic.

2. \( Z \) is not tight. There are distinct bi-interpretable strengthenings of \( Z \).

3. Every model of \( ZF \) is bi-interpretable with a transitive inner model of Zermelo set theory, with prescribed failures of replacement.
Mathias slim model technique

We use Mathias’s slim model construction [Mat01].
Mathias slim model technique

We use Mathias’s slim model construction [Mat01].

A class $C$ is *fruitful*, if

1. every $x \in C$ is transitive;
2. $\text{Ord} \subseteq C$;
3. $x \in C$ and $y \in C$ implies $x \cup y \in C$;
4. $x \in C$ and $y \subseteq P(x)$ implies $x \cup y \in C$. 
Mathias slim model technique

We use Mathias’s slim model construction [Mat01].

A class \( C \) is *fruitful*, if

1. every \( x \in C \) is transitive;
2. \( \text{Ord} \subseteq C \);
3. \( x \in C \) and \( y \in C \) implies \( x \cup y \in C \);
4. \( x \in C \) and \( y \subseteq P(x) \) implies \( x \cup y \in C \).

**Theorem (Mathias [Mat01, prop. 1.2])**

*If \( C \) is fruitful, then \( M = \bigcup C \) is a supertransitive model of Zermelo set theory with the foundation axiom.*
Interpretation

Interpretation in set theory

Interpretation in ZF

Interpretation in ZFC

Interpretation in Z

Zermelo set theory is neither solid nor tight

Mathias slim model technique

We use Mathias’s slim model construction [Mat01].

A class \( C \) is **fruitful**, if

1. every \( x \in C \) is transitive;
2. \( \text{Ord} \subseteq C \);
3. \( x \in C \) and \( y \in C \) implies \( x \cup y \in C \);
4. \( x \in C \) and \( y \subseteq P(x) \) implies \( x \cup y \in C \).

**Theorem (Mathias [Mat01, prop. 1.2])**

*If \( C \) is fruitful, then \( M = \bigcup C \) is a supertransitive model of Zermelo set theory with the foundation axiom.*

Key idea: construct fruitful classes by specifying allowed rate-of-growth \(|x \cap V_n|\).
Zermelo set theory is neither solid nor tight

**Slim model**

One such slim model $M$ has sets $x$ obeying rate of growth

$$\exists k \, \forall n \quad |\text{TC}(x) \cap V_n| \leq 2^{2^{\cdot\cdot\cdot 2^n}} \} k.$$
Zermelo set theory is neither solid nor tight

**Slim model**

One such slim model $M$ has sets $x$ obeying rate of growth

$$\exists k \forall n \quad |TC(x) \cap V_n| \leq 2^{2^{n^k}}.$$  

This does not include $V_\omega$ itself.
Slim model

One such slim model $M$ has sets $x$ obeying rate of growth

$$\exists k \forall n \quad |\text{TC}(x) \cap V_n| \leq 2^{2^{\cdots 2^n}}$$

This does not include $V_\omega$ itself.

This slim model $M$ is a transitive model of Zermelo with foundation, containing all ordinals, in which $V_\omega$ does not exist.
V is bi-interpretable with slim model M

We claim the original ZF model \( \langle V, \in \rangle \) is bi-interpretable with the slim model \( M \).
**V is bi-interpretable with slim model M**

We claim the original ZF model $\langle V, \in \rangle$ is bi-interpretable with the slim model $M$.

Fix $a \in M$. Build Zermelo tower:

$$\emptyset^{(a)} = a.$$
$$x^{(a)} = \{y^{(a)} \mid y \in x\}$$
$$V^{(a)} = \{x^{(a)} \mid x \in V\} \subseteq M$$

We replace all hereditary copies of $\emptyset$ in $x$ with $a$.

The map $x \mapsto x^{(a)}$ is isomorphism $\langle V, \in \rangle$ with $\langle V^{(a)}, \in \rangle$.

Can define $V^{(a)}$ inside $M$: all $\in$-descents pass through $a$.

So this is a bi-interpretation of $\langle V, \in \rangle$ with $\langle M, \in \rangle$. 
Zermelo set theory is neither solid nor tight

**V is bi-interpretable with slim model M**

We claim the original ZF model $\langle V, \in \rangle$ is bi-interpretable with the slim model $M$.

Fix $a \in M$. Build Zermelo tower:

- $\emptyset^{(a)} = a$
- $x^{(a)} = \{ y^{(a)} \mid y \in x \}$
- $V^{(a)} = \{ x^{(a)} \mid x \in V \} \subseteq M$

We replace all hereditary copies of $\emptyset$ in $x$ with $a$. 
Zermelo set theory is neither solid nor tight

V is bi-interpretable with slim model M

We claim the original ZF model \( \langle V, \in \rangle \) is bi-interpretable with the slim model \( M \).

Fix \( a \in M \). Build Zermelo tower:

\[
\emptyset(a) = a.
\]

\[
x(a) = \{ y(a) \mid y \in x \}
\]

\[
V(a) = \{ x(a) \mid x \in V \} \subseteq M
\]

We replace all hereditary copies of \( \emptyset \) in \( x \) with \( a \).

The map \( x \mapsto x(a) \) is isomorphism \( \langle V, \in \rangle \) with \( \langle V(a), \in \rangle \).
V is bi-interpretable with slim model M

We claim the original ZF model $\langle V, \in \rangle$ is bi-interpretable with the slim model $M$.

Fix $a \in M$. Build Zermelo tower:

$\emptyset^{(a)} = a.$

$x^{(a)} = \{ y^{(a)} \mid y \in x \}$

$V^{(a)} = \{ x^{(a)} \mid x \in V \} \subseteq M$

We replace all hereditary copies of $\emptyset$ in $x$ with $a$.

The map $x \mapsto x^{(a)}$ is isomorphism $\langle V, \in \rangle$ with $\langle V^{(a)}, \in \rangle$.

Can define $V^{(a)}$ inside $M$: all $\in$-descents pass through $a$. 

Bi-interpretation in set theory, Bristol 2020 Joel David Hamkins
V is bi-interpretable with slim model M

We claim the original ZF model $\langle V, \in \rangle$ is bi-interpretable with the slim model $M$.

Fix $a \in M$. Build Zermelo tower:

$$\emptyset^{(a)} = a.$$  
$$x^{(a)} = \{ y^{(a)} \mid y \in x \}.$$  
$$V^{(a)} = \{ x^{(a)} \mid x \in V \} \subseteq M$$

We replace all hereditary copies of $\emptyset$ in $x$ with $a$.

The map $x \mapsto x^{(a)}$ is isomorphism $\langle V, \in \rangle$ with $\langle V^{(a)}, \in \rangle$.

Can define $V^{(a)}$ inside $M$: all $\in$-descents pass through $a$.

So this is a bi-interpretation of $\langle V, \in \rangle$ with $\langle M, \in \rangle$. 

Zermelo set theory is neither solid nor tight
Zermelo set theory is neither solid nor tight

We’ve proved that every ZF model $\langle V, \in \rangle$ is bi-interpretable with a model $M$ of Zermelo set theory.

So Z is not solid.

Consider theories describing the situation. Let ZM assert Z plus the assertion that the Zermelo tower $V^{(\omega)}$ is a model of ZF, and that the universe $M$ is isomorphic to $M^{(\omega)}$ by our map.
Zermelo set theory is neither solid nor tight

We’ve proved that every ZF model $\langle V, \in \rangle$ is bi-interpretable with a model $M$ of Zermelo set theory.

So $Z$ is not solid.

Consider theories describing the situation. Let $ZM$ assert $Z$ plus the assertion that the Zermelo tower $V^{(\omega)}$ is a model of ZF, and that the universe $M$ is isomorphic to $M^{(\omega)}$ by our map.

These theories are different, but bi-interpretable, so $Z$ is not tight.
Zermelo set theory is neither solid nor tight

**Flexibility about which $V_{\lambda}$ is excluded**

The construction is flexible as to which $V_\alpha$ we will exclude from the slim model.
Flexibility about which $V_\lambda$ is excluded

The construction is flexible as to which $V_\alpha$ we will exclude from the slim model.

We can include $V_\omega$ and $V_\alpha$ for all $\alpha$ up to some desired limit ordinal $\lambda$, but $V_\lambda$ is excluded.
Model-by-model bi-interpretation

Consider bi-interpretation in models vs. theories.

**Definition**

Theories $T_1$, $T_2$ are *model-by-model* bi-interpretable if every model of one is bi-interpretable with a model of the other.

In effect we drop the uniformity requirement on the interpretation.

It could be different interpretations that work in some models than in others, with perhaps no uniform interpretation.
Theorem

There are theories $T_1$ and $T_2$ that are model-by-model bi-interpretable, but not bi-interpretable.

Proof.

Consider the theories

1. $T_1 = \text{ZF}$.
2. $T_2 = \text{ZF} \lor \text{ZM} = \{\alpha \lor \beta \mid \alpha \in \text{ZF}, \beta \in \text{ZM}\}$.

Every model of ZF is bi-interpretable with itself.
Theorem

There are theories $T_1$ and $T_2$ that are model-by-model bi-interpretable, but not bi-interpretable.

Proof.

Consider the theories

1. $T_1 = \text{ZF}$.
2. $T_2 = \text{ZF} \lor \text{ZM} = \{ \alpha \lor \beta \mid \alpha \in \text{ZF}, \beta \in \text{ZM} \}$.

Every model of $\text{ZF}$ is bi-interpretable with itself.

Conversely, every model of $T_2$ is either a model of $\text{ZF}$ or of $\text{ZM}$, which is bi-interpretable with a model of $\text{ZF}$.
Theorem

There are theories $T_1$ and $T_2$ that are model-by-model bi-interpretable, but not bi-interpretable.

Proof.

Consider the theories

1. $T_1 = \text{ZF}$.
2. $T_2 = \text{ZF} \lor \text{ZM} = \{\alpha \lor \beta \mid \alpha \in \text{ZF}, \beta \in \text{ZM}\}$.

Every model of ZF is bi-interpretable with itself.

Conversely, every model of $T_2$ is either a model of ZF or of ZM, which is bi-interpretable with a model of ZF.

But not bi-interpretable: let $M \models \text{ZM} + \neg \text{ZF}$, interpret $N \models \text{ZF}$, hence $T_2$, so interpret further $N^* \models \text{ZF}$. $N$ and $N^*$ bi-interpretable, hence isomorphic. But interpreting back to $T_1$ from $N$ or $N^*$ produces $M$ and $N$, not isomorphic. Contradiction.
<table>
<thead>
<tr>
<th>Interpretation</th>
<th>Interpretation in set theory</th>
<th>Interpretation in ZF</th>
<th>Interpretation in ZFC</th>
<th>Interpretation in Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>solid</td>
<td>solid</td>
<td>solid</td>
<td>solid</td>
<td>solid</td>
</tr>
<tr>
<td>tight</td>
<td>tight</td>
<td>tight</td>
<td>tight</td>
<td>tight</td>
</tr>
</tbody>
</table>

Zermelo set theory is neither solid nor tight

Summary

- Set theory has a robust mutual interpretation phenomenon.
Summary

- Set theory has a robust mutual interpretation phenomenon.
- But there is no nontrivial bi-interpretation for $\text{ZF}$ and stronger.
Summary

- Set theory has a robust mutual interpretation phenomenon.
- But there is no nontrivial bi-interpretation for ZF and stronger.
- The moral: by following the mutual interpretations of set theory, you can never go back home.
Summary

- Set theory has a robust mutual interpretation phenomenon.
- But there is no nontrivial bi-interpretation for ZF and stronger.
- The moral: by following the mutual interpretations of set theory, you can never go back home.
- Meanwhile, bi-interpretation occurs in weak set theories, such as ZFC$^-$ and Z.
Summary

- Set theory has a robust mutual interpretation phenomenon.
- But there is no nontrivial bi-interpretation for ZF and stronger.
- The moral: by following the mutual interpretations of set theory, you can never go back home.
- Meanwhile, bi-interpretation occurs in weak set theories, such as \( \text{ZFC}^- \) and \( Z \).
- Even \( H_{\omega_1} \) and \( H_{\omega_2} \) can be bi-interpretable.
Summary

- Set theory has a robust mutual interpretation phenomenon.
- But there is no nontrivial bi-interpretation for $\text{ZF}$ and stronger.
- The moral: by following the mutual interpretations of set theory, you can never go back home.
- Meanwhile, bi-interpretation occurs in weak set theories, such as $\text{ZFC}^-$ and $\text{Z}$.
- Even $H_{\omega_1}$ and $H_{\omega_2}$ can be bi-interpretable.
- Every $\text{ZF}$ model is bi-interpretable with a slim Zermelo inner model.
Thank you.


Joel David Hamkins
Oxford University
Zermelo set theory is neither solid nor tight

References I


<table>
<thead>
<tr>
<th>Interpretation in set theory</th>
<th>Interpretation in ZF</th>
<th>Interpretation in ZFC⁻</th>
<th>Interpretation in Z</th>
</tr>
</thead>
</table>

Zermelo set theory is neither solid nor tight.

## References II

### Joel David Hamkins.

**Different set theories are never bi-interpretable.** Mathematics and Philosophy of the Infinite. 2018.


### Leo Harrington.


### Adrian RD Mathias.

Zermelo set theory is neither solid nor tight

References III
