THE MODEL THEORY OF SET-THEORETIC MEREOLOGY

NOTRE DAME MATH LOGIC SEMINAR
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JOINT WORK WITH MAKOTO KIKUCHI, KOBE

MEREOLOGY

STUDY OF PART/WHOLE RELATION

$p \sqsubseteq q$ — p IS PART OF q.

RICH HISTORY
- CLASSICAL ROOTS IN PRESOCRATICS
- PLATO, ARISTOTLE
- MEDIEVAL ONTOLOGISTS, SCHOLASTICS
- FORMAL THEORY
  BRENTANO, HUSSERL (1901)
- LEŚNIEWSKI 1916 FOUNDATIONS OF GENERAL THEORY OF SETS
  1927-31 FOUNDATIONS OF MATHEMATICS
- DAVID LEWIS 1991 PARTS OF CLASSES

MEREOLOGY V. SET THEORY

- OFTEN CONTRASTED
- FORMAL THEORIES APPEAR AROUND SAME TIME
- MEREOLOGY — SET THEORY

  $p \sqsubseteq q$ — $p \in q$

  AXIOMS

  REFL $p \sqsubseteq p$

  TRAN $p \sqsubseteq q \sqsubseteq r \Rightarrow p \sqsubseteq r$

  ANTI-SYM $p \sqsubseteq q \sqsubseteq p \Rightarrow p = q$

  M. SUM $\forall p, q\ \exists\ p \sqcup q$

  DIFF $\forall p, q\ \exists\ p - q$.

  ATOMICITY EVERY p IS FUSION OF ATOMS

- SET THEORY HAS FOUND ENORMOUS SUCCESS AS A FOUNDATION OF MATHEMATICS.
  CAPACITY TO REPRESENT ARBITRARY MATH'L STRUCTURE

YET, MEREOLOGY HAS NOT. ONE CAN IMAGINE A RICH, DEEP THEORY OF MEREOLOGY, USED AS FoM.

QUESTION WHY HAS MEREOLOGY NOT SUCCEEDED AS A FOUNDATION OF MATHEMATICS?

SET-THEORETIC MEREOLOGY

- STUDY THE INCLUSION RELATION $p \subseteq q$.
- FOCUS OF LEWIS' MEREOLOGICAL CONCEPTION.

QUESTION
- CAN SET-THEORETIC MEREOLOGY SERVE AS FOUNDATION OF MATHEMATICS?
- CAN $\subseteq$ PROVIDE FAITHFUL REPRESENTATIONS OF ARB. MATH'L STRUCTURE?

AT BOTTOM: CAN WE GET BY WITH $\subseteq$ INSTEAD OF $\in$?

PROJECT GIVEN UNIVERSE OF SET THEORY $(V, \in)$

CONSIDER MEREOLOGICAL REDUCT $(V, \subseteq)$.

$\underbrace{\text{PURE MEREOLOGY}}_{\subseteq \text{ ONLY}}$ VS. $\underbrace{\text{AUGMENTED}}_{\text{e.g. SINGLETON } a \mapsto \{a\}}$.

DEFINE $u \subseteq v \leftrightarrow \forall x \quad x \in u \Rightarrow x \in v$

$y = \{x\} \longleftrightarrow \forall z \quad z \in y \Leftrightarrow z = x$

CONVERSELY $x \in y \Leftrightarrow \{x\} \subseteq y$.

CONCLUSION MEREOLOGY WITH SINGLETONS $=$ SET THEORY.

QUESTION (KIKUCHI) CAN THERE BE TWO DISTINCT MODELS OF SET THEORY WITH SAME INCLUSION?

ANSWER YES. FOR EVERY $(V, \in)$ THERE IS DEFINABLE $\in^* \neq \in$ s.t. $\subseteq^* = \subseteq$.

$(V, \in) \neq (V, \in^*)$ DISTINCT

$(V, \subseteq) = (V, \subseteq^*)$ SAME

PROOF: LET $\theta : V \to V$ BE ANY DEFINABLE (NONTRIV.) PERMUTATION.

LET $\tau(u) = \theta'' u = \{\theta(a) \mid a \in u\}$. THIS IS $\subseteq$-AUTOMORPHISM:

$u \subseteq v \Leftrightarrow \theta'' u \subseteq \theta'' v \Leftrightarrow \tau(u) \subseteq \tau(v)$.

DEFINE $a \in^* b$ IFF $\tau(a) \in \tau(b)$. $\quad (V, \in^*) \underset{\tau}{\cong} (V, \in)$

OBSERVE $u \subseteq^* v \Leftrightarrow \forall a \;\; a \in^* u \Rightarrow a \in^* v$

$\Leftrightarrow \forall a \;\; \tau(a) \in \tau(u) \Rightarrow \tau(a) \in \tau(v)$

$\Leftrightarrow \tau(u) \subseteq \tau(v)$.

$\Leftrightarrow u \subseteq v$.

SO $\subseteq^* = \subseteq$. $\boxtimes$

③

<u>COROLLARY</u> SET-THEORETIC MEREOLOGY IS NOT RIGID.

$\tau : (V, \subseteq) \cong (V, \subseteq)$.

CONTRAST: $(V, \in)$ IS RIGID FOR DEFINABLE MAPS.

<u>COROLLARY</u> $\in$ IS NOT DEFINABLE FROM $\subseteq$.

<u>Q</u> IS $\subseteq$ A FOUNDATION OF MATH?

<u>A</u> NO.

<u>THEOREM</u> SET-THEORETIC MEREOLOGY IS A DECIDABLE THEORY.

WARM-UP: CONSIDER $(HF, \subseteq)$.

ISOMORPHIC TO SET OF FINITE SUBSETS OF A CTBLE SET. $(P_{FIN}(\mathbb{N}), \subseteq)$.

ISO TO SQ-FREE NUMBERS UNDER $n \mid m$.

INTERPRETABLE IN $(\mathbb{N}, \cdot)$. DECIDABLE THEORY.

IN FACT:

<u>THEOREM</u> SET-THEORETIC MEREOLOGY, CONSIDERED AS $Th(V, \subseteq)$,
IS EXACTLY: ATOMIC, UNBOUNDED, RELATIVELY COMPLEMENTED
DISTRIBUTIVE LATTICE.

FINITELY AXIOMATIZABLE, COMPLETE, DECIDABLE.

PROOF: BY QE ~~[illegible]~~ IN STYLE OF TARSKI FOR BOOL. ALG.
& ERŠOV'S EXTENSION TO REL. COMPL. DISTRIB. LATTICES.

<u>FIRST</u> IF ~~[illegible]~~ $(W, \in^W)$ IS MODEL OF SET THEORY, THEN
$(W, \subseteq^W)$ IS SUCH AN ATOMIC, UNBDD ETC. LATTICE. ✓

ZACH McKENZIE HAS RESULTS ANALYZING MINIMAL SET THEORY NEEDED FOR THIS.

<u>SECOND</u> THEORY OF ATOMIC, UNBDD, REL. COMPLEMENTED, DISTRIB. LATTICE
ADMITS QE IN LANGUAGE WITH $\subseteq$, $x \cap y$, $x \cup y$, $x - y$, $\underbrace{|x| = n, \ |x| \geq n}_{\text{FOR } n \in \mathbb{N}}$.

PROOF: PROVE QE BY INDUCTION ON FORMULAS.

CONSIDER $\exists x \ \varphi(x, y_0, \ldots, y_n)$ WITH φ Q-FREE IN EXPANDED LANGUAGE.

ELIMINATE EQUALITY VIA $x = y \leftrightarrow x \subseteq y \ \& \ y \subseteq x$
ELIMINATE $\subseteq$ VIA $a \subseteq b \leftrightarrow |a - b| = 0$
ELIMINATE NEGATION $x \nsubseteq y \leftrightarrow |y \setminus x| \geq 1$.

$|t| \ngeq n \leftrightarrow \bigvee_{k<n} |t| = k$

$|t| \neq n \leftrightarrow |t| \geq n+1 \vee \bigvee_{k<n} |t| = k$

④

- REDUCE COMPLEX TERMS TO CELLS OF VENN DIAGRAM

$$|s \cup t| = n \longleftrightarrow \bigvee_{i+j+k=n} (|s| = i+j) \wedge (|s \cap t| = j) \wedge (|t| = j+k).$$

$$|s \cup t| \geq n \leftrightarrow \bigvee_{i+j+k=n} (|s| \geq i+j) \wedge (|s \cap t| \geq j) \wedge (|t| \geq j+k)$$

REDUCE TO CELL TERMS $\bigcap_{i<N} \pm x_i$ AT LEAST ONE VARIABLE APPEARS POSITIVELY

e.g. $(x_1 \cap x_2) - (x_3 \cup x_4 \cup x_5)$.

MUST ELIMINATE QUANTIFIER FROM $\exists x\, \varphi$

φ IS BOOL COMB OF SIZE ASSERTIONS ABOUT CELL TERMS, POSITIVE

USE DNF - & DISTRIBUTE OVER $\vee$

SO φ IS CONJUNCTION OF SIZE ASSERTIONS. ~~NOW CAN ELIMINATE~~

~~e.g. $\exists x$ $|x \cap c| \geq 3$ $\wedge$ ...~~

CAN GROUP BY CELL TERMS SEPARATELY, SINCE THEY ARE DISJOINT.

NOW ELIMINATE Q

e.g. $\exists x \ (|x \cap c| \geq 3 \wedge |x \cap c| \geq 7 \wedge |c - x| = 2 \wedge |d \cap x| \geq 7)$

$$\Longleftrightarrow |c| \geq 9 \wedge |d| \geq 7$$

SENTENCES ARE SETTLED TRIVIALLY. SO THE THEORY IS COMPLETE. $\boxtimes$

<u>CONCLUSION</u>
MEREOLOGY IS A DECIDABLE THEORY.

THIS IS WHY IT CANNOT SERVE AS A FOUNDATION OF MATHEMATICS.

MAIN PHILOSOPHICAL CONCLUSION

<u>COROLLARY</u> $(HF, \subseteq) \prec (V, \subseteq)$.

<u>QUESTION</u> GIVEN MODEL OF SET THEORY $(M, \in^M)$ FORM MEREOLOGICAL MODEL $(M, \subseteq^M)$.

WHICH MODELS ARISE THIS WAY?

OR... GIVEN $(M, \sqsubseteq)$ MODEL OF SET-THEORETIC MEREOLOGY, WHEN CAN WE FIND $\in^M$ s.t. $(M, \in^M) \models$ SET THEORY & $\sqsubseteq = \subseteq^M$?

HOW MUCH SET THEORY $Th(M, \in)$ DOES MEREOLOGY $(M, \subseteq)$ KNOW?

ANSWER MEREOLOGY DOESN'T KNOW ANY SET THEORY.

THEOREM ALL COUNTABLE MODELS OF SET THEORY $(M, \in^M) \models ZFC$ HAVE ISOMORPHIC MEREOLOGICAL REDUCTS $(M, \subseteq^M)$. ☆

WORKS WITH V. WEAK SET THEORIES, BUT AC CANNOT BE OMITTED.

CONSEQUENCE OF: ☆

THEOREM MEREOLOGICAL REDUCT $(M, \subseteq^M)$ OF CTBLE MODELS OF SET THEORY ARE PRECISELY THE CTBLE SATURATED MODELS OF SET-THEORETIC MEREOLOGY. ☆

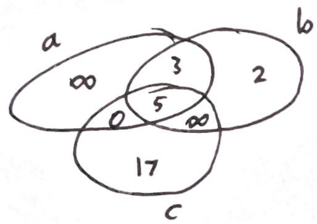
RECALL

THEOREM (LIPSHITZ/NADEL 1978) EVERY NONSTANDARD $(M, +, \cdot, 0, 1, <) \models PA$ HAS COMPUTABLY SATURATED REDUCT $(M, +)$.

FOR CTBLE M, $(M, +)$ IS DETERMINED UP TO ISOMORPHISM BY $SSy(M)$.

AT FIRST, WE EXPECTED THIS SITUATION FOR MEREOLOGY, BUT INSTEAD GOT ☆, A MUCH STRONGER RESULT.

LEMMA IF $p(a_1, \dots, a_n)$ IS A COMPLETE n-TYPE IN SET-THEORETIC MEREOLOGY, THEN $p(a_1, \dots, a_n)$ EXPRESSES EXACTLY THAT CELLS OF VENN DIAGRAM HAVE SOME SPECIFIC FINITE SIZE OR ARE INFINITE.



$|a - (b \cup c)| = \infty$

$|(a \cap b) - c| = 3$

ETC.

LEMMA A MODEL $(M, \subseteq)$ OF SET-THEORETIC MEREOLOGY IS $\aleph_0$-SATURATED IFF

① EVERY INFINITE ELT. OF M IS DISJOINT UNION OF TWO INFINITE ELTS.

& ② $\forall a \in M$ $\exists$ INFINITE ELT. DISJOINT FROM a.

PROOF OF ☆: CAN ASSEMBLE x REALIZING EACH PIECE OF THE VENN DIAGRAM. $\boxtimes$

<u>CONCLUSION</u> ~~ALL~~ ALL CTBLE MODELS OF ZFC HAVE ISOMORPHIC REDUCTS TO $\subseteq$.

MEREOLOGY KNOWS NO SET THEORY.

CAN'T TELL IF CH HOLDS OR FAILS

V=L

LC

ETC.

<u>BUT</u> WE NEEDED AC IN THE ARGUMENT.

CONSIDER A (ω-STANDARD) MODEL OF ZF WITH AN <u>AMORPHOUS</u> SET: ALL SUBSETS FINITE OR COFINITE.

SUCH A MODEL FAILS SATURATION CRITERION.

<u>QUESTION</u> HOW MANY NON-ISOMORPHIC MODELS $(M, \subseteq)$ ARISE FROM CTBLE $(M, \in^M) \models ZF$?

• CLASSES

MOVE TO GÖDEL-BERNAYS GBC

KELLEY-MORSE KM

<u>THEOREM</u> IF $(M, \in^M) \models GBC$, THEN $(M, \subseteq^M)$ IS $\aleph_0$-SATURATED OF CLASS-THEORETIC MEREOLOGY, AN $\aleph_0$-SATURATED INFINITE ATOMIC BOOLEAN ALGEBRA.

<u>COROLLARY</u> ALL CTBLE MODELS OF GBC HAVE ISO. INCLUSION RELATIONS.

• UNCTBLE MODELS

<u>THEOREM</u> FOR EVERY UNCTBLE κ, THERE ARE 2^κ MANY PAIRWISE NON-ISOMORPHIC MODELS OF SET-THEORETIC MEREOLOGY ARISING AS REDUCT $\subseteq$ OF MODEL OF SET THEORY.

SET-THEORETIC MEREOLOGY IS UNSTABLE — $\subseteq$ ITSELF IS AN ORDER RELATION.

<u>THEOREM</u> IF $(M, \sqsubseteq)$ IS MODEL OF SET-THEORETIC MEREOLOGY & SATURATED, THEN IT ARISES AS $\subseteq^M$ OF MODEL $(M, \in^M)$ OF ANY DESIRED CONSIS. SET THEORY.

PROOF USES <u>RESPLENDENCY</u>.

<u>THM</u> IF $\diamondsuit$ HOLDS AND ZFC CONSIST, THEN $\exists$ FAMILY OF 2^{ω_1} MANY ω_1-LIKE MODELS OF ZFC WITH PAIRWISE NONISOMORPHIC $\subseteq$ RELATIONS. INDEED, PAIRWISE NON-EMBEDDABLE.

PROOF BUILDS A TREE OF END-EXTENSIONS, USING $\diamondsuit$ TO ANTICIPATE & KILL OFF EMBEDDINGS.