Pointwise definable and Leibnizian extensions of models of arithmetic and set theory

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Joint work

This talk includes joint work with W. Hugh Woodin, Kameryn Williams, Victoria Gitman, as well as solo work.

Pointwise definability

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Meanwhile:

> *"I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it."*
>
> *–Horatio, age 8*

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Leibnizian models are thus precisely those that fulfill:

Leibniz principle on Identity of Indiscernibles

Indiscernible individuals are identical.

Goal Theorems

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The same method applies in set theory.

Goal Theorem 2

Every countable model of ZF has a pointwise definable end-extension. Can achieve $V = L$ in the extension, or any other theory, if true in an inner model of $V = \mathrm{HOD}$.

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The proofs are both flexible and soft.

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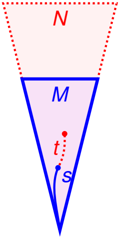

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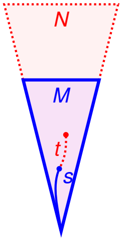

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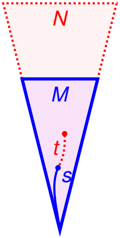

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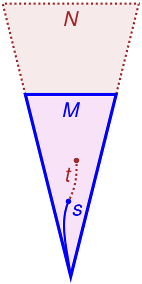
Proof proceeds by a highly self-referential algorithm, “the petulant child.”

Generalization to Σ_m-elementary extensions

The result generalizes ([Ham18]) to provide a Σ_{m+1}-definable finite sequence, with the universal extension property with respect to Σ_m-elementary end-extensions $M \prec_{\Sigma_m} N$.



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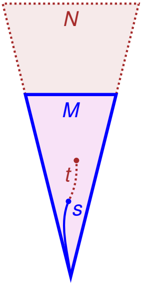

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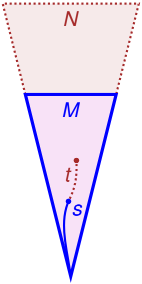

Again every model $M \models \mathrm{PA}$ can realize any desired extension t in an end-extension N.

But the difference now is that Σ_m truth is preserved between M and N.

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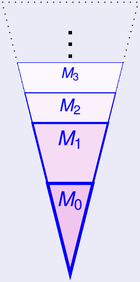

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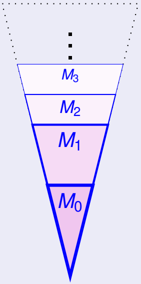

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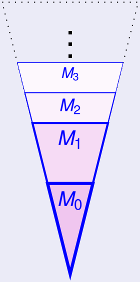

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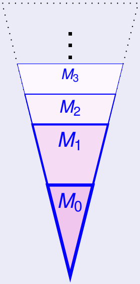

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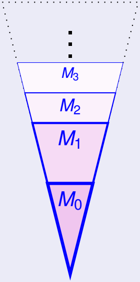

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Can arrange that every element becomes definable. So N is pointwise definable. □

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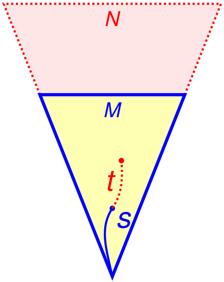

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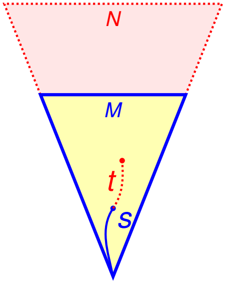

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The definition (complex, sophisticated) essentially looks for stages V_α that have no end-extension adding a next point a, even in any forcing extension, and when found, adds a anyway. “petulant child”

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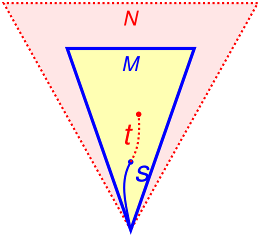

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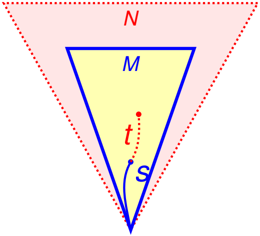

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In fact, can get $N \models \overline{\mathrm{ZFC}}$ for any theory true in some inner model W of M.

Generalization to Σ_m elementary end-extensions

In new work, I have been able to generalize to find a Σ_{m+1} definable sequence of ordinals

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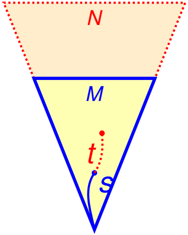

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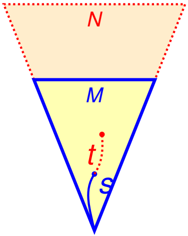

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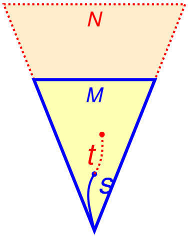

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If $V = \mathrm{HOD}$, can translate this to all objects, not just ordinals.

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This realizes a certain *resurrection* property: whatever is true in some inner model can become true again in an end-extension, even a pointwise definable end-extension.

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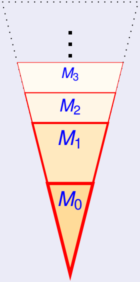

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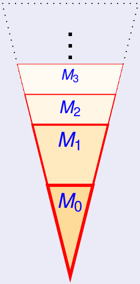

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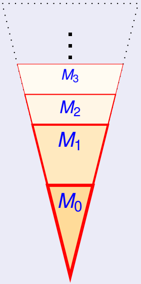

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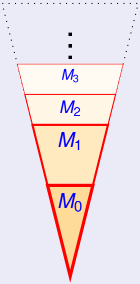

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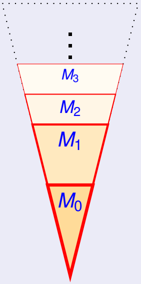

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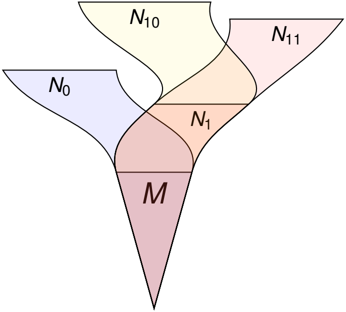
- Modal logic of end-extension potentialism is exactly S4

The tree of top-extensions



Radical-branching potentialism.

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Proof strategy. Given $M_0 \models \mathrm{PA}$ of size at most continuum, construct a progressively elementary tower

$$M_0 \prec M_1 \prec_{\Sigma_m} M_2 \prec M_3 \prec_{\Sigma_{m+1}} M_4 \prec M_5 \prec_{\Sigma_{m+2}} \cdots$$

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Proof strategy. Given $M_0 \models \mathrm{PA}$ of size at most continuum, construct a progressively elementary tower

$$M_0 \prec M_1 \prec_{\Sigma_m} M_2 \prec M_3 \prec_{\Sigma_{m+1}} M_4 \prec M_5 \prec_{\Sigma_{m+2}} \cdots$$

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Leibnizian extensions for PA

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- Odd stages, progressively elementary. Make those points definable.

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So we find $M \prec N$ with countably many new elements c_n that discern the elements of M.

Odd stages

At odd stages, we make the accumulating constants definable.

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And so the limit model is Leibnizian, as desired. $\square$

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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