Definability and the Math Tea argument:

Must there be numbers we cannot describe or define?

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The math tea argument

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> *"There must be real numbers we cannot describe or define, because there are uncountably many real numbers, but only countably many definitions."*

Does this argument withstand scrutiny?

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> *"I can describe any number. Let me show you: you tell me a number, and I'll tell you a description of it."*
>
> *–Horatio, age 8*

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No other elements are definable, because negation $x\mapsto -x$ is an automorphism.

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Thus, $\langle \mathbb{Z}, +, \cdot \rangle$ is *pointwise definable*: every individual is definable.

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Note that the order $<$ is definable from algebraic structure

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But only algebraic numbers are definable in $\langle\mathbb{R},+,\cdot,0,1,<\rangle$.

Theorem (Tarski)

In the ordered real field $\langle\mathbb{R},+,\cdot,0,1\rangle$, every formula $\varphi(x)$ is equivalent to a quantifier-free formula.

One begins to see this by recalling

$$\exists x\ ax^2+bx+c=0 \quad\iff\quad b^2-4ac\geq 0.$$

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Corollary

The field of real algebraic numbers $\mathbb{A}$ is an elementary substructure of $\langle\mathbb{R},+,\cdot,0,1,<\rangle$.

Leibnizian models

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Are these notions the same?

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This is a successful instance of the Math Tea argument.

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Note that $\langle \mathbb{Z}, <, A \rangle$ is rigid, even though it has no definable elements.

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In particular, every computable real number and much more is definable.

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In trying to define more objects, we are inevitably drawn to expand the language and to extend the structure.

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We are thereby pushed:

- to allow only countable languages, and
- to consider only structures that are themselves definable with respect to the set-theoretic background $\langle V, \in\rangle$.

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Well, it's complicated.

In a fixed structure

In a fixed structure $\mathcal{M}$ in a countable language, the math tea argument is fine: there are only countably many definitions, but uncountably many reals.

We simply associate each definable object r with a formula ψ_r that defines it. With access to such a definability map

$$\psi_r \mapsto r,$$

we may diagonalize against it to produce a real that is not definable.

Meta-mathematical obtacle

When defining reals r over the full set-theoretic universe $\langle V, \in\rangle$, however, a subtle meta-mathematical obstacle arises:

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The key subtlety is that if we lack the association of definition with object defined, we cannot undertake the diagonalization to produce the non-definable real.

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But that isn't true.

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Pointwise definable models with same theory are isomorphic. So these models are exactly all the pointwise definable models of ZFC.

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For continuum many such models, force to add a Cohen real $N[c]$, and then force $V = \mathrm{HOD}$ in $N[c][G]$ by coding into the GCH pattern, and make c definable. The definable elements of $N[c][G]$ include c and have pointwise definable Mostowski collapse. There is a perfect set of such c. $\square$

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The argument generalizes to show that the next-least ZFC-model L_β after L_α is also pointwise definable, and indeed pointwise definability is pervasive in the countable L-hierarchy.

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Every countable model of ZFC *has a pointwise definable class forcing extension.*

Proved by myself, Linetsky, and Reitz in [HLR13]. Mentioned independently by Enayat in [Ena05].

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  - Enumerate dense classes $D_0, D_1, D_2, \ldots$

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- First step. (Simpson) Find M-generic $U \subseteq \mathrm{Ord}^M$ via $\mathrm{Add}(\mathrm{Ord}, 1)$ such that $\langle M, \in^M, U \rangle$ is pointwise definable.
  - Enumerate elts of M as $a_0, a_1, a_2, \ldots$
  - Enumerate dense classes $D_0, D_1, D_2, \ldots$
  - Build U by meeting each D_n minimally, then coding a_n.

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- Second step. Force with self-encoding forcing to code U and G into GCH pattern of $M[G]$.
- Conclusion: in $M[G]$, every set is definable without parameters.

Extending to Gödel-Bernays set theory

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GBC has class comprehension, but only for first-order assertions. Conservative over ZFC.

Pointwise definable models of GBC

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Thus, even when we augment our ZFC model with a large family of non-definable classes, we may nevertheless make those classes (and all sets) first-order definable in an extension of the model.

In the end, we have a pure ZFC model, while retaining all original classes, and making them all definable without parameters.

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The result is a forcing extension $M[G]$ in which every set and class is first-order definable without parameters. $\square$

Definability map

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But this isn't always true.

The range of possibility

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Specifically, there is no formula $\mathrm{df}(x)$ in the language of set theory that is satisfied in any model $M \models \mathrm{ZFC}$ exactly by the definable elements. To see this, consider $\forall x\, df(x)$ in a pointwise definable model and elementary extensions.

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(ii) In some models, the class of definable elements is nevertheless definable.

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(iii) In others, the definable elements do not form a class.

Consider any nontrivial ultrapower of a pointwise definable model.

More possibilities

(iv) The definable elements may be a class, but not $\psi_r \mapsto r$.

This is true in a pointwise definable model.

(v) The definable elements can be a set, along with $\psi_r \mapsto r$.

True in V if there is γ with $V_\gamma \prec V$.

(vi) No model has a *definable* definability map $\psi_r \mapsto r$.

Diagonalize against it.

The surviving content of the math-tea argument: in any model with $\psi_r \mapsto r$, the definable reals do not exhaust all the reals.

Side-stepping Russell

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Reveals subtle definability aspect to Frege/Russell interaction.

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Returning to the math-tea argument. . .

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- But even if not, we might enlarge our universe to make this true.

And so ultimately, Horatio is right, but possibly only in an extension of the universe...

Thank you.

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References I

Ali Enayat. “Models of Set Theory with Definable Ordinals”. *Archive of Mathematical Logic* 44 (2005), pp. 363–385.

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