Some Set Theory of Kaufmann Models

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The Graduate Center, CUNY

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Preliminaries

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• Occasionally I'll consider extensions $\mathcal{L}^* \supseteq \mathcal{L}_{\mathrm{PA}}$. An $\mathcal{L}^*$-structure M^* is said to be a model of PA^* if its reduct to $\mathcal{L}_{\mathrm{PA}}$ is a model of PA and it satisfies the induction schema in the expanded language.

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• An important notion that will be key throughout this talk is that of end extension: if $M, N \models \mathrm{PA}$ then we say that N is an *elementary end extension* of M, denoted $M \prec_{end} N$ if $M \prec N$ and for all $x \in M$ and all $y \in N \setminus M$ $N \models x \leq y$.

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• The MacDowell-Specker theorem states that every model of PA (in fact PA* for a countable $\mathcal{L}^*$) has a proper elementary end-extension.

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• An important theorem is that for each standard n, there is (provably, in PA) a Σ_n formula $Tr_n(x, y)$ so that for all Σ_n formulas $\varphi(z)$ PA $\vdash \forall y[\varphi(y) \leftrightarrow Tr_n(\varphi, y)]$. Given a model $M \models$ PA let W_n^M denote the set of true Σ_n sentences from the point of view of M. I'll drop the superscript when M is clear from the context.

Satisfaction Classes

Since PA has the resources to discuss local truth, it makes sense to ask about global truth definitions. This justifies the following definition. Below, given a structure M and a set $X \subseteq M^2$ let, for $e \in M$, $(X)_e = \{x \in M \mid \langle x, e \rangle \in X\}$ be the projection onto the e^{th} slice.

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Definition

Let $M \models \text{PA}$. A set $S \subseteq M^2$ is called a *satisfaction class* if

- For all $x \in M$ $(S)_x$ is a set of formulas from the point of view of M.
- For all $n < \omega$ we have $(S)_n = W_n$.
- $(M, S) \models \text{PA}^*$ in the language expanded with a predicate for S.

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As a remark, the above is more technically called a *Partial Inductive* satisfaction class however, this is the only type we will be considering so I'll drop the extra qualifiers. It's also not the usual definition, but an equivalent one, which will be useful in this talk for proofs.

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Recursive saturation was first isolated by Schlipf in the 70's and studied extensively thereafter. It has an enormous number of equivalent characterizations and is one of the most useful ideas in model theory of arithmetic.

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2. (Kotlarski, Krajewski and Lachlan) If M is countable and recursively saturated, then M has a satisfaction class.

Note that part 2 requires the model to be countable, the starting point of today's talk is whether that assumption can be removed. We shall see the answer is "no", a result due to Kaufmann and Shelah.

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As an example, observe that if $M \models \mathrm{PA}$ is non-standard and $X \subseteq M$ is a cofinal sequence in order-type ω then X is a class. Since any cofinal ω-sequence cannot be definable, any model with such a sequence is not rather classless. In particular, no countable model is rather classless.

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• Using $\diamondsuit$ Kaufmann constructed such a model, with additional property of being ω_1-like: that is uncountable but every proper $\leq$-initial segment is countable.
• A *Kaufmann model* is an ω_1-like, recursively saturated, rather classless model of PA.

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Lemma (Kaufmann)

Suppose M is a countable, recursively saturated model of PA. *For any $A \subseteq M$ undefinable there is a countable, recursively saturated elementary end-extension $M \prec_{end} N$ in which A is not coded i.e. so that for no N-finite sequence $\bar{a}$ do we have $A = M \cap \bar{a}$.*

Kaufmann's Theorem

Sketch of Kaufmann's Theorem.

Fix a countable, recursively saturated model $M_0 \models \mathrm{PA}$. Let $\vec{A} = \langle A_\alpha \mid \alpha < \omega_1 \rangle$ a $\diamondsuit$ sequence.

- We will inductively define a sequence $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of countable, recursively saturated models so that for all α $M_\alpha \prec_{end} M_{\alpha+1}$ and if δ is a limit ordinal then $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$.
- The models will have universe some ordinal $\delta < \omega_1$. Note that there will necessarily be a club of δ so that M_δ has universe δ.

□

Kaufmann's Theorem

Continued.

• Since the limit stage is determined, we need to say what to do at successor stages. Suppose we have constructed M_α. If $|M_\alpha| = \alpha$ and A_α is undefinable let $M_{\alpha+1}$ be as in Kaufmann's lemma. Otherwise, let $M_{\alpha+1}$ be any countable, recursively saturated elementary end extension.

• Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$. Clearly this model is ω_1-like and recursively saturated. It remains to see that it's rather classless.

• Suppose $A \subseteq M$ is an undefinable class. Then there is a club of $\alpha < \omega_1$ so that $A \cap M_\alpha$ is an undefinable class in M_α. Therefore, by $\diamondsuit$ there is some α so that $A \cap M_\alpha = A_\alpha$. But then we ensured that A_α was not coded into $M_{\alpha+1}$ contradicting the assumption that A is a class.

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- For instance, using what we have observed so far, it's easy to see that if M is Kaufmann, then even though it has no satisfaction class, there is a club of elementary substructures $N \in [M]^\omega$ which do.

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• This is similar to the existence of an Aronszajn tree, an analogy I'll return to.

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Is there a Kaufmann model M and an ω_1-preserving forcing $\mathbb{P}$ so that forcing with $\mathbb{P}$ adds an undefinable class to M?

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I won't prove it in the interest of time, but the conclusing of 1 above is consistent also with CH and the conclusion of 2. is also consistent with the continuum arbitrarily large.

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• I need to show that, if $\Vdash_{\mathbb{P}}$ "$\check{M}$ has an undefinable class" then $\mathbb{P}$ collapses ω_1. □

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• Since M is ω_1-like this tree is an ω_1-tree i.e. it is uncountable, but every level is countable. If $t \in T^M_{fin}$ is of length $a \in M$ then it codes a subset of a. Thus, the a^{th}-level of the tree has size 2^a which externally is countable by ω_1-likeness.

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• Observe that if $A \subseteq M$ is a class which is not M-finite then its characteristic function is an uncountable branch through T^M_{fin}. Moreover this remains true in any forcing extension since we can't add new elements to M.

• Therefore all of the uncountable branches through T^M_{fin} are definable by rather classlessness, and if $\mathbb{P}$ adds an undefinable class, it must add a cofinal branch to T^M_{fin}.

• Since there are only $\aleph_1$ definable sets with parameters, T^M_{fin} has at most $\aleph_1$ uncountable branches. By a theorem of Baumgartner, $\mathsf{MA}_{\aleph_1}$ implies that if T is an ω_1-tree with at most $\aleph_1$ many cofinal branches then there T is *essentially special*: there is a function $f : T \to \omega$ so that for all $s, t, u \in T$ if $s \leq_T t, u$ and $f(s) = f(t) = f(u)$ then t and u are comparable.

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• To see why this is true, fix an essentially special tree T as witnessed by $f : T \to \omega$ and suppose $p \Vdash_{\mathbb{P}}$ "$\dot{b}$ is a new uncountable branch". Let $p \in G$ be generic over V and work in $V[G]$. Fix a cofinal, ω_1^V-indexed subset of b, say $\{b_\alpha \mid \alpha < \omega_1\}$ and consider the map $g : \omega_1^V \to \omega$ given by $\alpha \mapsto f(b_\alpha)$. If ω_1 isn't collapsed then $\omega_1 = \omega_1^V$ and so there is an $n < \omega$ so that $g^{-1}\{n\}$ is uncountable.

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• Choose $q \leq p$ $q \in G$ which decided which n. Without loss assume that q also decides some $\check{s} = b_\alpha \in g^{-1}\{n\}$ for some $\alpha < \omega_1$. Back in V, since $\dot{b}$ was forced to be new there are incomparable extensions $r_0, r_1 \leq q$ deciding some $t_i \in g^{-1}\{n\}$ for $i < 2$ with t_0 and t_1 incomparable. But this contradicts the definition of essentially special.

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Assume all ω_1-trees with $\aleph_1$ many cofinal branches are essentially special. Then given any ω_1-like model $M \models \mathrm{PA}$ with at most $\aleph_1$-many classes, any forcing $\mathbb{P}$ adding an undefinable class to M collapses ω_1.

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Corollary

Assume all ω_1-trees with $\aleph_1$ many cofinal branches are essentially special and there are no Kurepa trees. Then there is no ω_1-preserving forcing adding an undefinable class to an ω_1-like model of PA.

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• This tree has countable levels by ω_1-likeness. It has height cofinal in the model by recursive saturation. Finally any cofinal branch would generate a satisfaction class so it has none. Thus this is an M-Aronszajn tree.

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- For all $\alpha < \omega_1$, $M_\alpha \prec_{end} M_{\alpha+1}$ is countable and recursively saturated and if δ is a limit ordinal then $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$, AND

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- $S_\alpha \subseteq T^{M_\alpha}_{sat}$ intersects every level and for all α $S_\alpha \subseteq S_{\alpha+1}$, $S_{\alpha+1} \setminus S_\alpha \subseteq M_{\alpha+1} \setminus M_\alpha$ and if δ is a limit ordinal then $S_\delta = \bigcup_{\alpha<\delta} S_\alpha$.

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• If $\alpha = \delta + 1$ with δ a limit ordinal but A_δ is not a maximal antichain through S_δ then similarly let S_α be S_δ plus all extensions of elements from S_δ in $T_{sat}^{M_\alpha}$ to all of the levels in $M_\alpha \setminus M_\beta$. □

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• Fix a level $d \in M_{\delta+1} \setminus M_\delta$ and for each for each $t \in S_\delta$ pick an element s_t of $T_{sat}^{M_{\delta+1}}$ of height d which is above t and exactly one node of A_δ (this exists by recursive saturation and maximality of the antichain).

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• Let $S^-_{\delta+1} = S_\delta \cup \{s_t \mid t \in S_\delta\}$ and finally let S_δ be the downward closure of S^-_δ plus all extensions of elements from S^-_δ on all levels in $M_{\delta+1}$ above d

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• Finally let $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and let $S = \bigcup_{\alpha<\omega_1} S_\alpha$. The verification that M is a Kaufmann model is exactly the same as in Kaufmann's theorem.

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• Finally let $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and let $S = \bigcup_{\alpha<\omega_1} S_\alpha$. The verification that M is a Kaufmann model is exactly the same as in Kaufmann's theorem. The verification that S is a Souslin tree is exactly as in Jensen's theorem.
• To finish off the theorem, note by construction that S is a subtree of T^M_{sat} so any forcing adding an uncountable branch to S (such as S itself) adds an uncountable branch to T^M_{sat} and, as observed before, such a branch generates a satisfaction class for M. □

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• In general this question remains open and is the subject of ongoing reseach. However, these methods lend themselves to a nice application which (I think) is model theoretic: the study of strong logics in the context of these models.

The Logic $L_{\omega_1,\omega}(Q)$

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• We'll need to know how this logic interacts with forcing. First note that by Keisler's completeness theorem if a sentence ψ (from V) has a model in some forcing extension then it has a model in the V.

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• We'll need to know how this logic interacts with forcing. First note that by Keisler's completeness theorem if a sentence ψ (from V) has a model in some forcing extension then it has a model in the V.

• Also, it's easily proved by induction on formulas that if $\mathbb{P}$ is a forcing notion which does not collapse ω_1 and $\mathcal{A}$ is an $L_{\omega_1,\omega}(Q)$ structure, $\bar{a}$ is a tuple of elements from $\mathcal{A}$ and $\psi(\bar{x})$ is an $L_{\omega_1,\omega}(Q)$ formula, then $\mathcal{A} \models \psi(\bar{a})$ if and only if $\Vdash_{\mathbb{P}} \check{\mathcal{A}} \models \psi(\check{\bar{a}})$ i.e. $L_{\omega_1,\omega}(Q)$ truth cannot be changed by forcing that does not collapse ω_1.

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• By the first bullet point this model has a reduct to a Kaufmann model, hence there was a Kaufmann model in V all along.

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• Observe that being ω_1-like and recursively saturated are both easily expressible here so the question is really about rather classlessness.

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• Given Shelah's proof it's natural to ask whether the appeal to generic absoluteness is necessary. Put another way, might it be the case that already in V there is an $L_{\omega_1,\omega}(Q)$ sentence ψ so that every Kaufmann model M, or perhaps some expansion of M, satisfies ψ? Glossing over techincal details, the cartoon version of this question is simply, "Is the class of Kaufmann models axiomatizable in $L_{\omega_1,\omega}(Q)$?".

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• Observe that being ω_1-like and recursively saturated are both easily expressible here so the question is really about rather classlessness. To see this, note that

1. M is ω_1-like if and only if it satisfies $Qx(x = x) \wedge \forall y \neg Qx(x \leq y)$ and
2. M is recursively saturated if and only if it satisfies $\forall \bar{y} \bigwedge_{p(x,\bar{y}) \text{ a computable type}} (\bigwedge_{\Phi(x,\bar{y}) \text{ finite subset of } p(x,\bar{y})} \exists x \Phi(x, \bar{y}) \rightarrow \exists x \bigwedge_{\phi(x,\bar{y}) \in p(x,\bar{y})} \phi(x, \bar{y}))$

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Hence, roughly speaking, it's independent of ZFC whether or not you can axiomatize the class of Kaufmann models in $L_{\omega_1,\omega}(Q)$.

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Proof of Part 1.

Assume $MA_{\aleph_1}$. Let $\mathcal{L}' \supseteq \mathcal{L}_{PA}$ be the language of PA enriched with a single additional unary function symbol f. Since being an ω_1-like, recursively saturated model of PA is expressible in $L_{\omega_1,\omega}(Q)$ I need to prove that there is a sentence ψ, using the symbol f, so that a model M is Kaufmann if and only if there is a function f^M so that $\langle M, f^M \rangle \models \psi$.

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• The idea is that f will be a function witnessing the essential specialness of the tree T^M_{fin} with a little more. Specifically, first note that we can write that f is an essential specializing function as $(\forall x \bigvee_{n<\omega} f(x) = n) \wedge \forall s, t, u(s \leq_{fin} t, u \wedge f(s) = f(t) = f(u) \rightarrow (t \leq_{fin} u \vee u \leq_{fin} t))$.

□

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Proof of Part 1, Continued.

• The hard part is using this to say that M is rather classless. We need to say that the only uncountable branches are the (first order) definable ones.

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• The hard part is using this to say that M is rather classless. We need to say that the only uncountable branches are the (first order) definable ones. This is written as follows:

$$\forall s \bigvee_{n<\omega}(f(s) = n \wedge Qt(f(s) = f(t) = n \wedge s \lneq_{fin} t)) \rightarrow$$
$$\exists \bar{a} \bigvee_{\varphi \in L_{\mathrm{PA}}}[\forall y \varphi(y, \bar{a}) \leftrightarrow \exists t(s \leq_{fin} t \wedge t(y) = 1 \wedge f(t) = n)]$$

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• In English the above says the following:

For all s, if for some n $f(s) = n$ and there are uncountably many t so that $s \leq_{fin} t$ and $f(t) = n$ then there is an $\bar{a}$ and a formula $\varphi \in \mathcal{L}_{\mathrm{PA}}$ so that for all y $\varphi(y, \bar{a})$ if and only if $t(y) = 1$ for some t with $s \leq_{fin} t$ and $f(t) = n$.

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• We know that by $\mathrm{MA}_{\aleph_1}$ a model is Kaufmann if and only if its tree T^M_{fin} is essentially special and all of the uncountable branches are definable so let's see that this last part says that.

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• For the backward direction suppose M is ω_1-like, recursively saturated and has an expansion satisfying ψ. Let b be an uncountable branch through T^M_{fin}. We need to show that there is a formula φ and a tuple $\bar{a}$ so that for all $x \in M$, $b(x) = 1$ if and only if $M \models \varphi(x, \bar{a})$. □

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Proof of Part 1, Continued.

• Since b is uncountable, there is an s so that $s \in b$ and an $n < \omega$ so that $f(s) = n$ and there are uncountably many t above s so that $t \in b$ and $f(t) = n$ also.

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• By ψ then there is an $\bar{a}$ and a formula $\varphi \in \mathcal{L}_{PA}$ so that for all y $\varphi(y, \bar{a})$ if and only if there is a t above s with $t(y) = 1$ and $f(t) = n$. By the property of essentially specializing functions, if $s \leq_{fin} t$ and $f(t) = n$ then $t \subseteq b$. Therefore $b(y) = 1$ if and only if $\varphi(y, \bar{a})$ as required. □

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- The forward direction is similar. Let M be a Kaufmann model.
- By $\mathsf{MA}_{\aleph_1}$ there is an essential specializing function $f : T^M_{fin} \to \omega$. Let f^M be this f.
- We need to see that $\langle M, f^M \rangle \models \psi$. Fix $s \in T^M_{fin}$ and $n < \omega$ and suppose that $f^M(s) = n$ there are uncountably many t above s with $f^M(t) = n$. Then the set of these t must generate a cofinal branch b by essential specialness so we can define that branch as $b(x) = 1$ if and only if $M \models \varphi(x, \bar{a})$ by rather classlessness, hence ψ is satisfied.

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Proof of Part 2, Continued.

• However, the generic codes a satisfaction class, A_G. Consider a further expansion $\mathcal{L}''$ of $\mathcal{L}'$ by a single unary symbol A and let T_A be the $L_{\omega_1,\omega}(Q)$ theory in the language $\mathcal{L}''$ consisting of $Th(M')$ plus "A is a satisfaction class for the reduct to $\mathcal{L}_{\mathsf{PA}}$".

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• Therefore $N \equiv_L M$ but N has a satisfaction class, as required. □

Thank You!

References

James E. Baumgartner. *Applications of the Proper Forcing Axiom*
In Kenneth Kunen and Jerry E. Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 913 -959. North-Holland Pub. Co., 1984.

Matt Kaufmann. *A Rather Classless Model*
Proceedings of the American Mathematical Society, 62(2):330-333, 1977.

H. Jerome Keisler. *Logic with the Quantifier "There Exist Uncountably Many"*
Ann. Math. Logic, 1(1):1-93, 1970.

James H. Schmerl. *Recursively Saturated, Rather Classless Models of Peano Arithmetic*
Springer, Berlin. pp. 268-282, 1981.

Saharon Shelah. *Models with Second-Order Properties. II. Trees with No Undefined Branches*
Ann. Math. Logic, 14(1):73-87, 1978.

Corey Switzer. *Destructibility and Axiomatizability of Kaufmann Models*
In Preparation.