A tutorial in set-theoretic geology

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Part of the work I shall discuss is collaborative work with Gunter Fuchs, College of Staten Island of CUNY, and Jonas Reitz, New York City Tech, CUNY.

A preprint of our joint paper, "Set-theoretic geology," which introduces the topic, is available at http://arxiv.org/abs/1107.4776.

Philosophy of set theory

A fascinating debate is currently underway in the emerging area known as the philosophy of set theory:

- **Universe view**. The philosophical position that there is an absolute set-theoretic background, in which set-theoretic questions will have their final answers. The task of set theory is to discover those answers.
- **multiverse view**. The philosophical position that there are numerous distinct but often closely-related concepts of set, giving rise to distinct set-theoretic universes. The task of set theory is to investigate these universes and their interconnections.

Set-theoretic geology

Part of the evidence for the multiverse position is what set-theorists have discovered is the enormous diversity of set-theoretic possibility.

On the multiverse perspective, a major part of the goal of set theory is to understand the structural relations between all these various set-theoretic worlds.

Set-theoretic geology aims to fulfill the part of this goal as it arises in consideration of forcing, one of the principal set-theoretic methods for building alternative set-theoretic worlds.

In set-theoretic geology, we seek to study the set-theoretic universe in the context of all its grounds and forcing extensions.

Forcing

Forcing is a set-theoretic method (Cohen, 1962) for constructing a larger model of set theory from a given model.

Begin with a ground model $V \models \mathrm{ZFC}$ and poset $\mathbb{P} \in V$. Adjoin an ideal "generic" element G, a V-generic filter $G \subseteq \mathbb{P}$, and with it construct the forcing extension $V[G]$, akin to a field extension.

$$V \subseteq V[G]$$

Objects of $V[G]$ are constructible algebraically from objects in V and the new object G. The ground model V has a surprising degree of access to the objects and truths of $V[G]$.

Forcing has been used to construct diverse models of set theory.

A new perspective

Forcing is naturally viewed as a method of building *outer* as opposed to *inner* models of set theory.

Nevertheless, a simple switch in perspective enables forcing as a method of producing inner models as well. Namely, we look for how the universe V itself might have arisen via forcing.

A transitive class $W \subseteq V$ is a *ground* if the universe was obtained by set forcing over W. That is, if $V = W[G]$ for some W-generic filter $G \subseteq \mathbb{P} \in W$.

This change in viewpoint leads to *set-theoretic geology*.

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A new fundamental question

Set-theoretic geology begins with a theorem of Laver (2004), which answers a fundamental question about forcing that could have been asked decades earlier, but which was not.

Question

Is the ground model V definable in its forcing extensions $V[G]$?

It turns out that it is.

Geology begins by recognizing the ground

Theorem (Laver 2007, independently Woodin)

The universe V is a definable class in every set-forcing extension $V[G]$.

There is a first-order formula φ in the language of set theory and parameter $r \in V$ such that

$$x \in V \qquad \Longleftrightarrow \qquad V[G] \models \varphi(x, r).$$

The proof uses my methods on approximation and covering, and if there is time later I shall explain the definition in detail.

The Ground Axiom

Upon learning of Laver's theorem, Jonas Reitz and I formulated the Ground Axiom. The idea was to try to undo forcing, to do it 'backwards'.

Definition (Hamkins,Reitz)

The *Ground Axiom* is the assertion that the universe V has no nontrivial grounds.

That is, V satisfies the ground axiom GA if there is no transitive inner model $W \models \mathrm{ZFC}$ such that $V = W[G]$ for some nontrivial W-generic filter $G \subseteq \mathbb{P} \in W$.

In other words, the ground axiom asserts that the universe V was not obtained by forcing.

GA is first order

The ground axiom asserts that the universe is not a nontrivial forcing extension of any transitive inner model.

In other words, GA asserts that whenever W is a transitive proper class model of ZFC and $G \subseteq \mathbb{P}$ is W-generic for nontrivial forcing $\mathbb{P}$, then $V \neq W[G]$.

Although this formulation of the ground axiom is *prima facie* second-order, since it quantifies over inner models, nevertheless Reitz proved that the ground axiom is first-order axiomatizable.

Theorem (Reitz)

The ground axiom is first-order expressible in set theory.

Grounds

Definition

A transitive class W is a *ground* of V if $W \models \mathrm{ZFC}$ and $V = W[G]$ for some W-generic $G \subseteq \mathbb{P} \in W$.

Laver's theorem shows that each ground W is definable in V from parameter r:

$$x \in W \quad \text{if and only if} \quad V \models \varphi(x, r).$$

So by varying the parameter r, we reach all possible grounds.

Possibly some parameters r do not succeed in defining a ground. Reitz observed that whether $\varphi(x, r)$ defines a ground is a first-order property of r.

The grounds form a parameterized family

Theorem

There is a parameterized family $\{ W_r \mid r \in V \}$ *such that*

1. *Every* W_r *is a ground of* V *and* $r \in W_r$.
2. *Every ground of* V *is* W_r *for some* r.
3. *The relation "*$x \in W_r$*" is first order.*

This reduces second-order statements about grounds to first-order statements about parameters.

For example, the relation "$V = W_r[G]$ by W_r-generic filter $G \subseteq \mathbb{P} \in W_r$" is first order in arguments $(r, G, \mathbb{P})$.

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Reducing Second to First order

Thus, we may quantify over grounds by quantifying over the parameters used to define them.

Conclusion

The ground axiom is equivalent to the first-order assertion $\forall r\ V = W_r$.

Second-order questions about grounds W_r become first-order questions about the index r.

Natural models of GA

The ground axiom holds in many canonical models of set theory:

- The constructible universe L.
- Extensions $L[0^\sharp]$, $L[\mu]$, $L[\vec{E}]$.
- Many other canonical inner models of large cardinals, including many instances of the core model.

These models are among the most highly regular models of set theory that are known. They exhibit numerous highly attractive structural features, such as the GCH, diamond, $V = \mathrm{HOD}$, condensation and so on.

Consequences of GA

Since all the known models of GA were highly structured models, satisfying GCH diamond, and so on, it was natural to inquire to what extent these regularity features were consequences of GA.

Test Question

Does the ground axiom imply CH?

After all, the only way we know how to violate CH is by forcing, and under GA the universe is not a forcing extension, so an affirmative answer seems reasonable.

Obtaining GA in an extension

Nevertheless, GA does not settle CH.

Theorem (Reitz)

Every model of ZFC *has an extension, preserving any desired* V_α*, which is a model of* GA*.*

It follows that the ground axiom is compatible with any set-theoretic behavior that can occur inside any V_α of any model of set theory.

This includes CH, $\neg$CH, $\Diamond$, $\neg\Diamond$ and so on. Every Σ_2 statement that is consistent with ZFC is consistent with ZFC + GA.

Thus, the ground axiom has essentially NO regularity consequences.

Forcing GA

Theorem (Reitz)

Every model of ZFC *has an extension, preserving any desired* V_α*, which is a model of* GA*.*

The paradoxical situation is that although GA asserts that the universe is not obtained by forcing, Reitz obtains GA in a forcing extension, but by using class forcing.

The point is that GA is concerned only with set forcing extensions, for which $\mathbb{P}$ is a set, and we can perform proper class forcing whose resulting extension is not obtainable by set forcing over any model.

Reitz's method is to force a very strong version of $V = \mathrm{HOD}$.

Making sets definable

Suppose x is an arbitrary set of natural numbers. Perhaps x is not definable. Can we make it definable in a forcing extension?

Yes. Easton's theorem gives us complete control over the GCH pattern on regular cardinals. So we may find a forcing extension $V[G]$ in which

$$n \in x \qquad \Longleftrightarrow \qquad 2^{\aleph_n} = \aleph_{n+1}.$$

Thus, the set x becomes definable in the forcing extension.

We may now iterate this idea to make every set definable from ordinal parameters.

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Continuum coding axiom

Definition

The continuum coding axiom CCA is the assertion that every set of ordinals is coded into the GCH pattern on a block of regular cardinals.

That is, whenever $x \subseteq \gamma$, then there is an ordinal λ such that $\alpha \in x \iff 2^{\aleph_{\lambda+\alpha+1}} = \aleph_{\lambda+\alpha+2}$ for all $\alpha < \gamma$.

Theorem (Folklore)

There is a class forcing extension $V[G]$ in which the CCA holds.

Forcing the CCA

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Proof.

Traditional method: bookkeeping iteration.

Simpler method: generic coding. Let $\mathbb{P}$ be the Easton-support forcing iteration, whose conditions may decide generically at stage α, using the lottery sum, either to force the GCH at $\aleph_{\alpha+1}$ or to force $\neg\mathrm{GCH}$ at $\aleph_{\alpha+1}$. The generic filter decides which.

It is dense that any set $x \subseteq \mathrm{ORD}$ that is added is subsequently coded into the GCH pattern "generic bookkeeping".

Consequently, $V[G] \models \mathrm{CCA}$.

Theorem (Reitz)

The continuum coding axiom implies the ground axiom.

Proof.

Suppose $V \models \mathrm{CCA}$ and $V = W[g]$ for some $g \subseteq \mathbb{P} \in W$. Since the GCH pattern is not affected above $|\mathbb{P}|$, it follows that every set in V is coded into W, and so $V \subseteq W$ and so $V = W$. $\square$

By starting the forcing above α, Reitz obtains:

Corollary

Every model V of set theory has a class forcing extension $V[G]$, preserving any desired V_α, which is a model of GA.

Consequently, GA has no regularity consequences such as CH or diamond.

An amusing application: MA+GA is consistent

Martin's axiom MA is the assertion that every c.c.c. partial order $\mathbb{P}$ and small family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ has a filter $F \subseteq \mathbb{P}$ meeting every $D \in \mathcal{D}$.

One usually achieves MA via long iterated forcing, adding all the various witnessing filters.

Furthermore, MA is customarily conceived of as asserting precisely that a lot of c.c.c. forcing has already been performed.

Nevertheless, Reitz's argument shows one can also have GA:

Theorem (Reitz)

Every model of ZFC *has an extension that is a model of* $\mathrm{ZFC} + \mathrm{GA} + \mathrm{MA}$.

GA and V=HOD

Thus, we can obtain GA by forcing strong version of $V = \mathrm{HOD}$.

Question

Is GA consistent with $V \neq \mathrm{HOD}$?

The answer is yes.

Theorem (Hamkins,Reitz,Woodin)

Every model of set theory has an extension which is a model of GA *plus* $V \neq \mathrm{HOD}$.

Will give details later. Sketch: first extend to $V \subseteq \overline{V} \models \mathrm{CCA}$; then add $\overline{V}[G]$ a Cohen subset to every regular cardinal. By homogeneity, this achieves $V \neq \mathrm{HOD}$. Use details of approximation and cover for GA.

Upward amalgamation

Let us turn now to an upward-oriented question.

To what extent may we amalgamate forcing extensions? Are the forcing extensions of a given model upward directed?

If $V[G]$ and $V[H]$ are two forcing extensions of V, must there be a common extension $V[K]$?

There are meta-mathematical issues with formulating the question. After all, if our only method of referring to $V[G]$ and $V[H]$ together is by means of a common extension $V[K]$ in which they both already exist, then the issue becomes moot.

The toy model perspective can be illuminating.

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Non-amalgamation

Theorem (Woodin)

If W is a countable transitive model of ZFC, then there are W-generic Cohen reals c and d such that $W[c]$ and $W[d]$ have no common extension to a model $\bar{W} \models \mathrm{ZFC}$ with the same ordinals.

Proof.

W is countable. Build c and d in stages. Fix a "bad" real z, such as a real coding all of W. Fix D_n dense sets in W. Let c_0 meet D_0, and $d_0 = 00\cdots0$ up to same length of c_0, followed by 1, followed by $z(0)$, and then extended to meet D_0. Extend c_0 to c_1 by adding $0s$ to length of d_0, then 1, then meet D_1, etc. This codes z into $c \oplus d$. $\square$

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Extending to large finite families

The argument generalizes to add three reals c, d and e, any two of which can be amalgamated, but not all three. And so on.

Theorem

If $W \models \mathrm{ZFC}$ is countable, then for every n there are W-generic Cohen reals $c_1, \ldots, c_n$ such that any proper subfamily of the extensions $W[c_1], \ldots, W[c_n]$ is amalgamable, but the whole family is not.

Proof.

The proof is similar. Enumerate the dense sets D_n. Fix forbidden real z. Build c_i in steps, extending all but one each time, adding 0s to the other, and coding $z(k)$ there. □

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Infinite version

Does it generalize to infinitely many extensions? If a family of forcing extensions is finitely amalgamable, can one find a common forcing extension of them all?

One shouldn't ask for too much.

If W is countable, we may build extensions $W[G_n]$ that collapse more and more cardinals of W, so that $\bigcup_n W[G_n]$ collapses all the cardinals of W. Thus, the $W[G_n]$ have no common extension $M[H]$.

So one wants to consider only forcing extensions of uniformly bounded size in W.

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So one wants to consider only forcing extensions of uniformly bounded size in W.

No extension with the sequence

Another way to ask for too much:

If W is countable, let z be any real that cannot be added by forcing over W, such as a real coding all of W. Let $W[d_n]$ be mutually generic Cohen reals. Modify d_n on the first bit only to agree with $z(n)$, producing c_n. Thus, each c_n is W-generic Cohen real, and any finitely many are mutually generic. But no extension $M[G]$ has $\langle c_n \mid n < \omega \rangle$, since from this sequence we could build z.

So we should not expect the *sequence* of generic objects in the common extension.

Finding a common extension

With the right goal it turns out that one CAN find a common extension:

Theorem

If W is a countable model of ZFC *and*

$$W \subseteq W[G_0] \subseteq W[G_1] \subseteq W[G_2] \subseteq \cdots$$

is a countable tower of forcing extensions, with forcing of bounded size in W, then there is a common forcing extension $W[H]$ above them all.

Thus, any finitely amalgamable family of forcing extensions is fully amalgamable.

Proof. Suppose (special case) increasing tower of extensions

$$W \subseteq W[g_0] \subseteq W[g_0 \times g_1] \subseteq W[g_0 \times g_1 \times g_2] \subseteq \cdots ,$$

with $g_n \subseteq \mathbb{Q}_n$ mutually W-generic. We seek a single common extension $W[H]$ containing them all.

The proof method is "hiding the generics." Pick $\delta > |\mathbb{Q}_n|$, let $\theta = 2^\delta$. Let $\mathbb{P} = \prod_{\alpha<\theta} \mathbb{R}_\alpha$ be the finite support product of all size $< \delta$ posets, with unbounded repetition. $\mathbb{P}$ is δ-c.c. Since W is countable, there is W-generic $H \subseteq \mathbb{P}$. In fact, $\bigcup_{n<\omega} W[g_0 \times g_1 \times \cdots \times g_n]$-generic.

Key step. Externally, let $\langle \theta_n \mid n < \omega \rangle$ cofinal in θ, with $\mathbb{R}_{\theta_n} = \mathbb{Q}_n$. Modify H by surgery to H^* with g_n at θ_n. H^* is still a filter on $\mathbb{P}$. It is also W-generic, since every antichain $A \subseteq \mathbb{P}$ in W is bounded in θ, hence involves only finitely many θ_n.

But $g_n \in W[H^*]$, so $W[g_0 \times \cdots \times g_n] \subseteq W[H^*]$ for every $n < \omega$, as desired. $\square$

Geology

Let us turn now to the development of set-theoretic geology.

We shall dig a bit deeper underground, under the grounds, hitting bedrock or solid bedrock, eventually uncovering the mantle, the inner mantles, the generic mantle, the inner generic mantles and ultimately the outer core.

Let's explore the underground...

Bedrock models

Suppose that the universe is a forcing extension of a ground. This ground may be a forcing extension of a deeper ground. And that ground may be a forcing extension of a still-deeper ground.

Do we eventually hit bedrock?

To hit bedrock would mean to have a ground model W that cannot be further reduced to a deeper ground. In other words, a minimal ground.

A *bedrock* is a ground that is minimal among all grounds. Equivalently, a bedrock of V is a ground of V that satisfies the ground axiom.

A bottomless model

Theorem (Reitz)

Every model of set theory V has a class-forcing extension $V[G]$ having no bedrock.

Proof.

By forcing if necessary, assume GCH in V. Let $\mathbb{P}$ be the class product forcing $\Pi_\alpha \mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ generically chooses to force the GCH or its negation at $\aleph_{\alpha+1}$, and consider the extension $V[G]$. Every set of ordinals $x \in V$ will be coded into the GCH pattern of $V[G]$. Meanwhile, the tail extensions $V[G^\alpha]$, where $G^\alpha \subseteq \mathbb{P} \restriction [\alpha, \infty)$ are ground models of $V[G]$. If $V[G] = W[h]$ where $h \subseteq \mathbb{Q} \in W$ is W-generic, then $V \subseteq W$ by the coding, and above $|\mathbb{Q}|$, W will contain all the tails of G, and so $V[G^\alpha] \subseteq W \subseteq V[G]$ for large enough α. So $V[G^{\alpha+1}]$ is a still-deeper ground. $\square$

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Unique bedrock

Suppose that the universe V has a bedrock, that is, a ground W that cannot be further decomposed as a forcing extension.

Open Question

Is the bedrock unique when it exists?

We don't know.

For all the models in which we are able to calculate the answer, the answer is yes.

Solid bedrock

Consider the collection of all grounds of V under the inclusion relation.

- A *bedrock* is a minimal ground.
- A *solid bedrock* is smallest ground, that is, a ground contained in all other grounds.

Note that the solid bedrock is unique when it exists.

CCA implies solid bedrock

Observation

If the continuum coding axioms CCA holds, then V is a solid bedrock in all its forcing extensions.

Proof.

Suppose that the CCA holds in V. Then every set in V is coded into the GCH pattern of V. Set forcing $V[G]$ preserves this coding above the size of the forcing. Similarly, the coding is preserved to any ground $W \subseteq V[G]$. Thus, every set in V is coded in W, and consequently $V \subseteq W$. So V is a solid bedrock in $V[G]$. $\square$

How many grounds?

Theorem

If there is a solid bedrock, then there are only set many grounds. That is, there is a set I such that every ground is W_r for some $r \in I$.

Proof.

If M is a solid bedrock, then M is a ground of $V = M[G]$ and every other ground W is trapped $M \subseteq W \subseteq \mathrm{M}[G] = V$. From this it follows that $W = M[G \cap \mathbb{B}_0]$ for some complete subalgebra $\mathbb{B}_0 \subseteq \mathbb{B}$. There are only a set of possible $\mathbb{B}_0$. $\square$

Common ground

Open Question

Do any two grounds contain a common ground?

That is, for any r and s, is there t with $W_t \subseteq W_r \cap W_s$?

In other words, are the grounds downward directed?

This question could have been asked forty years ago.

I place it at the foundation of any serious investigation of forcing. If we are to claim any serious understanding of forcing, we must know the answer to this fundamental question.

Set-directedness

Because we are also able to quantify over any *set* of indices r for grounds W_r, it is also sensible and natural to seek a common ground below any set of grounds.

Question

Are the grounds downward set-directed? That is, for any set A, is there t with

$$W_t \quad \subseteq \quad \bigcap_{r \in A} W_r \ ?$$

This is true, of course, in any model with a solid bedrock. But it is also true in Reitz's bottomless model, where there is no bedrock.

Downward directedness hypotheses

Definition

1. The Downward Directed Grounds Hypothesis DDG asserts that the grounds are downward directed.

   For every r and s there is t such that $W_t \subseteq W_r \cap W_s$.

2. The Strong DDG asserts that they are downward set-directed.

   For every A there is t with $W_t \subseteq \bigcap_{r \in A} W_r$.

The strong DDG holds in every model for which we are able to determine the answer. Also, the strong DDG holds if $V = L[A]$.

Meanwhile, Woodin has a candidate counterexample model.

The Mantle

Burrowing deeper underground, the principal new concept of set-theoretic geology is the *mantle*.

Definition

The *mantle* M is the intersection of all grounds.

Thus, the mantle removes whatever forcing might have been performed when forming the universe.

Mantle is definable

Theorem

The mantle is a first-order definable transitive class, containing all ordinals.

Proof.

We have the first-order definable parameterization of grounds W_r. So we may define the mantle by $x \in \mathrm{M}$ if and only if $\forall r\ x \in W_r$.

And the mantle is easily seen to be transitive and contain all ordinals. $\square$

Ancient Paradise

The analysis of the mantle engages with an interesting philosophical view:

Ancient Paradise. This is the philosophical view that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If we could sweep the accumulated material away, we should find an ancient paradise.

The mantle, of course, wipes away an entire strata of forcing.

So the ancient paradise view suggests that the mantle may be highly regular.

Every model is a mantle

Unfortunately, our initial main theorem tends to refute this perspective:

Theorem (Fuchs, Hamkins, Reitz)

Every model of ZFC *is the mantle of another model of* ZFC*.*

By sweeping away the accumulated sands of forcing, what we find is not a highly regular ancient core, but rather: an arbitrary model of set theory.

Conclusion: we will not be able to prove any highly regular structural features of the mantle.

The proof will come later.

Reducing Second to First order

As mentioned earlier, the parameterized family $\{ W_r \mid r \in V \}$ of grounds reduces 2nd order properties about grounds to 1st order properties about parameters.

- The Ground Axiom holds if and only if $\forall r\, W_r = V$.
- W_r is a bedrock if and only if $\forall s\, (W_s \subseteq W_r \implies W_s = W_r)$.
- W_r is a solid bedrock if and only if $\forall s\, (W_r \subseteq W_s)$.
- The mantle is defined by $\mathrm{M} = \{ x \mid \forall r\, (x \in W_r) \}$.
- The DDG asserts $\forall r, s\, \exists t\; W_t \subseteq W_r \cap W_s$.
- The strong DDG asserts $\forall A$ *existst* $W_t \subseteq \bigcap_{r \in A} W_r$.

So all of our questions about the nature and structure of the grounds and of the mantle are first-order questions in the language of set theory, expressible in ZFC.

The Mantle under Directedness

Under the downward directed grounds hypothesis, the mantle is well behaved.

Theorem

1. *If the* DDG *holds, then the mantle is constant across the grounds, and* $\mathrm{M} \models \mathrm{ZF}$.
2. *If the Strong* DDG *holds, then* $\mathrm{M} \models \mathrm{ZFC}$.

The hypothesis in (2) can be weakened.

Let's describe the proof.

DDG implies mantle is invariant to grounds

Theorem

If the grounds are downward directed, that is, if the DDG *holds, then the mantle is constant across grounds.*

Proof.

Suppose the grounds of V are downward directed. Fix a ground W. Any ground of W is a ground of V; so the mantle of V is contained in the mantle of W. Conversely, if a is not in the mantle of V, then $a \notin W'$ some ground W', and so $a \notin W \cap W'$. By directedness, there is a ground $\bar{W} \subseteq W \cap W'$. But $\bar{W}$ is a ground of W and $a \notin \bar{W}$, and so a is not in the mantle of W. So the mantle is constant among the ground models of V. $\square$

ZF in the mantle

Theorem

If the mantle is constant across the grounds, then it is a model of ZF*.*

Proof.

If the mantle is constant across the grounds, then it is definable in every ground, so it satisfies ZF by the intersection theorem below. □

Intersection Theorem

If $\mathcal{W}$ is family of ZFC models, all with same ordinals and $\cap\mathcal{W}$ is a class in every $W \in \mathcal{W}$, then $\cap\mathcal{W} \models \mathrm{ZF}$.

Intersection Theorem

If $\mathcal{W}$ is a family of ZFC models with same ordinals and $\cap\mathcal{W}$ is a class in every $W \in \mathcal{W}$, then

1. $\cap\mathcal{W} \models \mathrm{ZF}$.
2. If $\mathcal{W}$ is *locally realized*, meaning every $y \in \cap\mathcal{W}$ has $W \in \mathcal{W}$ with $P(y)^{\cap\mathcal{W}} = P(y)^W$, then $\cap\mathcal{W} \models \mathrm{ZFC}$.

Proof.

$\bigcap\mathcal{W}$ is transitive, contains ORD and closed under Gödel operations. Remains only to show *almost universal*: every $A \subseteq \bigcap\mathcal{W}$ has $A \subseteq B \in \bigcap\mathcal{W}$. Use $B = V_\alpha \cap (\bigcap\mathcal{W})$ for large α.

For ZFC, consider any set y in $\bigcap\mathcal{W}$, realize $P(y \times y)^{\cap\mathcal{W}}$ in W. So y has well-orders in W that survive to $\bigcap\mathcal{W}$. $\square$

Intersecting models of ZFC

There are some interesting boundary cases:

- Descending ORD-length sequences $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_\alpha \supseteq \cdots$ have $\cap_\alpha M_\alpha \models \mathrm{ZFC}$.
- It is not true for set-length sequences: the intersection of first ω_1 iterates of a normal ultrapower does not model ZFC.
- Similarly, $\mathrm{HOD}^\omega = \cap_n \mathrm{HOD}^n$ may not satisfy ZFC.
- The intersection of two ZFC models, even two grounds, need not satisfy ZFC.

There are some very interesting meta-mathematical issues with HOD^n sequence.

Intersection of two grounds

Observation

The intersection of two grounds may not model ZFC.

Proof.

Suppose $G \subseteq \mathbb{P}$ is L-generic for the forcing to make $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$. Use $\mathbb{P} = \Pi_n \mathbb{Q}_n$, where $\mathbb{Q}_n = \mathrm{Add}(\aleph_n, \aleph_{n+2})$. Let c be a Cohen real over $L[G]$, and let $V = L[G][c]$. Consider grounds $L[G]$ and $L[G \restriction c]$. Same cardinals. Note that $L[G] \cap L[G \restriction c]$ has every G_n for $n \in c$, and so GCH holds at $\aleph_n$ if and only if $n \in c$. But c does not exist in $L[G] \cap L[G \restriction c]$. $\square$

In fact, there is no largest ZF model inside $L[G] \cap L[G \restriction c]$.

ZFC in the mantle

Theorem

If the grounds are downward set-directed, that is, if the strong DDG *holds, then the mantle is a model of* ZFC.

Proof.

Under finite directedness, we've already established that the mantle $\mathrm{M} \models \mathrm{ZF}$. To get choice, use locally downward set-directed, which follows from strong DDG.

The point is that any $y \in \mathrm{M}$ has $P(y \times y)^{\mathrm{M}}$ realized in some ground W_r, and so all the well-orderings of y in W_r survive to the mantle M. □

Set many grounds

Observation

The mantle is a ground $\iff$ the universe has a solid bedrock.

Earlier we saw this implies there are only set many grounds.

Question

Is the solid bedrock axiom equivalent to the assertion that there are only set many grounds?

A strong counterexample to this would be a model V having only set many grounds, but no minimal ground. Any such model would of course also be a counterexample to downward set-directedness and the generic strong DDG hypothesis.

The generic mantle

Earlier, we defined the mantle to be the intersection of all grounds of V.

Define now that the *generic mantle* gM is the intersection of all grounds of all forcing extensions of V.

This is the intersection of a larger collection of grounds, and so

$$\mathrm{gM} \subseteq \mathrm{M}.$$

Theorem

The generic mantle is a definable transitive class, contains all ordinals, is invariant under forcing, and is a model of ZF.

Proof.

Clearly transitive and contains all ordinals.

Invariant under forcing: easy to see $\mathrm{gM}^V \subseteq \mathrm{gM}^{V[G]}$. If $x \notin \mathrm{gM}^V$, then $x \notin W$ some ground $W \subseteq V[H]$. So there is a condition q forcing $\check{x} \notin \dot{W}_{\dot{r}}$. Now we may assume H is $V[G]$-generic, so $V[G][H]$ has a ground omitting x. So $x \notin \mathrm{gM}^{V[G]}$. Thus, $\mathrm{gM}^V = \mathrm{gM}^{V[G]}$.

Now argue $\mathrm{gM} \models \mathrm{ZF}$ by the intersection theorem, since it is intersection of ZFC models in each of which it is a class. $\square$

The generic multiverse

The generic mantle of a model is naturally considered in a context that includes all its forcing extensions, their grounds, subsequent forcing extensions, and so on.

The *generic multiverse* is the family of universes obtained by closing under forcing extensions and grounds.

There are various philosophical motivations to study the generic multiverse.

The generic multiverse

Woodin introduced the generic multiverse essentially to reject it, to defeat a certain multiverse view of truth: the idea that to be True means to be true in the generic multiverse.

We don't hold that view of truth, but nevertheless find the generic multiverse to be a natural context for set-theoretic investigation. Indeed, it is a principal focus for geology.

The generic multiverse naturally partitions the larger multiverse of models of set theory into equivalence (meta)classes, consisting of models reachable from one another by passing to forcing extensions and ground models.

Formalization of generic multiverse

Because the generic multiverse concept is clearly second-order or higher-order, however, there are certain difficulties of formalization and meta-mathematical issues that need to be addressed. This is particularly true when one wants to consider the generic multiverse of the full set-theoretic universe V, rather than merely the generic multiverse of a toy countable model.

The standard approaches to second-order set theory, after all, such as Gödel-Bernays set theory or Kelly-Morse set theory, do not seem to provide a direct account of the generic multiverse of V, whose forcing extensions are of course not directly available, even as GBC or KM classes.

But many multiverse questions are first order

Nevertheless, many generic multiverse questions are first-order.

We already treated grounds, bedrocks, solid bedrocks, the ground axiom, the mantle, the generic mantle, and so on in first-order set theory.

We have a first-order manner of treating truth in the forcing extensions of V. It is first-order to state that φ holds in some forcing extension of V or all forcing extensions of V.

Thus, also first-order to assert that φ holds in some forcing extension of a ground of some forcing extension.

The toy model formalization

Sometimes we seek the entire generic multiverse.

Toy model perpsective. Analogous to countable-transitive-model approach to forcing. Use a countable $W \models \mathrm{ZFC}$; consider all forcing extensions, grounds of W, as constructed in V.

The toy model generic multiverse of W depends on the background in which it is computed.

The toy model approach is used in:

- Countable-transitive-model approach to forcing.
- Our previous non-amalgamation result $W[c]$, $W[d]$.
- Formalization of IMH.

Generic multiverse possibility

We may view the generic multiverse as a Kripke model of possibility. Thus, φ is *multiverse possible*, written $\Diamond_m \varphi$, if it holds somewhere in the generic multiverse.

A *multiverse path* is $\langle U_0, \ldots, U_n \rangle$, where each U_{i+1} is either a ground or forcing extension of U_i.

Woodin has argued that $\Diamond_m \varphi$ if and only if φ holds in a model reachable by a multiverse path of length three, specifically, in a forcing extension of a ground of a forcing extension.

This implies that $\Diamond_m \varphi$ is first-order expressible.

Directedness implies that two steps suffice.

Generic mantle is forcing invariant

Corollary

The generic mantle is constant across the generic multiverse. Indeed, it is the intersection of the generic multiverse.

Proof.

Since the generic mantle is invariant by set forcing, it is preserved from ground to extension and vice versa. Thus, all models in the generic multiverse have the same generic mantle. Hence, the generic mantle is contained within the intersection of the generic multiverse. Conversely, it is the intersection of part of the generic multiverse. So the generic mantle is the intersection of the generic multiverse. □

Generic mantle = largest forcing-invariant class

Corollary

The generic mantle is the largest forcing-invariant class.

Proof.

We showed it is forcing-invariant above. Any other class that is definable and invariant by forcing will be preserved to forcing extensions and grounds, and will therefore be contained within the generic mantle. So the generic mantle is the largest such forcing-invariant class. □

The generic mantle is thus a highly canonical object, the largest forcing-invariant definable class. It should become a central focus of attention for those set-theorists interested in forcing and models of set theory.

Inclusion of models

Lemma

Any generic ground of V contained in V is a ground of V.

Proof.

If W is a ground of $V[G]$ and $W \subseteq V \subseteq V[G]$, then by general forcing facts, W is a ground of V. $\square$

Question

If W is in the generic multiverse of V and $W \subseteq V$, must W be a ground of V?

Is the $\subseteq$ relation the same as the "is a ground model of" relation? Yes, if grounds are dense in generic multiverse.

DDG in all grounds

Theorem. The following are equivalent

1. The DDG holds: the grounds of V are downward directed.
2. The DDG holds in some forcing extension of V.
3. The DDG holds in every ground of V.

Proof.

$3 \implies 1 \implies 2$ are easy. For $2 \implies 3$, suppose grounds of $V[G]$ are downward directed and W and W' are grounds of U, a ground of V. So W, W' are grounds of $V[G]$, and so by 2 there is ground $\bar{W}$ of $V[G]$ with $\bar{W} \subseteq W \cap W'$. Since $\bar{W} \subseteq U \subseteq V[G]$, it follows that $\bar{W}$ is a ground of U, and so 3 holds. $\square$

Theorem. The following are equivalent

1. The generic DDG holds: in every forcing extension, the grounds are downward directed.
2. The grounds of V are downward directed and dense below generic grounds.
3. The grounds of V are downward directed and dense below grounds of ground extensions.

Proof.

($1 \implies 3$) If W is a ground of $W_r[G]$, then $W = W_t^{W_r[G]}$ some $t \in W_r[G]$, having name $\dot{t}$. If G is V-generic, form $V[G]$ and apply (1) to get $W_s^{V[G]}$ below V and W. So there is condition p forcing this. Argue that $W_s^{W_r} \subseteq W$ even when G is not V-generic. ($3 \implies 2$) immediate. ($2 \implies 1$) Use grounds of V. $\square$

The generic DDG

Corollary

If the generic DDG holds, then:

1. the mantle is the same as the generic mantle;
2. the class of ground extensions is closed under forcing extensions and grounds;

Proof.

If the grounds are dense below the generic grounds, then the mantle is the same as the generic mantle. Also, the class of ground extensions is closed under forcing extensions and grounds. □

Generic multiverse

So if the DDG holds in all extensions, then the collection of ground extensions $W_r[G]$ is closed under forcing extensions and grounds.

Theorem

If DDG *holds in all extensions, then the generic multiverse of V consists precisely of the ground extensions.*

In other words, every model in the generic multiverse can be reached in two steps: first go down to a ground W_r, and then go up to a forcing extension of that ground $W_r[G]$.

Up-down may not suffice

Consider the dual two-step, first go to a forcing extension $V[G]$, and then to a ground $W_r^{V[G]}$. That is, go "up-and-then-down."

Observation

If ZFC is consistent, there is a toy-model generic multiverse not exhausted by the generic grounds.

Proof.

If $V[c]$ and $V[d]$ are not amalgamable, then $V[d]$ is not a generic ground of $V[c]$, but they have the same multiverse. □

Theorem. The following are equivalent

1. The generic strong DDG holds. That is, in every forcing extension of V, the grounds are downward set-directed.
2. The grounds of V are downward set-directed and dense below the generic grounds.
3. The grounds of V are downward set-directed and dense below the grounds of every ground extension.

Proof.

$1 \implies 3$ not hard. $3 \implies 2$ immediate. For $2 \implies 1$, basically get below the grounds of any extension and apply strong DDG in V (but there are subtleties). $\square$

Generic mantle has ZFC

Theorem

If the generic strong local DDG *holds, then the generic mantle is a model of* ZFC.

Proof.

We already know $\mathrm{gM} \models \mathrm{ZF}$. Use strong local DDG to argue for AC, but there are subtleties. $\square$

Solid generic bedrock

Question

When does the universe have a generic solid bedrock? That is, when is the generic mantle also a ground?

This phenomenon is not universal, in light of the following.

Corollary

There is a class extension $V[G]$ with the generic strong local DDG*, but no bedrock and no generic bedrock.*

Proof.

Use Reitz's bottomless model. It continues to be bottomless in every set forcing extension. □

If V is constructible from a set

Theorem

If V is constructible from a set, then this is true throughout the generic multiverse.

Proof.

Being $L[x]$ is clearly preserved by forcing extensions. Suppose W is a ground of $V = L[x] = W[G]$, where $G \subseteq \mathbb{P} \in W$. Pick name $x = (\dot{x})_G$. Let $a \in W$ be set of ordinals coding transitive closure $t = \mathrm{TC}(\{\mathbb{P}, \dot{x}\})$; consider $L[a]$. Note $L[a] \subseteq W$. Also, G is $L[a]$-generic, and so $L[a] \subseteq W \subseteq L[a][G]$. This implies that W is a forcing extension of $L[a]$, and hence constructible from a set. $\square$

Bedrocks in $L[a]$

Theorem

If $V = L[a]$ and there is no bedrock, then this is true throughout the generic multiverse.

Proof.

Suppose V is constructible from a set and there is no bedrock. Thus, there are class many grounds. This remains true in any set-forcing extension, and so there is no bedrock in any extension. Also, it is preserved to grounds. So it is true throughout the generic multiverse. □

The Generic HOD

HOD is the class of hereditarily ordinal definable sets.

$$\mathrm{HOD} \models \mathrm{ZFC}$$

The *generic* HOD, introduced by Fuchs, is the intersection of all HODs of all forcing extensions.

$$\mathrm{gHOD} = \bigcap_G \mathrm{HOD}^{V[G]}$$

The original motivation was to identify a very large canonical forcing invariant class.

The Generic HOD

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The original motivation was to identify a very large canonical forcing invariant class.

The Generic HOD

Facts

1. gHOD is constant across the generic multiverse.
2. The HODs of all forcing extensions are downward set-directed.
3. Consequently, gHOD is locally realized and $\mathrm{gHOD} \models \mathrm{ZFC}$.
4. The following inclusions hold.

$$\begin{array}{ccccc} & \mathrm{HOD} & & & \\ & \cup & & & \\ \mathrm{gHOD} & \subseteq & \mathrm{gM} & \subseteq & M \end{array}$$

$V = L[a]$ implies generic strong DDG

Theorem

If $V = L[a]$, then the generic strong DDG holds.

Proof.

Consider $\mathrm{HOD}^{V^{\mathrm{Coll}(\omega,\alpha)}}$. By homogeneity, these are contained in HOD^V. They are downward set-directed. Vopěnka's theorem, that every set is generic over HOD, implies that these models are grounds of V, since we need only add a. They are dense below the grounds, since if $V = W[g]$ via $g \subseteq \mathbb{P} \in W$, then absorb $\mathbb{P}$ into $\mathrm{Coll}(\omega, \alpha)$ for large enough α, and observe that $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^W \subseteq W$, as desired. Similarly dense in generic grounds. $\square$

Theorem

If $V = L[a]$, then the following are equivalent:

1. *There are only set many grounds.*
2. *The bedrock axiom.*
3. *The solid bedrock axiom.*

Proof.

($1 \implies 2$) Apply strong DDG. ($2 \implies 3$) Suppose that W is a bedrock. It must be $W = \mathrm{HOD}^{V^{\mathrm{Coll}(\omega,\alpha)}}$ some large α. But $\mathrm{HOD}^{V^{\mathrm{Coll}(\omega,\beta)}}$ are dense below grounds, so W is a solid bedrock. ($3 \implies 1$) proved generally. $\square$

The structure of all grounds

Theorem

1. *The collection of grounds between a fixed ground W and the universe V is an upper semi-lattice.*
2. *If the grounds of V are downward directed, then the grounds of V are an upper semi-lattice.*
3. *The grounds of V need not be a complete upper semi-lattice, even if $V = L[a]$.*
4. *The grounds of V need not be a lattice, even when the grounds are downward set-directed, and even if the universe is constructible from a set.*

(1) and (2) are soft. (3) and (4) follow from fact that two grounds may have no largest common ground.

Separating the notions

$$\begin{array}{ccccc} \mathrm{HOD} & & & & \\ \cup & & & & \\ \mathrm{gHOD} & \subseteq & \mathrm{gM} & \subseteq & M \end{array}$$

To what extent can we control and separate these classes?

We answer with our main theorems.

Realizing V as the Mantle

Theorem (Fuchs, Hamkins, Reitz)

If $V \models \mathrm{ZFC}$, then there is a class extension $\overline{V}$ in which

$$V = \mathrm{M}^{\overline{V}} = \mathrm{gM}^{\overline{V}} = \mathrm{gHOD}^{\overline{V}} = \mathrm{HOD}^{\overline{V}}$$

In particular, as mentioned earlier, every model of ZFC is the mantle and generic mantle of another model of ZFC.

It follows that we cannot expect to prove ANY regularity features about the mantle or the generic mantle.

Forcing definability

The initial idea goes back to McAloon (1971), to make sets definable by forcing.

For an easy case, consider an arbitrary real $x \subseteq \omega$. It may not happen to be definable in V.

With an infinite product, we can force the GCH to hold at $\aleph_n$ exactly when $x(n) = 1$.

In the resulting forcing extension $V[G]$, the original real x is definable, without parameters.

Proof sketch of first separation theorem

We start in $V \models \mathrm{ZFC}$ and want $V[G]$ with $V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{HOD}^{V[G]}$.

Let $\mathbb{Q}_\alpha$ generically decide whether to force GCH or $\neg\mathrm{GCH}$ at $\aleph_\alpha$ (*). Let $\mathbb{P} = \Pi_\alpha \mathbb{Q}_\alpha$, with set support. Consider $V[G]$ for generic $G \subseteq \mathbb{P}$.

Every set in V becomes coded unboundedly into the continuum function of $V[G]$. Hence, definable in $V[G]$ and all extensions.

So $V \subseteq \mathrm{gHOD}$. Consequently $V \subseteq \mathrm{gHOD} \subseteq \mathrm{gM} \subseteq \mathrm{M}$ and $V \subseteq \mathrm{HOD}$.

Every tail segment $V[G^\alpha]$ is a ground of $V[G]$. Also, $\cap_\alpha V[G^\alpha] = V$. Thus, $\mathrm{M} \subseteq V$. Consequently, $V = \mathrm{gHOD} = \mathrm{gM} = \mathrm{M}$.

$\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^{V[G^\alpha]}$, since $\mathbb{P} \restriction \alpha$ is densely almost homogeneous.

So $\mathrm{HOD}^{V[G]} \subseteq V$.

In summary, $V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{HOD}^{V[G]}$, as desired.

Theorem (Fuchs, Hamkins, Reitz)

Other combinations are also possible.

1. *Every model of set theory V has an extension $\overline{V}$ with*

$$V = \mathrm{M}^{\overline{V}} = \mathrm{gM}^{\overline{V}} = \mathrm{gHOD}^{\overline{V}} = \mathrm{HOD}^{\overline{V}}$$

2. *Every model of set theory V has an extension W with*

$$V = \mathrm{M}^{W} = \mathrm{gM}^{W} = \mathrm{gHOD}^{W} \qquad \textit{but} \quad \mathrm{HOD}^{W} = W$$

3. *Every model of set theory V has an extension U with*

$$V = \mathrm{HOD}^{U} = \mathrm{gHOD}^{U} \qquad \textit{but} \quad \mathrm{M}^{U} = U$$

4. *Lastly, every V has an extension Y with*

$$V = \mathrm{HOD}^{Y} = \mathrm{gHOD}^{Y} = \mathrm{M}^{Y} = \mathrm{gM}^{Y}$$

Second separation theorem: mantles low, HOD high

For the 2nd separation theorem, we want $V[G]$ with

$$V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} \qquad \text{but} \qquad \mathrm{HOD}^{V[G]} = V[G]$$

- Balance the forces on M, gM, gHOD and HOD.
- Force to $V[G]$ where every set in V is coded unboundedly in the GCH pattern.
- Also ensure that G is definable, but not robustly.
- The proof uses *self-encoding forcing*:

  Add a subset $A \subseteq \kappa$. Then code this set A into the GCH pattern above κ. Then code THOSE sets into the GCH pattern, etc. Get extension $V[G_{(\kappa)}]$ in which $G_{(\kappa)}$ is definable.

Keeping HODs low, Mantles high

For the 3rd separation, we keep the HODs low and the Mantle high, $V = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]}$ but $\mathrm{M}^{V[G]} = V[G]$.

Such a model $V[G]$ will of course be a model of the Ground Axiom plus $V \neq \mathrm{HOD}$. Recall

Theorem (Hamkins,Reitz,Woodin)

Every $V \models \mathrm{ZFC}$ has a class forcing extension $V[G] \models \mathrm{GA} + V \neq \mathrm{HOD}$.

We modified the argument to obtain:

Theorem

If $V \models \mathrm{ZFC}$, then there is a class extension $V[G]$ in which

$$V = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]} \qquad \textit{but} \qquad \mathrm{M}^{V[G]} = V[G]$$

Mantles high, HODs high

Lastly,

Theorem

If $V \models \mathrm{ZFC}$, then there is $V[G]$ in which

$$V[G] = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]}$$

This is possible by forcing the Continuum Coding Axiom CCA.

The Inner Mantles

When the mantle M is a model of ZFC, we may consider the mantle of the mantle, iterating to reveal the *inner mantles*:

$$\mathrm{M}^1 = \mathrm{M} \qquad \mathrm{M}^{\alpha+1} = \mathrm{M}^{\mathrm{M}^\alpha} \qquad \mathrm{M}^\lambda = \bigcap_{\alpha<\lambda} \mathrm{M}^\alpha$$

Continue as long as the model satisfies ZFC.

The *Outer Core* is reached if M^α has no grounds, $M^\alpha \models \mathrm{ZFC} + \mathrm{GA}$.

Conjecture. Every model of ZFC is the αth inner mantle of another model, for arbitrary $\alpha \leq \mathrm{ORD}$.

Philosophical view: ancient paradise?

Large cardinals across the multiverse

We now turn to the question of set-theoretic features that may hold throughout the generic multiverse.

Large cardinal indestructibility, the question of whether a large cardinal property is preserved to a forcing extension, has been a focus of study for decades.

Let us enlarge the problem to the question of whether a large cardinal exhibits its large cardinal property throughout a substantial portion of the generic multiverse.

Supercompactness

Theorem

If κ is supercompact, then there is a forcing extension $V[G]$ in which κ remains supercompact, becomes indestructible by $<\kappa$-directed closed forcing, and the Ground Axiom holds.

Thus, in this model, the supercompactness of κ is indestructible both upward by $<\kappa$-directed closed forcing and (vacuously) downward to ground models.

Question

Can we arrange that κ is supercompact throughout the $<\kappa$-directed closed multiverse?

No for supercompact cardinals (and measurables)

Theorem

No supercompact cardinal κ is indestructible throughout the $<\kappa$-directed closed multiverse. For every cardinal κ, there is a $<\kappa$-directed closed generic ground in which κ is not measurable.

Proof.

The forcing $\mathbb{P}$ to add a slim κ-Kurepa tree always destroys the measurability of κ. But $\mathbb{P}$ can be absorbed into the collapse forcing $\mathbb{P} * \mathrm{Coll}(\kappa, 2^\kappa) \cong \mathrm{Coll}(\kappa, 2^\kappa)$. So the extension by $\mathbb{P}$ is a $<\kappa$-directed closed generic ground of V in which κ is not measurable. $\square$

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But Yes for weakly compact cardinals

Theorem

If κ is supercompact, then there is a class extension $V[G]$ such that κ is weakly compact throughout the $<\kappa$-closed multiverse of $V[G]$.

The proof uses:

Lemma

If κ is weakly compact and this is indestructible by $<\kappa$-closed forcing, then κ retains this property throughout the $<\kappa$-closed generic multiverse.

There are a large number of similar such questions in this area.

Approximation and cover properties

The definition of V inside $V[G]$ makes use of the following key ideas:

Definition (Hamkins)

1. $W \subseteq V$ has δ *cover property* if every $A \subseteq W$ with $A \in V$, $|A|^V < \delta$ is covered $A \subseteq B$ by some $B \in W$ with $|B|^W < \delta$.
2. $W \subseteq V$ has δ *approximation property* if every $A \subseteq W$ with $A \in V$ and all small approximations $A \cap B$ in W, whenever $|B|^W < \delta$, is already in the ground model $A \in W$.

Uniqueness of grounds

Lemma (Hamkins)

If $V \subseteq V[G]$ and $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ with $\mathbb{P}$ nontrivial and $\Vdash \dot{\mathbb{Q}}$ is $\leq |\mathbb{P}|$-strategically closed, then $V[G]$ has the δ cover and approximation properties for $\delta = |\mathbb{P}|^+$.

Lemma (Laver,Hamkins)

If $W, W' \subseteq V$ have δ cover and approximation properties, $P(\delta)^W = P(\delta)^{W'}$, $(\delta^+)^W = (\delta^+)^{W'} = (\delta^+)^V$, then $W = W'$.

Laver had first proved the lemma for small forcing, and I extended it to cover and approximation property.

This lemma provides the definition of W inside the forcing extension $W[G]$, using parameter $P(\delta)^W$.

$\mathrm{GA} + V \neq \mathrm{HOD}$

Theorem (Hamkins,Reitz,Woodin)

Every model of set theory has an extension which is a model of GA *plus* $V \neq \mathrm{HOD}$.

First extend to $V \subseteq \overline{V} \models \mathrm{CCA}$; then add $\overline{V}[G]$ a Cohen subset to every regular cardinal. By homogeneity, this achieves $V \neq \mathrm{HOD}$. Details of approximation and cover establish GA.

Questions

Set-theoretic geology is a young area, and there are a large number of open questions.

- To what extent does the mantle satisfy ZF or ZFC?
- Are the grounds or generic grounds downward directed? downward set-directed? locally?
- Is the bedrock unique when it exists?
- Is $\mathrm{gM} = \mathrm{M}$? Is $\mathrm{gM} = \mathrm{gHOD}$?
- Does the generic mantle satisfy ZFC?
- Does inclusion = ‘ground model of’ in the generic multiverse?

Time permitting, let's briefly discuss some issues in the philosophy of set theory regarding the question of the continuum hypothesis, and how it is treated by the universe perspective in comparison with the multiverse view.

Case study: the Continuum Hypothesis

The continuum hypothesis (CH) is the assertion that every set of reals is either countable or equinumerous with $\mathbb{R}$.

This was a major open question from the time of Cantor, and appeared at the top of Hilbert's famous list of open problems at the dawn of the 20th century.

The continuum hypothesis is now known to be neither provable nor refutable from the usual ZFC axioms of set theory.

Gödel proved that CH holds in the constructible universe L.

Cohen proved that L has a forcing extension $L[G]$ with $\neg$CH.

CH in the Multiverse

More important than mere independence, both CH and $\neg$CH are forceable over any model of set theory. Every V has:

- $V[\vec{c}]$, collapsing no cardinals, such that $V[\vec{c}] \models \neg\mathrm{CH}$.
- $V[G]$, adding no new reals, such that $V[G] \models \mathrm{CH}$.

That is, both CH and $\neg$CH are easily forceable. We can turn CH on and off like a lightswitch.

We have a deep understanding of how CH can hold and fail, densely in the multiverse, and we have a rich experience in the resulting models. We know, in a highly detailed manner, whether one can obtain CH or $\neg$CH over any model of set theory, while preserving any number of other features of the model.

The CH is settled

The multiverse perspective is that the CH question is settled.

The answer consists of our detailed understanding of how the CH both holds and fails throughout the multiverse, of how these models are connected and how one may reach them from each other while preserving or omitting certain features.

Fascinating open questions about CH remain, of course, but the most important essential facts are known.

In particular, I shall argue that the CH can no longer be settled in the manner that set theorists formerly hoped it might be.

Traditional Dream solution for settling CH

Set theorists traditionally hoped to settle CH this way:

Step 1. Produce a set-theoretic assertion Φ expressing a natural 'obviously true' set-theoretic principle. (e.g. AC)

Step 2. Prove that Φ determines CH.

That is, prove that $\Phi \implies \mathrm{CH}$,

or prove that $\Phi \implies \neg\mathrm{CH}$.

And so, CH would be settled, since everyone would accept Φ and its consequences.

Dream solution will never be realized

I argue that this template is now unworkable.

The reason is that because of our rich experience and familiarity with models having CH and $\neg\mathrm{CH}$, the mere fact of Φ deciding CH immediately casts doubt on its naturality. So we cannot accept such a Φ as obviously true.

In other words, success in the second step exactly undermines the first step.

Let me present two examples illustrating how this plays out.

Freiling: "a simple philosophical 'proof' of $\neg$CH"

The Axiom of Symmetry (Freiling JSL, 1986)

Asserts that for any function f mapping reals to countable sets of reals, there are x, y with $y \notin f(x)$ and $x \notin f(y)$.

Freiling justifies the axiom on pre-theoretic grounds, with thought experiments throwing darts. The first lands at x, so almost all y have $y \notin f(x)$. By symmetry, $x \notin f(y)$.

> *"Actually [the axiom], being weaker than our intuition, does not say that the two darts have to do anything. All it claims is that what heuristically will happen every time, can happen."*

Thus, Freiling carries out step 1 in the template.

Freiling carries out step 2

Theorem (Freiling)

The axiom of symmetry is equivalent to $\neg\mathrm{CH}$.

Proof.

If CH, let $f(r)$ be the set of predecessors of r under a fixed well-ordering of type ω_1. So $x \in f(y)$ or $y \in f(x)$ by linearity.

If $\neg\mathrm{CH}$, then for any ω_1 many x_α, there must be $y \notin \bigcup_\alpha f(x_\alpha)$, but $f(y)$ contains at most countably many x_α. $\square$

Thus, Freiling exactly carries out the template.

Was his proposal received as a solution of CH? No.

Objections to Symmetry

Many mathematicians, ignoring Freiling's pre-reflective appeal, objected from a perspective of deep experience with non-measurable sets and functions, including extreme violations of Fubini. For them, the pre-reflective arguments simply fell flat.

We have become skeptical of naive uses of measure precisely because we know the pitfalls; we know how badly behaved sets and functions can be with respect to measure concepts.

Because of our detailed experience, we are not convinced that AS is intuitively true. Thus, the reception follows my prediction.

And similarly for other dream solutions of CH.

Another example using the dream template

Consider the following set-theoretic principle:

The powerset size axiom PSA

PSA asserts that whenever a set is strictly larger than another in cardinality, then it also has strictly more subsets:

$$\forall x, y \qquad |x| < |y| \Rightarrow |P(x)| < |P(y)|.$$

Set-theorists understand this axiom very well.

Powerset size axiom: $|x| < |y| \Rightarrow |P(x)| < |P(y)|$

How is this axiom received in non-logic mathematical circles?

Extremely well!

To many mathematicians, this principle is *Obvious*, as natural and appealing as AC. Many are surprised to learn it is not a theorem. (Ask your colleagues!)

Meanwhile, set theorists do not agree. Why not? In part, because we know how to achieve all kinds of crazy patterns $\kappa \mapsto 2^\kappa$ via Easton's theorem. Cohen's $\neg\mathrm{CH}$ model violates it; Martin's axiom violates it; Luzin's hypothesis violates it. PSA fails under many of the axioms, such as PFA, MM that are often favored particularly by set-theorists with the universe view.

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Powerset size axiom

So we have a set-theoretic principle

- which many mathematicians find to be obviously true;
- which expresses an intuitively clear pre-reflective principle about the concept of size;
- which set-theorists know is safe and (relatively) consistent;

is almost universally rejected by set-theorists when proposed as a fundamental axiom.

We are too familiar with the ways that PSA can fail, and have too much experience working in models where it fails.

But imagine an alternative history, in which PSA is used to settle a prominent early question and is subsequently adopted as a fundamental axiom.

Similarly, with CH

I claim that the dream template will not settle CH, because as soon as we know that a proposed principle Φ implies CH or implies $\neg\mathrm{CH}$, we cannot accept Φ as obvious.

Conclusion. The dream template is unworkable.

We simply have too much experience in and familiarity with the CH and $\neg\mathrm{CH}$ worlds. We therefore understand deeply how Φ can fail in worlds that seem perfectly acceptable set-theoretically.

Other attempts to settle CH

More sophisticated attempts to settle CH do not rely on this traditional template.

Woodin has advanced arguments to settle CH based on Ω-logic, and based on Ultimate-L.

To the extent that an argument aims to settle CH, what the Multiversist desires is an explicit explanation of how our experience in the CH or in the $\neg$CH worlds was somehow illusory, as it seems it must be for the argument to succeed.

Since we have an informed, deep understanding of how it could be that CH holds or fails, even in worlds close to any given world, it is difficult to regard these worlds as imaginary.

Thank you

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