The natures of proof

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I shall survey several views of proof, placing the Novaes dialogical nature of proof into context

Question. What is proof?

- Mathematicians frankly often not able to answer well.

  What is a proof, really? Mathematicians are sometimes excused from jury duty, it is said, because according to the prosecutors, they do not know what it means to prove something "beyond a reasonable doubt"

  …A proof is any sufficiently detailed convincing mathematical argument that logically establishes the conclusion of a theorem from its premises. – Hamkins 2020

Proof as prose

- Compare with: 2-column proofs in high-school geometry
- Euclid's Elements is written in elegant prose.
- Contemporary research writing is an open, flexible format, essay style. Rather like Euclid.

Formal proof

- Formal deduction. Can be checked. Provides validation/certainty of correctness.
- Formal proof is often mentioned as a principal goal of mathematical activity
- Mathematicians often view all proof as in-principle formalizable
- But most proofs are not ever formalized…
- However, formal proof is increasingly realized in computer era
- Alexander Paseau
  - Replacing a proof argument with an atomized formal version can reduce credence
  - Whatever insightful ideas might have existed in the original proof are of necessity mixed in with innumerable banal atomic deductive steps in the formal proof; the key idea of the proof is lost in a sea of uninteresting details.
  - "Atomization is not in itself epistemically valuable"

Computer-verified proof

- Major goal: to find formal foundation simultaneously:
  - Supporting formal proof, with automated proof checking
  - Close to human reasoning
  - Close to how human mathematicians represent their ideas

- Vision of future described by some: all mathematicians formally validating their own proofs.
- Incorrect in my view. Proof validation will be a speciality
    - Experts will formalize and validate the proofs of others.
    - Formal proof is not close enough to human reasoning
    - Formal reasoning is anathema to some mathematical thinkers.

Computer-aided proof

- Not the same as computer-verified proof
- Use of proof assistants/verifiers plays different role in process
- Proof assistants often used to help check and guide mathematical analysis, not just as verifier.
- Mathematicians use proof assistants to test conjectures, as experiment.
- Practitioners describe a process of back-and-forth.
- Rather like a dialogue

Bill Thurston

- describes a disparaging view of formalism and proof
- as opposed to mathematical thinking, often impossible to communicate.
- Proof is for convincing others, not for thinking oneself.
- Rattles off seven ways of understanding the meaning of the derivative of a function—familiar conceptions using infinitesimals, slopes of tangent lines, epsilon formalism, linear approximation, and so on—and he imagines the list continuing, “there is no reason for it ever to stop.” He imagines item 37 on the list: the derivative of a real-valued function on a domain is the Lagrangian section of a certain cotangent bundle.

  This is a list of different ways of thinking about or conceiving of the derivative, rather than a list of different logical definitions. Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions. Thurston 1994.

- Counterpoint: formalization as a sharpening of mathematical ideas
    - Divergent inequivalent concepts of “wriggliness”

  The process of formalization is not typically one of translating fully formed mathematical ideas into a sterile language, with punctuation and balanced parentheses being the main concern. Rather, the process of formalization is quite commonly also one of sharpening our ideas, giving substance and further meaning to initially vague conceptions or proof ideas; it is at the very heart of mathematical activity: figuring out and saying precisely what you mean. An initial idea about wriggliness admits dozens of precise formulations, each one a slightly different expression of that vague thought, and not all equivalent, although some of them may be. We might place this list next to Thurston's. –Hamkins 2020
- In Thurston's account, tensions between formalization and mathematical understanding run to the very core of mathematics; formalization can erase mathematical meaning.

The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community's standard of valid proofs. Nonetheless, large computer programs, even when they have been very carefully written and very carefully tested, always seem to have bugs…. When one considers how hard it is to write a computer program even approaching the intellectual scope of a good mathematical paper, and how much greater time and effort have to be put into it to make it ``almost'' formally correct, it is preposterous to claim that mathematics as we practice it is anywhere near formally correct.

- But Thurston does not find mathematics unreliable. Rather, his point is that the reliability of mathematics arises not from formal proof, but from mathematical understanding.

Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It is just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas.

- Thurston's focus on communication seems to resonate superficially with proof-as-dialogue.
- Yet, Thurston's image is not of a mathematician in dialogue, but rather of a mathematician struggling to express his ideas in terms understandable to others at all
- Thinking versus explaining
  How big a gap is there between how you think about mathematics and what you say to others?
  …I've been fascinated by the phenomenon the question addresses for a long time. We have complex minds evolved over many millions of years, with many modules always at work. A lot we don't habitually verbalize, and some of it is very challenging to verbalize or to communicate in any medium. Whether for this or other reasons, I'm under the impression that mathematicians often have unspoken thought processes guiding their work which may be difficult to explain, or they feel too inhibited to try. One prototypical situation is this: there's a mathematical object that's obviously (to you) invariant under a certain transformation. For instance, a linear map might conserve volume for an "obvious" reason. But you don't have good language to explain your reason---so instead of explaining, or perhaps after trying to explain and failing, you fall back on computation. You turn the crank and without undue effort, demonstrate that the object is indeed invariant. – Thurston 2016

Mike Shulman

- Not particularly worried about errors in formal proof

  While working on a recent project I discovered no fewer than nine mistaken theorem statements (not just mistakes in proofs of correct theorems) in published or almost-published literature, including several by well known experts (and two by myself). However, in all nine cases it was simple to strengthen the hypothesis or weaken the conclusion in such a way as to make the theorem true, in a way that sufficed for all the applications I know of. I would argue that this is because the mistaken statements were based on correct ideas, and the mistakes were simply in making those ideas precise. Or to put it differently, mathematicians get our intuitions from "well-behaved" objects: sometimes that intuition can be wrong for "pathological" objects we didn't know about, but in such cases we simply alter the definitions to exclude the pathological ones from consideration.

- Shulman is claiming that the process of formalization can be the source of error, and this is not particularly worrisome, if the underlying mathematical ideas are robust

Proof as instructions

- Much of Euclid is written as construction instructions.
- Sometimes quite weird by contemporary standards. E.g.
  Proposition 10. To bisect a given line segment.
- Proof as recipe – Fenner Tanswell & Matthew Inglis
  - The language of proofs includes abundance of imperatives
  - Let, consider, suppose, define, observe, compute,…
  - Study of math papers on the arxiv, finding huge numbers of imperatives
  - Proof as instructions for what we are to do.

The dialogical nature of proof, Novaes

- Extremely refreshing view
- Very welcome expansion of our view of what proof is.
- Clarifying many aspects of proof and the role of proof in mathematics
- Very deserving of the Lakatos award

A few criticisms

1. How are we to reconcile proof-as-dialogue with the fact that proofs are commonly written by one person, in isolation, and not subsequently modified?
    - Proof as monologue?
    - The dialogue seems often at best imaginary
    - Is the dialogue dispensible? If so, how can it be a core feature of proofs?
2. Tension between proof-as-dialogue with the idea of a finished or completed proof.
    - Proof as dialogue seems inherently unfinished
    - But we don't ordinarily view proofs as unfinished.
3. Tension between proof-as-dialogue with Wittgenstein's idea that it must be possible to reproduce a proof exactly, even if there are differences.
    - My diagram might not be perfectly identical to yours, but Wittgenstein says I have still copied your proof exactly even so.
    - Related to the identity problem. What does it mean to say that two proofs are the same? When are two algorithms are the same?
4. Tension between proof-as-dialogue as a linear notion vs. proof as an argument that can be revisited at previous points.
    - Like A-series vs. B-series view of time.
        - Is the proof a completed thing, visited by the dialogue at various points?
        - Or does the dialogue traverse through the proof?
    - A dialogue proceeds linearly even if topics are revisited.
    - But the proof is there already completed
        - one can talk about the proof nonlinearly, going back and forth to different steps
    - So the proof and the proof dialogue are not the same thing?
5. Does Novaes conflate the proof with the dialogues that naturally arise in connection with it?
    - The proof is not the same thing as the dialogue we might have about it
    - The book ≠ the literary criticism of it
    - The film ≠ the file review on Siskel and Ebert
    - Why should proof = the proof dialogue?