Pluralism in Set Theory: Does Every Mathematical Statement Have a Definite Truth Value?

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The theme

The theme of this talk is the question:

> *Does every mathematical problem have a definite answer?*

I shall be particularly interested in this question as it arises in the case of set theory:

> *Does every set-theoretic assertion have a definite truth value?*

Set theory as Ontological Foundation

A traditional view in set theory is that it serves as an ontological foundation for the rest of mathematics, in the sense that other abstract mathematical objects can be construed fundamentally as sets. On this view, mathematical objects—functions, real numbers, spaces—are sets. Being precise in mathematics amounts to specifying an object in set theory. In this way, the set-theoretic universe becomes the realm of all mathematics.

Having a common foundation was important for the unity of mathematics.

A weaker position remains compatible with structuralism. Sets provide objects fulfilling the desired structural roles of mathematical objects, which therefore *can* be viewed as sets.

The Set-Theoretical Universe

These sets accumulate transfinitely to form the universe of all sets. This cumulative universe is regarded by set theorists as the domain of all mathematics.

The orthodox view among set theorists thereby exhibits a two-fold realist or Platonist nature:

- First, mathematical objects (can) exist as sets, and
- Second, these sets enjoy a real mathematical existence, accumulating to form the universe of all sets.

A principal task of set theory, on this view, is to discover the fundamental truths of this cumulative set-theoretical universe. These truths will include all mathematical truths.

Uniqueness of the universe

On this traditional Platonist view, the set-theoretic universe is unique: it is the universe of *all* sets.

In particular, on this view every set-theoretic question, such as the Continuum Hypothesis and others, has a definitive final answer in this universe.

With the ontological problem thus solved, what remains is the epistemological problem: how shall we discover these final set-theoretic truths?

The Universe View

Let me therefore describe as the *universe view*, the position that:

There is a unique absolute background concept of set, instantiated in the cumulative universe of all sets, in which set-theoretic assertions have a definite truth value.

Thus, the universe view is one of determinism for set-theoretic truth, and hence also for mathematical truth.

Isaacson: The reality of mathematics

In support of this view, Isaacson sharply distinguishes between *particular* vs. *general* mathematical structure.

Two fundamentally different uses of axioms.

- Axioms express our knowledge about a particular structure, such as the natural numbers $\mathbb{N}$, real numbers $\mathbb{R}$.
- Axioms define a general class of structures, such as class of groups, fields, orders.

Axioms for particular structures often have character of self-evident truths. Typically characterize the structure up to isomorphism. Categoricity. $\langle \mathbb{N}, S \rangle$ satisfies Peano's axioms; $\mathbb{R}$ is a complete ordered field.

Axioms for general structures are more like definitions.

Isaacson: mathematical experience

Particular structures are found by mathematical experience, and then characterized as unique.

> *"If the mathematical community at some stage in the development of mathematics has succeeded in becoming (informally) clear about a particular mathematical structure, this clarity can be made mathematically exact... usually by means of a full second-order language. Why must there be such a characterisation? Answer: if the clarity is genuine, there must be a way to articulate it precisely. If there is no such way, the seeming clarity must be illusory. Such a claim is of the character as the Church-Turing thesis,... for every particular structure developed in the practice of mathematics, there is [a] categorical characterization of it."(p. 31, Reality of Mathematics...)*

Isaacson: informal rigour

We come to know particular mathematical structures by informal rigour, establishing their coherence:

> *"The something more that is needed to represent true sentences of arithmetic as logical consequences of the second-order axioms of arithmetic is. . . the informal rigour by which we have come to understand these second-order axioms, and thereby to see that they are coherent. It is a development of mathematical understanding through informal rigour and not some further derivation that is needed.. . . We must reflect on our conceptual understanding of a given particular mathematical structure as it has developed to see how it is that truths of e.g. arithmetic are those that hold in the structure of the natural numbers which we have succeeded in characterizing." (p. 33, Reality of Mathematics...)*

For Isaacson, the point then is that the cumulative universe of set theory is a particular mathematical structure, characterized in second-order logic.

Martin's categoricity argument

Donald Martin argues that there is at most one structure meeting the concept of set.

Assuming what he calls the 'Uniqueness Postulate', asserting that every set is determined uniquely by its members (Note: this is an essentially anti-structuralist position), any two structures meeting the "weak" concept of set must agree. They will have the same ordinal stages of construction and will construct same sets at each stage.

Even without the UP, he argues that any two structures meeting what he calls the 'strong' concept of set are isomorphic, by a unique isomorphism.

Martin on definite truth

Although Martin's position aligns with important parts of the universe view, he declines a full endorsement:

> *"My theorems imply, in particular, that the existence of a structure meeting the strong concept of set is sufficient to give truth values to all set-theoretic statements. Independently of everything I have said in this paper, I have a strong feeling that whether or not the continuum hypothesis has a truth-value is an open question. I do not think this question can be settled by arguments as simple—as cheap, if you will—as the ones I have given in the paper. If my feeling is right, then it is epistemically possible that not all sentences of set theory have truth values. One way this possibility might be explained is by admitting that there may be no structure meeting the strong concept of set."*
>
> *("Multiple universes of sets and indeterminate truth values" Topoi 20: 5–16, 2001)*

Mathematical support for the universe view

The universe view is often combined with consequentialism as a criterion for truth.

For example, set theorists point to the increasingly stable body of regularity features flowing from the large cardinal hierarchy, such as determinacy consequences and uniformization results in the projective hierarchy for sets of reals.

Because these regularity features are mathematically desirable and highly explanatory, the large cardinal perspective seems to provide a coherent unifying theory.

This is taken as evidence for the truth of those axioms.

Main challenge for the universe view

A difficulty for the Universe view. The central discovery in set theory over the past half-century is the enormous range of set-theoretic possibility. The most powerful set-theoretical tools are most naturally understood as methods of constructing alternative set-theoretical universes, universes that seem fundamentally set-theoretic.

forcing, ultrapowers, canonical inner models, etc.

Much of set-theory research has been about constructing as many different models of set theory as possible. These models are often made to exhibit precise, exacting features or to exhibit specific relationships with other models.

An imaginary alternative history

Imagine that set theory had followed a different history:

- Imagine that as set theory developed, theorems were increasingly settled in the base theory.
- ...that the independence phenomenon was limited to paradoxical-seeming meta-logic statements.
- ...that the few true independence results occurring were settled by missing natural self-evident set principles.
- ...that the basic structure of the set-theoretic universe became increasingly stable and agreed-upon.

Such developments could have constituted evidence for the Universe view.

But the actual history is not like this...

Actual history: an abundance of universes

Over the past half-century, set theorists have discovered a vast diversity of models of set theory, a chaotic jumble of set-theoretic possibilities.

Whole parts of set theory exhaustively explore the combinations of statements realized in models of set theory, and study the methods supporting this exploration.

Would you like CH or $\neg\mathrm{CH}$? How about $\mathrm{CH} + \neg\Diamond$? Do you want $2^{\aleph_n} = \aleph_{n+2}$ for all n? Suslin trees? Kurepa trees? Martin's Axiom?

Set theorists build models to order.

Category-theoretic nature

As a result, the fundamental object of study in set theory has become: the model of set theory.

We have L, $L[0^\sharp]$, $L[\mu]$, $L[\vec{E}]$; we have models V with large cardinals, forcing extensions $V[G]$, ultrapowers M, cut-off universes L_δ, V_α, H_κ, universes $L(\mathbb{R})$, HOD, generic ultrapowers, boolean ultrapowers, etc. Forcing especially has led to a staggering variety of models.

Set theory has discovered an entire cosmos of set-theoretical universes, connected by forcing or large cardinal embeddings, like lines in a constellation filling a dark night sky.

Set theory now exhibits a category-theoretic nature.

A challenge to the universe view

The challenge is for the universe view to explain this central phenomenon, the phenomenon of the enormous diversity of set-theoretic possibilities.

The universe view seems to be fundamentally at odds with the existence of alternative set theoretic universes. Will they be explained away as imaginary?

In particular, it does not seem sufficient, when arguing for the universe view, to identify a particularly robust or clarifying theory, if the competing alternatives still appear acceptably set-theoretic. It seems that one must still explicitly explain (or explain away) the pluralistic illusion.

The Multiverse View

A competing position accepts the alternative set concepts as fully real.

The Multiverse view. The philosophical position holding that there are many set-theoretic universes.

The view is that there are numerous distinct concepts of set, not just one absolute concept of set, and each corresponds to the universe of sets to which it gives rise.

Diverse set concepts

The various concepts of set are simply those giving rise to the universes we have been constructing.

A key observation

From any given concept of set, we are able to generate many new concepts of set, relative to it.

From a set concept giving rise to a universe W, we describe other universes L^W, HOD^W, $L(\mathbb{R})^W$, K^W, forcing extensions $W[G]$, $W[H]$, ultrapowers, and so on.

Each such universe amounts to a new concept of set described in relation to the concept giving rise to W.

A philosophical enterprise becomes mathematical

Many of these concepts of set are closely enough related to be analyzed together from the perspective of a single set concept.

So what might have been a purely philosophical enterprise—comparing different concepts of set—becomes in part a mathematical one.

And the subject known as the philosophy of set theory thus requires a pleasing mix of (sometimes quite advanced) mathematical ideas with philosophical matters.

Multiverse view is realism

The multiverse view is a brand of *realism*. The alternative set-theoretical universes arise from different concepts of set, each giving rise to a universe of sets fully as real as the Universe of sets on the Universe view.

The view in part is that our mathematical tools—forcing, etc.—have offered us glimpses into these other mathematical worlds, providing evidence that they exist.

A Platonist may object at first, but actually, this IS a kind of Platonism, namely, Platonism about universes, second-order realism. Set theory is mature enough to adopt and analyze this view mathematically.

Plenitudinous Platonism

The multiverse view has strong affinities with Mark Balaguer's view:

> "The version of platonism that I am going to develop in this book—I will call it *plenitudinous platonism*, or alternatively, *full-blooded platonism* (FBP for short)—differs from traditional versions of platonism in several ways, but all of the differences arise out of one bottom-level difference concerning the question of *how many* mathematical objects there are. FBP can be expressed very intuitively, but also rather sloppily, as the view that *all possible mathematical objects exist*." (p. 5)

The analogy with Geometry

Geometry studied concepts—points, lines, planes—with a seemingly clear, absolute meaning; but those fundamental concepts shattered via non-Euclidean geometry into distinct geometrical concepts, realized in distinct geometrical universes.

The first consistency arguments for non-Euclidean geometry presented them as simulations within Euclidean geometry (e.g. ‘line’ = great circle on sphere).

In time, geometers accepted the alternative geometries more fully, with their own independent existence, and developed intuitions about what it is like to live inside them.

Today, geometers have a deep understanding of these alternative geometries, and no-one now regards the alternative geometries as illusory.

Set theory – geometry

Geometers reason about the various geometries:

- externally, as embedded spaces.
- internally, by using newly formed intuitions.
- abstractly, using isometry groups.

Extremely similar modes of reasoning arise with forcing:

- We understand the forcing extension from the perspective of the ground model, via names and the forcing relation.
- We understand the forcing extension by jumping inside: "Argue in $V[G]$"
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Isaacson on analogy with geometry

"The independence of the fifth postulate reflects the fact... that there are different geometries, in one of which the fifth postulate holds (is true), in others of which it is false. It makes no sense to ask whether the fifth postulate is really true or not. Whether it holds or not is a matter of which geometry we are in. The truth or falsity of the fifth postulate is not an open question, and is not something that can be overcome by finding a new axiom to settle it. By contrast, the independence of the continuum hypothesis does not establish the existence of a multiplicity of set theories. In a sense made precise and established by the use of second-order logic, there is only one set theory of the continuum. It remains an open question whether in that set theory [CH holds or not]." (p. 38)

The Continuum Hypothesis

The Continuum Hypothesis (CH) is the assertion that every set of real numbers is either countable or equinumerous with $\mathbb{R}$.

This was a major open question from the time of Cantor, and appeared at the top of Hilbert's famous list of open problems at the dawn of the 20th century.

- (1938) Gödel proved that CH holds in the constructible universe L.
- (1962) Cohen proved that L has a forcing extension $L[G]$ with $\neg\mathrm{CH}$.

Thus, the Continuum Hypothesis is now known to be formally independent of the axioms of set theory. It is neither provable nor refutable in ZFC.

The dream solution template for CH

Set theorists yearn for a definitive solution to CH, what I call the dream solution:

Step 1. Produce a set-theoretic assertion Φ expressing a natural ‘obviously true’ set-theoretic principle. (e.g. AC)

Step 2. Prove that Φ determines CH.

That is, prove that $\Phi \to \mathrm{CH}$,

or prove that $\Phi \to \neg\mathrm{CH}$.

And so, CH would be settled, since everyone would accept Φ and its consequences.

Dream solution will never be realized

I argue that the dream solution is now unworkable.

I shall argue that our rich experience in worlds having CH and others having $\neg\mathrm{CH}$, worlds that seem fully set-theoretic to us, will prevent us from ever accepting a principle Φ as obviously true, if it decides CH.

In other words, success in the second step exactly undermines the first step.

My prediction is that any specific dream solution proposal will be rejected from a position of deep mathematical experience with the contrary.

CH in the Multiverse

More important than mere independence, both CH and $\neg$CH are easily forceable over any model of set theory.

Every set-theoretic universe has a forcing extension in which CH holds, and another in which it fails. We can turn CH on and off like a lightswitch.

We have a deep understanding of how CH can hold and fail, densely in the multiverse, and we have a rich experience in the resulting models. We know, in a highly detailed manner, whether one can obtain CH or $\neg$CH over any model of set theory, while preserving any number of other features of the model.

These are places we've been. These universes feel fully set-theoretic. We can imagine living out a full mathematical life inside almost any one of them.

Turning the tables on Isaacson

So it is not merely that CH is formally independent and we have no additional knowledge. Rather, we have an informed, deep understanding of how CH could be true and how CH could be false, and how to build such worlds from one another.

So if a proposed axiom Φ settles CH, then we will not look upon it as natural, since we already know very well how it can fail. It would be like someone having an axiom implying that only Brooklyn existed, while we already know about Manhattan and the other boroughs.

Thus, the dream solution will not succeed.

The CH is settled

The multiverse perspective is that the CH question is settled.

The answer consists of our detailed understanding of how the CH both holds and fails throughout the multiverse, of how these models are connected and how one may reach them from each other while preserving or omitting certain features.

Fascinating open questions about CH remain, of course, but the most important essential facts are known.

Freiling: "a simple philosophical 'proof' of $\neg\mathrm{CH}$"

The Axiom of Symmetry (Freiling JSL, 1986)

Asserts that for any function f mapping reals to countable sets of reals, there are x, y with $y \notin f(x)$ and $x \notin f(y)$.

Freiling justifies the axiom on pre-theoretic grounds, with thought experiments throwing darts. The first lands at x, so almost all y have $y \notin f(x)$. By symmetry, $x \notin f(y)$.

> *"Actually [the axiom], being weaker than our intuition, does not say that the two darts have to do anything. All it claims is that what heuristically will happen every time, can happen."*

Thus, Freiling carries out step 1 in the template.

Freiling carries out step 2

Next, Freiling carries out step 2, by proving that the axiom of symmetry is equivalent to $\neg\mathrm{CH}$.

> *Proof: If CH, let $f(r)$ be the set of predecessors of r under a fixed well-ordering of type ω_1. So $x \in f(y)$ or $y \in f(x)$ by linearity. Conversely, if $\neg\mathrm{CH}$, then for any ω_1 many x_α, there must be $y \notin \bigcup_\alpha f(x_\alpha)$, but $f(y)$ contains at most countably many x_α.* $\Box$

Thus, Freiling exactly carries out the template.

Was his proposal received as a solution of CH? No.

Objections to Symmetry

Many mathematicians, ignoring Freiling's pre-reflective appeal, objected from a perspective of deep experience with non-measurable sets and functions, including extreme violations of the Fubini theorem property. For them, the pre-reflective arguments simply fell flat.

We have become skeptical of naive uses of measure precisely because we know the pitfalls; we know how badly behaved sets and functions can be with respect to measure concepts.

Because of our detailed experience, we are not convinced that AS is intuitively true. Thus, the reception follows my prediction.

Another example using the dream template

Consider the following set-theoretic principle:

The powerset size axiom (PSA) asserts:

Strictly larger sets have strictly more subsets.

In other words,

$$\forall x, y \qquad |x| < |y| \quad \Rightarrow \quad |P(x)| < |P(y)|.$$

Set-theorists understand this axiom very well.

Powerset size axiom: $|x| < |y| \Rightarrow |P(x)| < |P(y)|$

How is this axiom received in non-logic mathematical circles?

Extremely well!

To many mathematicians, this principle is *Obvious*, as natural and appealing as AC. Many are surprised to learn it is not a theorem.

Meanwhile, set theorists do not agree. Why not? In part, because they know how to achieve all kinds of crazy patterns $\kappa \mapsto 2^\kappa$ via Easton's theorem. Cohen's $\neg\mathrm{CH}$ model violates it; Martin's axiom violates it; Luzin's hypothesis violates it. PSA fails under many of the axioms, such as PFA, MM that are often favored particularly by set-theorists with the universe view.

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Powerset size axiom

So we have a set-theoretic principle

- which many mathematicians find to be obviously true;
- which expresses an intuitively clear pre-reflective principle about the concept of size;
- which set-theorists know is safe and (relatively) consistent;

is almost universally rejected by set-theorists when proposed as a fundamental axiom.

We are too familiar with the ways that PSA can fail, and have too much experience working in models where it fails.

But imagine an alternative history, in which PSA is used to settle a prominent early question and is subsequently adopted as a fundamental axiom.

The ontology of forcing

The central dispute is whether there are universes outside V, taken under the Universe view to consist of all sets. A special case captures the essential debate:

Question

Do forcing extensions of the universe exist?

On the Universe view, forcing extensions of V are illusory. On the Multiverse view, V is an introduced constant, referring to the universe currently under consideration.

Ontological status of generic filters

Those who take V as the unique universe of all sets object:

"There are no V-generic filters"

Surely we all agree that for nontrivial forcing notions, there are no V-generic filters **in** V.

But isn't the objection like saying:

"There is no square root of -1"

Of course, $\sqrt{-1}$ does not exist in the reals $\mathbb{R}$. One must go to a field extension, the complex numbers, to find it.

Similarly, one must go to the forcing extension $V[G]$ to find G.

Imaginary objects

Historically, $\sqrt{-1}$ was viewed with suspicion, deemed imaginary, but useful.

Eventually, mathematicians realized how to simulate the complex numbers $a + bi \in \mathbb{C}$ inside the real numbers, representing them as pairs (a, b), and thereby gain access to the complex numbers from a world having only real numbers.

The case of forcing is similar. We have a measure of access from any universe to its forcing extensions. I have described the *Naturalist account of forcing* in part to account for this.

Multiverse for universe adjudication

The Universe view is simulated inside the Multiverse by fixing a particular universe V and declaring it to be the absolute set-theoretical background, restricting the multiverse to the worlds below V.

This in effect provides absolute background notions of countability, well-foundedness, etc., and there are no V-generic filters in the restricted multiverse...

Thus, the multiverse view seems capable of adjudicating the Universe view discussion.

Multiverse mathematics

The multiverse perspective leads one to view the task of set theory not as the search for the one final True set theory, but rather to to explore all the various interesting set theories that we know of, to find new ones, and to discover how they are related.

It has therefore lead to several new mathematical topics, which remain strongly engaged with the philosophical issues.

- Modal Logic of forcing. Study the multiverse as a Kripke model of possible worlds.
- Set-theoretic geology. Study the structure of all ground models of the universe.

References

I have several articles exploring aspects of the multiverse debate.

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These and other articles, including several on the modal logic of forcing and set-theoretic geology, are available on my web page http://jdh.hamkins.org.

Thank you.

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