# Generalizations of the Kunen Inconsistency

Joel David Hamkins

New York University, Philosophy

The City University of New York, Mathematics
College of Staten Island of CUNY
The CUNY Graduate Center

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# The Kunen Inconsistency

Many large cardinal axioms assert the existence of a nontrivial elementary embedding $j : V \to M$.

As the axioms becomes stronger, $M$ exhibits increasing affinity with $V$.

Reinhardt, taking the natural limit of this trend, proposed a nontrivial elementary embedding $j : V \to V$.

Shortly after it was proposed, Kunen refuted the existence of such embeddings $j$.

# The Kunen inconsistency

**Theorem (The Kunen Inconsistency, 1971)**

*There is no nontrivial elementary embedding $j : V \to V$. Consequently, there are no Reinhardt cardinals.*

The theorem has been generalized by many mathematicians: Woodin, Foreman, Harada, Zapletal, Suzuki, and others.

In this talk, I shall present several known results along with some new generalizations.

The talk could have been called "generalizations-of-generalizations" of the Kunen inconsistency.

# A sample of the generalizations

1. There is no $j : V[G] \to V$; nor $j : V \to V[G]$.
2. More generally, there is no $j$ between two ground models.
3. There is no $j : M \to N$, if $M, N$ eventually stationary correct.
4. There is no $j : V \to \mathrm{HOD}$.
5. There is no $j : V \to \mathrm{HOD}^\eta$, no $j : V \to \mathrm{gHOD}$, no $j : V \to \mathrm{gHOD}^\eta$.
6. There are no such $j$ added by set forcing.
7. If $j : V \to M$ is elementary, then $V = \mathrm{HOD}(M)$.
8. There is no $j : \mathrm{HOD} \to V$.
9. There is no $j : M \to V$, if $M$ is definable.
10. There is no $j : \mathrm{HOD} \to \mathrm{HOD}$ definable from parameters.

Other results: iterated $\mathrm{HOD}^\eta$, the generic-HOD and $\mathrm{gHOD}^\eta$, generic embeddings, definable embeddings, and $\neg\mathrm{AC}$ results.

# Dispelling a few meta-mathematical clouds

Let me begin by clearing up a few meta-mathematical issues.

The first is that the Kunen inconsistency is explicitly a second-order claim

> "there is no $j$ such that..."

Since $j$ is a proper class of some kind, this is explicitly quantifying over classes.

How are we to formalize the assertion in set theory?

# A second-order claim

To be sure, many large cardinal notions have second-order definitions, with first-order equivalent formulations.

Example: measurable cardinals, supercompact cardinals.

**Observation**

Reinhardt cardinals have no consistent first-order formulation.

**Proof.**

Suppose they do. Let $\kappa$ be the least Reinhardt cardinal. So there is $j : V \to V$ with critical point $\kappa$. By elementarity, $j(\kappa)$ is also the least Reinhardt cardinal, a contradiction. □

Similarly, no consistent first order property can imply that $\kappa$ is Reinhardt.

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Similarly, no consistent first order property can imply that $\kappa$ is Reinhardt.

# Classes in ZFC

The traditional approach to classes in ZFC is via the first-order formulas that might define them. All talk of classes is a substitute for formulas.

With this approach, the Kunen inconsistency becomes a theorem scheme, asserting of each formula $\psi$ that for no parameter $z$ does the relation $\psi(x, y, z)$ define a function

$$y = j(x)$$

that is an elementary embedding from $V$ to $V$.

Our view is that this does not convey the full power of the Kunen inconsistency.

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# Formalize in GBC

A stronger result is obtained by formalizing the Kunen inconsistency in second-order set theory, such as Gödel-Bernays or Kelly-Morse.

Kunen himself used Kelly-Morse set theory:

> *It is intended that our results be formalized within the second order Morse-Kelley set theory..., so that statements involving the satisfaction predicate for class models can be expressed. (Kunen, 1971)*

But actually, GBC suffices, a fragment of $\mathrm{ZFC}(j)$.

# A second cloud

In GBC, how do we formalize the claim that $j$ is elementary?

Naïvely, this is a scheme:

$$\forall x\,[\varphi(\vec{x}) \longleftrightarrow \varphi(j(\vec{x}))].$$

But a scheme does not serve our purpose, since the assertion that $j$ is elementary appears negatively in the theorem, and the negation of a scheme is not expressible as a scheme.

Kunen addressed the issue by using KM, in which first-order satisfaction is expressible.

# Elementarity in GBC

In the weaker theory GBC, one can use Gaifman's observation:

**Lemma (Gaifman)**

*If $j : M \to N$ is $\Delta_0$-elementary and cofinal, where $M$ and $N$ satisfy ZF, then $j$ is fully elementary.*

Note that when the models have the same ordinals, then $\Sigma_1$-elementary embeddings are cofinal.

The conclusion of the lemma is a scheme, but the hypothesis is a first-order assertion.

Note that in KM formalization, we get full elementarity internally to the theory. For example, we can induct on $\Sigma_n$-elementarity.

In GBC, this induction takes pace in the meta-theory.

# Formalizing the Kunen inconsistency

To summarize:

We formalize and prove the Kunen inconsistency in GBC as the claim that there is no class $j$ which is a nontrivial $\Sigma_1$-elementary embedding $j : V \to V$.

# Generalizations of the Kunen inconsistency

Let's begin now to prove various generalizations of the Kunen inconsistency.

Begin with the case of an elementary embedding

$$j : V[G] \to V,$$

which is a very natural case to consider. From the forcing extension $V[G]$, such an embedding maps the universe into a transitive inner model, which might seem initially like an ordinary large cardinal situation.

# Embeddings $j : V[G] \to V$

**Theorem (Woodin)**

*In any set-forcing $V[G]$, there is no $j : V[G] \to V$.*

**Proof.**

Suppose $j : V[G] \to V$ via $\mathbb{P}$. Find $\lambda > |\mathbb{P}|$, $\kappa = \mathrm{cp}(j)$ with $j(\lambda) = \lambda$ and hence $j(\lambda^+) = \lambda^+$. In $V[G]$ partition $\mathrm{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha<\kappa} S_\alpha$ into stationary sets. Let $S^* = j(\vec{S})(\kappa)$, a stationary subset of $(\mathrm{Cof}_\omega \lambda^+)^V$ in $V$, disjoint from every $j(S_\alpha)$. Let $C = \{\, \beta < \lambda^+ \mid j\,"\,\beta \subseteq \beta \,\}$, club subset of $\lambda^+$. Find club $D \subseteq C$ with $D \in V$. So $\exists \beta \in D \cap S^*$. Observe $\mathrm{cof}(\beta) = \omega$ and $j\,"\,\beta = \beta$, and hence $j(\beta) = \beta$. Also, $\beta \in S_\alpha$ some $\alpha$, so $\beta = j(\beta) \in j(S_\alpha)$ and in $S^*$, contradiction. □

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An equivalent formulation of the theorem is:

**Corollary**

*If $j : V \to M$ is a nontrivial elementary embedding in $V$, then $V$ is not a set-forcing extension of $M$.*

In other words, if $j : V \to M$, then $M$ is not a ground of $V$.

# Converse embeddings $j : V \to V[G]$

## Theorem (Woodin)

*In any set-forcing extension $V[G]$, there is no nontrivial elementary embedding $j : V \to V[G]$.*

**Proof.**

Suppose $j : V \to V[G]$. Let $\kappa = \text{cp}(j)$. We may find $\lambda \gg \kappa, |\mathbb{P}|$ with $j(\lambda) = \lambda$ (but more subtle than before), and hence $j(\lambda^+) = \lambda^+$ and $j(\lambda^{++}) = \lambda^{++}$. Note that $\text{Cof}_{\lambda^+}$ is absolute between $V$ and $V[G]$. Partition $\text{Cof}_{\lambda^+} \lambda^{++} = \bigsqcup_{\alpha<\kappa} S_\alpha$ into stationary sets in $V$. Let $S^* = j(\vec{S})(\kappa)$, stationary subset of $\text{Cof}_{\lambda^+} \lambda^{++}$ in $V[G]$, disjoint from every $j(S_\alpha)$. Let $C = \{ \beta < \lambda^{++} \mid j\,"\,\beta \subseteq \beta \}$, club subset of $\lambda^{++}$ in $V[G]$. So $\exists \beta \in S^* \cap C$. Since $\text{cof}(\beta) = \lambda^+$, it follows $j(\beta) = \beta$. So $\beta \in S_\alpha$ some $\alpha$ and so $\beta = j(\beta) \in j(S_\alpha)$ and $\beta \in S^*$, a contradiction. $\square$

# Converse embeddings $j : V \to V[G]$

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# No $j$ between grounds

Both theorems are instances of the following:

**Theorem**

*If $M$ and $N$ are set-forcing ground models of $V$, then there is no nontrivial elementary embedding $j : M \to N$.*

In other words, if $M$ and $N$ have a common set-forcing extension $M[G] = N[H] = V$, then there is no $j : M \to N$ there.

**Corollary (Suzuki 1998)**

*In no set-forcing extension $V[G]$ is there a nontrivial elementary embedding $j : V \to V$.*

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# Generalizing to stationary-correct

A class model $M \subseteq V$ is *stationary correct* to $V$ at $\delta$ if every stationary subset of $\delta$ in $M$ remains stationary in $V$.

**Theorem**

*If $M, N \models \mathrm{ZFC}$ are eventually stationary-correct to $V$, then there is no nontrivial elementary embedding $j : M \to N$.*

The proof is to push harder on the previous arguments, and pay attention to some delicate details, but it works out. Essentially: suppose $j : M \to N$; find very large $\lambda$ with $j(\lambda) = \lambda$. Partition $\mathrm{Cof}_{\lambda^+} \lambda^{++}$ into stationary sets in $M$. Let $S^* = j(\vec{S})(\kappa)$ stationary in $N$. Find $\beta = j(\beta)$ in both $j(S_\alpha)$ and in $S^*$.

# There is no nontrivial elementary $j : V \to \mathrm{HOD}$

**Theorem (Woodin)**

*There is no nontrivial elementary embedding $j : V \to \mathrm{HOD}$.*

**Proof.**

Suppose $j : V \to \mathrm{HOD}$. Find $\lambda > \kappa = \mathrm{cp}(j)$ with $j(\lambda) = \lambda$, $j(\lambda^+) = \lambda^+$. Partition $\mathrm{Cof}_\omega\, \lambda^+ = \bigsqcup_{\alpha<\lambda^+} S_\alpha$ into stationary sets. Let $\vec{T} = j(\vec{S})$.

Claim: $\xi \in \mathrm{ran}(j)$ iff $T_\xi$ is stationary in $V$. If $\xi = j(\alpha)$, then $S_\alpha$ and $j(S_\alpha) = T_{j(\alpha)}$ agree on club $C = \{\, \beta < \lambda^+ \mid j\,"\,\beta \subseteq \beta \,\}$. Conversely, if $T_\xi$ stationary, then $\exists \beta \in C \cap T_\xi$, and so $\beta \in S_\alpha$ some $\alpha$ and $\beta = j(\beta) \in j(S_\alpha) = T_{j(\alpha)}$. So $\xi = j(\alpha)$.

Thus, $j\,"\,\lambda^+ \in \mathrm{HOD}$. Consequently, $C \in \mathrm{HOD}$. Complete argument as before: $\exists \beta \in C \cap T_\kappa$, but $\beta \in S_\alpha$ some $\alpha$ and $\beta = j(\beta) \in j(S_\alpha)$ and in $T_\kappa$, a contradiction. □

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Thus, $j \text{ "} \lambda^+ \in \mathrm{HOD}$. Consequently, $C \in \mathrm{HOD}$. Complete argument as before: $\exists \beta \in C \cap T_\kappa$, but $\beta \in S_\alpha$ some $\alpha$ and $\beta = j(\beta) \in j(S_\alpha)$ and in $T_\kappa$, a contradiction. □

## Theorem

*If $j : V \to M$, then $V = \mathrm{HOD}(M)$.*

## Proof.

Suppose $j : V \to M$. Find large $\lambda$ with $j(\lambda) = \lambda$, $j(\lambda^+) = \lambda^+$. Partition $\mathrm{Cof}_\omega \lambda^+ = \bigsqcup_{\alpha<\lambda^+} S_\alpha$ into stationary sets. Observe again $\xi \in \mathrm{ran}(j)$ if and only if $T_\xi$ is stationary in $V$. Thus, $j \restriction \lambda^+$ is definable in $V$ from $\vec{T}$, which is an element of $M$. Since any $A \subseteq \gamma$ is definable from $j(A)$ and $j\,"\,\gamma$, it follows that $A \in \mathrm{HOD}(M)$, and hence $V = \mathrm{HOD}(M)$. □

The theorem shows for any $j : V \to M$ that $j \restriction A$ is definable in $V$ using parameters from $M$.

## Corollary

*If $j : V \to M$ and $M \subseteq \mathrm{HOD}$, then $V = \mathrm{HOD}$.*

# An improved version

The methods can be pushed to attain the following:

**Theorem**

*If $j : M \to N$ and $M$ is eventually stationary correct to $V$, then $M \subseteq \mathrm{HOD}(N)$ and $j \restriction A \in \mathrm{HOD}(N)$ for any $A \in M$.*

**Corollary**

*There is no generic embedding $j : V \to \mathrm{HOD}$. That is, in no set-forcing extension $V[G]$ is there a nontrivial elementary embedding $j : V \to \mathrm{HOD}^V$.*

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# Consider the iterated HODs

One may attempt naively to iterate the HOD construction:

- $\mathrm{HOD}^0 = V$
- $\mathrm{HOD}^{n+1} = \mathrm{HOD}^{\mathrm{HOD}^n}$
- $\mathrm{HOD}^\omega = \bigcap_{n<\omega} \mathrm{HOD}^n$.

But there are meta-mathematical complications. In fact, we have no *uniform* definition of the $\mathrm{HOD}^n$.

Harrington (1974), with related work of McAloon, shows consistent that every $\mathrm{HOD}^n$ can exist, but $\mathrm{HOD}^\omega$ is not a class.

But some models have structure allowing a uniform account. Define that "$\mathrm{HOD}^\eta$ exists" to mean that we have a class $H$ for which $H^0 = V$, $H^{\alpha+1} = \mathrm{HOD}^{H^\alpha}$ and $H^\gamma = \bigcap_{\alpha<\gamma} H^\alpha$ up to $\eta$.

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There is little reason to expect $\mathrm{HOD}^\eta$ definable, even when $\eta$ is.

# There is no $j : V \to \mathrm{HOD}^\eta$

**Corollary**

*Assume $\mathrm{HOD}^\eta$ exists.*

1. *There is no $j : V \to \mathrm{HOD}^\eta$.*
2. *If $M \subseteq \mathrm{HOD}^\eta$ is eventually stationary correct to $\mathrm{HOD}^\eta$, then there is no $j : V \to M$.*
3. *Indeed, no such $j$ exists in any $V[G]$.*

**Proof.**

For 1, we've already done the work: if $j : V \to \mathrm{HOD}^\eta$, then by the previous $V = \mathrm{HOD}(\mathrm{HOD}^\eta)$ and so $V = \mathrm{HOD}$ and so $\mathrm{HOD}^\eta = V$. So we reduce to $j : V \to V$, a contradiction. □

# Consider the generic HOD

The *generic* HOD, introduced by Fuchs, is the intersection of the HODs of all forcing extensions.

$$\mathrm{gHOD} = \bigcap \mathrm{HOD}^{V[G]}$$

Collapse forcing suffices. $\mathrm{gHOD} \models \mathrm{ZFC}$, and is invariant by set forcing. The $\mathrm{gHOD}$ can be far smaller than HOD and also than the *mantle*, the intersection of all grounds.

**Theorem**

*If $j : V \to N$, then $V = \mathrm{gHOD}(N)$. If $j : M \to N$ and $M$ is eventually stationary correct to $V$, then $M \subseteq \mathrm{gHOD}(N)$ and $j \restriction A \in \mathrm{gHOD}(N)$ every $A \in M$.*

**Corollary**

*If $j : V \to M$ and $M \subseteq \mathrm{gHOD}$, then $V = \mathrm{gHOD}$.*

# There is no $j : V \to \mathrm{gHOD}$

## Corollary

1. *There is no nontrivial elementary* $j : V \to \mathrm{gHOD}$.
2. *If* $\mathrm{gHOD}^\eta$ *exists, then there is no* $j : V \to \mathrm{gHOD}^\eta$.
3. *If* $M \subseteq \mathrm{gHOD}^\eta$ *is eventually stationary correct to* $\mathrm{gHOD}^\eta$, *then there is no* $j : V \to M$.

## Corollary

*For any class* $A$, *if* $\mathrm{HOD}[A]^\eta$ *exists, then there is no* $j : V \to \mathrm{HOD}[A]^\eta$ *and no* $j : V \to M$ *for any* $M \subseteq \mathrm{HOD}[A]^\eta$ *eventually stationary correct to* $\mathrm{HOD}[A]^\eta$. *Similarly for* $\mathrm{gHOD}[A]^\eta$. *And no such* $j$ *exists in any* $V[G]$.

# There is no $j : V \to \mathrm{gHOD}$

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3. *If* $M \subseteq \mathrm{gHOD}^\eta$ *is eventually stationary correct to* $\mathrm{gHOD}^\eta$, *then there is no* $j : V \to M$.

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*For any class* $A$, *if* $\mathrm{HOD}[A]^\eta$ *exists, then there is no* $j : V \to \mathrm{HOD}[A]^\eta$ *and no* $j : V \to M$ *for any* $M \subseteq \mathrm{HOD}[A]^\eta$ *eventually stationary correct to* $\mathrm{HOD}[A]^\eta$. *Similarly for* $\mathrm{gHOD}[A]^\eta$. *And no such* $j$ *exists in any* $V[G]$.

# Embeddings $j : \mathrm{HOD} \to V$

Let me turn now to the case of embeddings $j : \mathrm{HOD} \to V$ and other definable classes.

The arguments will have a very different character, and will not rely on any result in infinite combinatorics, such as Solovay's stationary partition result.

Instead, we shall extend the embedding $\mathrm{HOD} \to V$ into an infinite inverse system of embeddings

$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V,$$

and then analyze the nature of the inverse limit. The overall argument is soft, but details run into subtle meta-mathematical issues, which we resolve.

# The Kunen inconsistency under $V = \mathrm{HOD}$

There is a very easy proof when $V = \mathrm{HOD}$:

**Theorem**

*If $V = \mathrm{HOD}$, then there is no nontrivial $j : V \to V$.*

**Proof.**

Let $\lambda = \sup\langle\kappa_n \mid n < \omega\rangle$. It follows that $j(\lambda) = \lambda$. Let $s = \langle\alpha_n\rangle_n$ be HOD-least $\omega$-sequence with $\lambda = \sup(s)$. Since $s$ is definable from $\lambda$, it follows that $j(s) = s$ and hence also $j(\alpha_n) = \alpha_n$. But $j$ has no fixed points between $\kappa$ and $\lambda$, a contradiction. □

The argument needs only a definable well-ordering of $[\lambda]^\omega$.

# There is no $j : \mathrm{HOD} \to V$

**Theorem (Hamkins,Kirmayer,Perlmutter)**

*There is no nontrivial elementary embedding $j : HOD \to V$.*

**Proof**

Suppose $j : \mathrm{HOD} \to V$. Extend to an inverse system

$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V$$

The subtle issue about uniform presentation of $\mathrm{HOD}^n$ is resolved by proving $\mathrm{HOD}^n = \mathrm{dom}(j^n)$. Let $<^{n+1}$ be the canonical well-ordering of $\mathrm{HOD}^{n+1}$ definable in $\mathrm{HOD}^n$. Let $<^0 = j(<^1)$. Thus, $j(<^{n+1}) = <^n$. Define $\vec{x} = \langle x_n \mid n < \omega \rangle$ is *j-coherent*, if $j(x_{n+1}) = x_n$ for all $n < \omega$. . . .

# There is no $j : \mathrm{HOD} \to V$

**Theorem (Hamkins,Kirmayer,Perlmutter)**

*There is no nontrivial elementary embedding $j : HOD \to V$.*

**Proof**

Suppose $j : \mathrm{HOD} \to V$. Extend to an inverse system

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$$\cdots \longrightarrow \mathrm{HOD}^n \longrightarrow \cdots \longrightarrow \mathrm{HOD}^2 \longrightarrow \mathrm{HOD} \longrightarrow V$$

Claim. Every $j$-coherent sequence is constant.

Proof: Suppose $\vec{x} = \langle x_n \mid n < \omega \rangle$ is nonconstant $j$-coherent, with $\in$-minimal $x_0$. It follows all $x_n$ same rank. Also, $x_{n+1} \neq x_n$. Let $a_n$ be $<^n$-least in $x_{n+1} \triangle x_n$. It follows $\langle a_n \mid n < \omega \rangle$ is $j$-coherent, nonconstant, lower rank, a contradiction.

Claim. There is a non-constant $j$-coherent sequence.

Proof: Let $y_n$ be the $<^n$-least element of $\mathrm{HOD}^n \setminus \mathrm{HOD}^{n+1}$. It follows by the $j$-coherence of the relations $<^n$ that $j(y_{n+1}) = y_n$, and so this sequence is $j$-coherent. Since $y_0 \in V \setminus \mathrm{HOD}$ and $y_1 \in \mathrm{HOD} \setminus \mathrm{HOD}^2$, it follows that $y_0 \neq y_1$, and so this sequence is not constant.

This contradiction proves there is no $j : \mathrm{HOD} \to V$. □

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This contradiction proves there is no $j : \mathrm{HOD} \to V$. $\square$

# A generalization

**Theorem**

*Suppose $j : M \to N$, where $M \subseteq N \models \mathrm{ZF}$ and $M$ is b-definable in $N$ with $j(b) = b$, and $A$ is b-definable class in $N$ with tcl(A) has b-definable well-ordering in $N$. Then $A^M = A^N$.*

The proof uses a similar idea, expanding to inverse system

$$\cdots \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^2 \longrightarrow M^1 \longrightarrow M^0,$$

and considers $j$-coherent sequences, establishing first that they are all constant, and second, under the assumption that $A^M \neq A^N$, that there is a nonconstant $j$-coherent sequence.

# A surprising level of agreement

## Corollary

*If $j : M \to N$ is elementary for $M \subseteq N \models \mathrm{ZF}$ and $M$ is $b$-definable in $N$ for parameter $b = j(b)$, then $M$ and $N$ have*

1. *the same cardinals and the same cofinality function,*
2. *the same continuum function,*
3. *the same $\Diamond^*_\kappa$ pattern and*
4. *and the same large cardinals of any particular kind.*
5. *Furthermore, $\mathrm{HOD}^M = \mathrm{HOD}^N$ and $\mathrm{gHOD}^M = \mathrm{gHOD}^N$ and more.*

# There is no $j : \mathrm{HOD}^2 \to \mathrm{HOD}$

## Corollary

*If $M \subsetneq N \models \mathrm{ZF}$, with $M$ definable in $N$ and $M \subseteq \mathrm{HOD}^N$, then there is no $j : M \to N$.*

## Proof.

If such $j : M \to N$, then $\mathrm{HOD}^M = \mathrm{HOD}^N$. Thus, $M \subseteq \mathrm{HOD}^N = \mathrm{HOD}^M \subseteq M$ and so $M = \mathrm{HOD}^M$, and consequently $N = \mathrm{HOD}^N$ and so $M = N$, contradiction. □

## Corollary

*There is no $j : \mathrm{HOD}^2 \to \mathrm{HOD}$, if different, and no $j : \mathrm{HOD}^n \to \mathrm{HOD}^m$ for $m < n$, if different. Similarly no $j : \mathrm{gHOD}^2 \to \mathrm{gHOD}$ etc.*

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*If $M \subsetneq N \models \mathrm{ZF}$, with $M$ definable in $N$ and $M \subseteq \mathrm{HOD}^N$, then there is no $j : M \to N$.*

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If such $j : M \to N$, then $\mathrm{HOD}^M = \mathrm{HOD}^N$. Thus, $M \subseteq \mathrm{HOD}^N = \mathrm{HOD}^M \subseteq M$ and so $M = \mathrm{HOD}^M$, and consequently $N = \mathrm{HOD}^N$ and so $M = N$, contradiction. □

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**Corollary**

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# A sweeping general result

The proof method leads to a sweeping result:

**Theorem**

*If $M$ is a definable transitive class, then there is no nontrivial elementary embedding $j : M \to V$.*

This is a GBC scheme. The nonexistence of $j : \mathrm{HOD} \to V$ is a special case, generalizing to no $j : \mathrm{HOD}^n \to V$.

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The theorem is a consequence of the following general formulation.

# Definable domain

**Theorem**

*If $j : M \to N$ and $M \subseteq N$ and $N$ is eventually stationary correct to $V$, then $M$ is not definable in $N$ from parameters fixed by $j$.*

The proof uses the stationary partition methods we saw earlier, but making critical use of the fact that if $j : M \to N$ and $M$ is definable in $N$ via parameters fixed by $j$, then $\mathrm{Cof}_\omega^M = \mathrm{Cof}_\omega^N$.

**Corollary**

*If $M$ is a definable class in $V$, then in no set-forcing extension $V[G]$ is there a nontrivial elementary $j : M \to V$.*

For example, there is no generic $j : \mathrm{HOD} \to V$ or $j : \mathrm{gHOD} \to V$.

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For example, there is no generic $j : \mathrm{HOD} \to V$ or $j : \mathrm{gHOD} \to V$.

# Nontrivial $j : \mathrm{HOD} \to \mathrm{HOD}$?

It is open whether there can be $j : \mathrm{HOD} \to \mathrm{HOD}$.

The following corollary may be a way to produce such $j$.

**Corollary**

*Do not assume* AC. *If $j : M \to V$ is a nontrivial elementary embedding from a transitive proper class $M$ that is definable in $V$ from parameters fixed by $j$, then there is a nontrivial elementary embedding from* HOD *to* HOD.

**Proof.**

By earlier theorem, $\mathrm{HOD}^M = \mathrm{HOD}^V$, and so $j \restriction \mathrm{HOD} : \mathrm{HOD} \to \mathrm{HOD}$ is the desired embedding. □

# Definable embeddings

Let's turn now to the case where the embedding $j$ is not merely a GBC class, but a first-order definable class (with parameters).

In this case, many of the arguments admit of soft proofs, requiring neither any results from infinite combinatorics nor the axiom of choice.

# Definable embeddings

An embedding $j : M \to N$ is definable in $V$ using parameter $p$, when there has been provided a first-order formula $\varphi(x, y, z)$, such that $j(x) = y$ if and only if $\varphi(x, y, p)$ holds in $V$.

For a given formula $\varphi$, the question whether a given parameter $p$ succeeds in $\varphi(\cdot, \cdot, p)$ defining a nontrivial elementary embedding $j : V \to V$ is a first-order expressible property of $p$.

Similarly, for a given formula $\varphi$, the collection of ordinals $\kappa$ which arise as the critical point of such an embedding is definable.

# The Kunen inconsistency for definable $j$

The case of definable embeddings is easy to refute:

**Theorem (Folklore, Suzuki)**

*Assume only* ZF. *There is no nontrivial elementary embedding $j : V \to V$ that is definable from parameters.*

**Proof.**

Suppose $j(x) = y \iff \varphi(x, y, p)$. Choose $p$ so that the resulting critical point $\kappa$ is as small as possible, using this $\varphi$. So $j(\kappa)$ is also like that, a contradiction. □

This is essentially the same as the classical observation that Reinhardt cardinals are not first-order.

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This is essentially the same as the classical observation that Reinhardt cardinals are not first-order.

# Generic definable embeddings

**Theorem**

*Do not assume AC. There is no $j : M \to V$ definable in any set-forcing extension $V[G]$, allowing $M \subseteq V[G]$.*

**Proof.**

Suppose $j : M \to V$ is defined in $V[G]$ by $\varphi(\cdot, \cdot, b)$. So $\exists q \in G$ forcing $\varphi(\cdot, \cdot, \dot{b})$ defines an embedding. Assume without loss that $\kappa$ is smallest possible critical point arising this way using $\varphi$, any $\mathbb{Q}$. So $\kappa$ is definable in $V$ without parameters. But $j : M \to V$ is elementary and $\kappa \notin \mathrm{ran}(j)$, a contradiction. □

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# Definable generic embeddings

We immediately deduce the following as special cases:

## Corollary

*Do not assume AC.*

1. *There is no $j : M \to V$ definable with parameters in $V$.*
2. *There is no $j : V \to V$ definable with parameters in $V[G]$.*
3. *There is no $j : V[G] \to V$ definable with parameters in $V[G]$.*
4. *There is no $j : V \to V[G]$ definable with parameters in $V[G]$.*

# There is no definable $j : \mathrm{HOD} \to \mathrm{HOD}$

## Theorem

*Do not assume* AC*. There is no nontrivial* $j : \mathrm{HOD} \to \mathrm{HOD}$ *definable in* $V$ *from parameters.*

**Proof.**

Formally a ZF scheme. Suppose $j : \mathrm{HOD} \to \mathrm{HOD}$ defined by $j(x) = y \iff V \models \varphi(x, y, b)$. (Perhaps $b \notin \mathrm{HOD}$.) Let $\kappa = \mathrm{cp}(j)$. By reflection, there is a definable class club $C$ of $\gamma$ with $\varphi$ and $\exists y \varphi(x, y, z)$ absolute $V_\gamma$ to $V$. So $\gamma \in C \implies j \mathrel{\text{"}} \gamma \subseteq \gamma$. Let $\delta = \omega^{\text{th}}$ in $C$ above $\kappa$ and $\rho(b)$. In particular, $j \mathrel{\text{"}} \delta \subseteq \delta$ and $\mathrm{HOD} \models \mathrm{cof}(\delta) = \omega$, and so $j(\delta) = \delta$ and hence $j((\delta^+)^{\mathrm{HOD}}) = (\delta^+)^{\mathrm{HOD}}$. Let $\gamma = (\delta^+)^{\mathrm{HOD}}$-th element of $C$. So $j \mathrel{\text{"}} \gamma \subseteq \gamma$ and $\mathrm{HOD} \models \mathrm{cof}(\gamma) = (\delta^+)^{\mathrm{HOD}}$, and so $j(\gamma) = \gamma$.

This is now enough to run the stationary-partition argument using $(\mathrm{Cof}_\omega \gamma)^{\mathrm{HOD}} = \bigsqcup_{\alpha<\kappa} S_\alpha$ etc. □

# There is no definable $j : \mathrm{HOD} \to \mathrm{HOD}$

## Theorem

*Do not assume* AC. *There is no nontrivial $j : \mathrm{HOD} \to \mathrm{HOD}$ definable in $V$ from parameters.*

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Formally a ZF scheme. Suppose $j : \mathrm{HOD} \to \mathrm{HOD}$ defined by $j(x) = y \iff V \models \varphi(x, y, b)$. (Perhaps $b \notin \mathrm{HOD}$.) Let $\kappa = \mathrm{cp}(j)$. By reflection, there is a definable class club $C$ of $\gamma$ with $\varphi$ and $\exists y \varphi(x, y, z)$ absolute $V_\gamma$ to $V$. So $\gamma \in C \implies j\,"\,\gamma \subseteq \gamma$. Let $\delta = \omega^{\text{th}}$ in $C$ above $\kappa$ and $\rho(b)$. In particular, $j\,"\,\delta \subseteq \delta$ and $\mathrm{HOD} \models \mathrm{cof}(\delta) = \omega$, and so $j(\delta) = \delta$ and hence $j((\delta^+)^{\mathrm{HOD}}) = (\delta^+)^{\mathrm{HOD}}$. Let $\gamma = (\delta^+)^{\mathrm{HOD}}$-th element of $C$. So $j\,"\,\gamma \subseteq \gamma$ and $\mathrm{HOD} \models \mathrm{cof}(\gamma) = (\delta^+)^{\mathrm{HOD}}$, and so $j(\gamma) = \gamma$.

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# No definable $j : \mathrm{HOD} \to \mathrm{HOD}$

Note that the proof that there is no definable $j : \mathrm{HOD} \to \mathrm{HOD}$ is much simpler in the case where $j$ is definable without parameters or with ordinal parameters, for in this case one gets directly that $j \upharpoonright \theta \in \mathrm{HOD}$ for every ordinal $\theta$, and this is enough to complete the argument.

Indeed, when $j : \mathrm{HOD} \to \mathrm{HOD}$ is definable in $V$ using no parameters or using ordinal parameters, then HOD satisfies $\mathrm{ZFC}(j)$ and so we have directly an instance of the Kunen inconsistency by restricting to $\langle \mathrm{HOD}, \in, j \rangle$.

But the full proof treats the case $j$ is definable using parameters not necessarily in HOD.

# Open questions

The main open question in this area is whether the Kunen inconsistency requires $\mathrm{AC}$.

## Question

Is it consistent with $\mathrm{ZF}(j)$ that $j : V \to V$ is a nontrivial elementary embedding of the universe to itself?

We are naturally also interested in the corresponding question for each of the generalizations of the Kunen inconsistency whose current proofs use $\mathrm{AC}$.

For example, in the $\neg\mathrm{AC}$ context, can there be nontrivial elementary embeddings $j : V[G] \to V$ or $j : V \to V[G]$ for a set-forcing extension $V[G]$?

# Can there be $j : \mathrm{HOD} \to \mathrm{HOD}$?

The second main question is:

**Question**

Is it consistent that there is a nontrivial elementary embedding $j : \mathrm{HOD} \to \mathrm{HOD}$?

We ask in the $\mathrm{GBC}$ context, but it is also sensible to drop $\mathrm{AC}$.

There are numerous other questions. To what extent do the theorems we have mentioned about embeddings arising in set-forcing extensions also apply to class forcing? Or to certain kinds of class forcing? Or to other non-forcing extensions? To what extent do the theorems on $\mathrm{HOD}$ generalize to other natural definable classes?

Thank you.

Preprint available at http://boolesrings.org/hamkins

Joel David Hamkins

New York University
and
The City University of New York