Boldface resurrection & the strongly uplifting, the superstrongly unfoldable, and the almost-hugely unfoldable cardinals

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Mathematics, Philosophy, Computer Science

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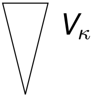
J. D. Hamkins, T. A. Johnstone, “Strongly uplifting cardinals and the boldface resurrection axioms,” under review, arxiv.org/abs/1403.2788.

J. D. Hamkins, T. A. Johnstone, “Resurrection axioms and uplifting cardinals,” Archive for Mathematical Logic 53:3-4(2014), p. 463–485.

Recall the uplifting cardinals

Definition (Hamkins, Johnstone)

An inaccessible cardinal κ is *uplifting*, if there are arbitrarily large inaccessible cardinals θ with $V_\kappa \prec V_\theta$.



In consistency strength, "ORD is Mahlo" $<$ uplifting $<$ Mahlo

Pseudo-uplifting: do not require the target θ to be inaccessible.

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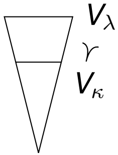

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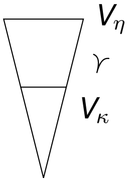

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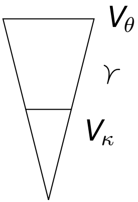

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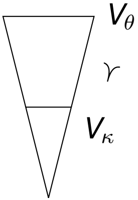

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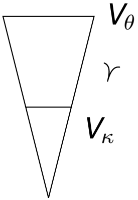

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Connection with forcing axioms

We introduced the uplifting cardinals precisely to prove:

Theorem (Hamkins, Johnstone)

The following are equiconsistent over ZFC:

1. There is an uplifting cardinal.
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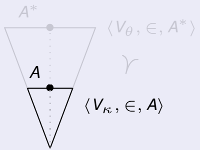
Strongly uplifting cardinals

We now generalize this concept to a boldface notion.

Definition (Hamkins, Johnstone)

A cardinal κ is *strongly uplifting*, if for every $A \subseteq \kappa$ there are unboundedly many inaccessible cardinals θ and A^* with

$$\langle V_\kappa, \in, A\rangle \prec \langle V_\theta, \in, A^*\rangle$$



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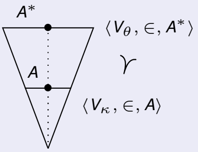

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Surprising strength on the target for free

Suppose every $A \subseteq \kappa$ has arbitrarily large θ and A^* with

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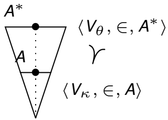

Claim. Without loss, θ is inaccessible, w. compact and more.

strongly uplifting = pseudo strongly uplifting
= strongly uplifting with indescribable targets

Proof.

Fix $A \subseteq \kappa$. Let $C = \{ \delta < \kappa \mid \langle V_\delta, \in, A \cap \delta \rangle \prec \langle V_\kappa, \in, A \rangle \}$, club. Get $\langle V_\kappa, \in, A, C \rangle \prec \langle V_\theta, \in, A^*, C^* \rangle$. Since $\kappa \in C^*$ and is inaccessible, weakly compact, indescribable, etc. in V_θ, there are many such $\gamma \in C^*$. It follows that $\langle V_\kappa, \in, A \rangle \prec \langle V_\gamma, \in, A^* \cap \gamma \rangle$ for such γ. □

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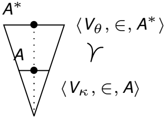

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$$\begin{aligned}\text{strongly uplifting} &= \text{pseudo strongly uplifting}\\ &= \text{strongly uplifting with indescribable targets}\end{aligned}$$

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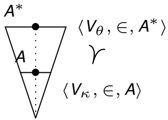

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Absolute to L

Theorem

Every strongly uplifting cardinal is strongly uplifting in L.

Proof.

Fix any $A \subseteq \kappa$ with $A \in L$. Let C be club of $\delta < \kappa$ with $\langle L_\delta, \in, A \cap \delta \rangle \prec \langle L_\kappa, \in, A \rangle$. Extend $\langle V_\kappa, \in, A, C \rangle \prec \langle V_\theta, \in, A^*, C^* \rangle$ as before. It follows that $\langle L_\kappa, \in, A \rangle \prec \langle L_\gamma, \in, A^* \cap \gamma \rangle$ and $A^* \cap \gamma \in L$ for $\gamma \in C^*$. □

Embedding characterizations

We have many familiar large cardinal embedding characterizations, even very low in the large cardinal hierarchy.

- A cardinal κ is measurable if it is the critical point of an elementary embedding $j : V \to M$.
- κ is weakly compact if for every $A \subseteq \kappa$ there is $M \models \mathrm{ZFC}$ with $A \in M$ and $j : M \to N$ with critical point κ.
- κ is θ-unfoldable if there is such $j : M \to N$ with $j(\kappa) \geq \theta$.
- κ is strongly θ-unfoldable if also $V_\theta \subseteq N$.

Strong unfoldability is in essence a transfinite continuation of total indescribability.

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A few new large cardinal concepts

In the same vein, consider the following weak analogues of superstrongness and almost hugeness.

Definition (Hamkins, Johnstone)

- A cardinal κ is *weakly superstrong* if for every $A \subseteq \kappa$ there is $j : M \to N$, critical point κ with $A \in M \models \mathrm{ZFC}$ and $V_{j(\kappa)} \subseteq N$.
- The cardinal κ is *weakly almost huge* if for every $A \subseteq \kappa$ there is $j : M \to N$, critical point κ with $A \in M \models \mathrm{ZFC}$ and $N^{<j(\kappa)} \subseteq N$.

Extending to arbitrarily large targets

In the unfoldability style, it is natural to ask for arbitrarily large targets.

Definition (Hamkins, Johnstone)

- A cardinal κ is *superstrongly unfoldable* (or *weakly superstrong with arbitrarily large targets*), if for every $A \subseteq \kappa$ there is $j : M \to N$ with critical point κ, $A \in M \models \mathrm{ZFC}$, $j(\kappa)$ arbitrarily large and $V_{j(\kappa)} \subseteq N$.
- A cardinal κ is *almost-hugely unfoldable* (or *weakly almost huge with arbitrarily large targets*), if for every $A \subseteq \kappa$ there is $j : M \to N$ with critical point κ, $A \in M \models \mathrm{ZFC}$, $j(\kappa)$ arbitrarily large and $N^{<j(\kappa)} \subseteq N$.

A surprising equivalence

Nevertheless, these notions are actually equivalent, and both are equivalent to our previous notion of strongly uplifting!

Theorem

The following large cardinal concepts are equivalent.

1. κ *is strongly uplifting.*
2. κ *is superstrongly unfoldable.*
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Let's sketch the proof.

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The nontrivial implication:

κ strongly uplifting $\implies$ almost-hugely unfoldable.

Proof.

Fix $A \subseteq \kappa$ and $M \models \mathrm{ZFC}$ with $A \in M \supseteq M^{<\kappa}$. Find $\pi : \langle M, \in\rangle \cong \langle \kappa, E\rangle$. By strong uplifting, $\langle V_\kappa, \in, E\rangle \prec \langle V_\gamma, \in, E^*\rangle$ with γ inaccessible. Since E is well-founded and γ is regular, E^* is well-founded. Mostowski collapse $\tau : \langle \gamma, E^*\rangle \cong \langle N, \in\rangle$. Let $j = \tau \circ \pi : M \to N$. Note $j \restriction \kappa = \mathrm{id}$ since if α is coded by ξ with respect to E, then it is also coded by ξ with respect to E^*, and so $j(\alpha) = \alpha$. Similarly, $j(\kappa) = \gamma$, and so $\mathrm{cp}(j) = \kappa$. Since $\langle V_\kappa, \in, E\rangle$ sees that every set is coded via E, it follows by elementarity that every $x \in V_\gamma$ is coded via E^*, and so $V_{j(\kappa)} = V_\gamma \subseteq N$, giving superstrongness. Similarly, $M^{<\kappa} \subseteq M$ implies that $\langle V_\kappa, \in, E\rangle$ believes that $\langle \kappa, E\rangle$ is closed under $<\kappa$-sequences, and so for $\langle V_\gamma, \in, E^*\rangle$, which implies $N^{<j(\kappa)} \subseteq N$, giving almost hugeness. □

Weakly superstrong = weakly almost huge

Similarly, the local concepts are also equivalent.

Theorem

A cardinal κ is weakly superstrong if and only if κ is weakly almost huge.

As a result, many techniques from the superstrongness and almost hugeness context have fruitful analogues with the corresponding weak notions.

General phenomenon:

weak versions of strongness and supercompactness coincide
weak superstrongness and weak almost hugeness coincide

Quick upper bound

Theorem

$0^\sharp$ *exists* $\implies$ *every Silver indiscernible is strongly uplifting in L.*

Proof.

If $\kappa < \delta$ are Silver indiscernibles, let $j : L \to L$ have $j(\kappa) = \delta$. It follows that $\langle L_\kappa, \in, A\rangle \prec \langle L_\delta, \in, j(A)\rangle$ witnesses strong uplifting. □

Consequently, if there is a Ramsey cardinal, for example, then $\aleph_1$ is weakly superstrong, weakly almost huge and more in L.

A better upper bound

A cardinal δ is *subtle*, if for any $\langle B_\eta \mid \eta < \delta\rangle$ with $B_\eta \subseteq \eta$ and any club $C \subseteq \delta$ there are $\kappa < \eta$ in C such that $B_\kappa = B_\eta \cap \kappa$.

Theorem

If δ is a subtle cardinal, then the strongly uplifting cardinals in V_δ are stationary.

Proof.

If not, fix club $C \subseteq \delta$ with no $\kappa \in C$ strongly uplifting in V_δ. Fix $\theta_\kappa, A_\kappa \subseteq \kappa$ so $\langle V_\kappa, \in, A_\kappa\rangle$ has no extension $\langle V_\gamma, \in, A^*\rangle$ any $\gamma > \theta_\kappa$ in V_δ. Thin the club. Let $B_\kappa \subseteq \kappa$ code diagram of $\langle V_\kappa, \in, A_\kappa\rangle$. Since δ subtle, $\exists \kappa < \eta$ in C with $B_\kappa = B_\eta \cap \kappa$. So $\langle V_\kappa, \in, A_\kappa\rangle \prec \langle V_\eta, \in, A_\eta\rangle$, contradiction. □

Laver functions

Theorem

Every superstrongly unfoldable cardinal has an ordinal-anticipating Laver function.

A Laver function is $\ell:\kappa\to\kappa$ such that for any α and any $A\subseteq\kappa$ there is $j:M\to N$, critical point κ, with $\ell,A\in M\models\mathrm{ZFC}$ and $j(\kappa)$ as large as desired, $V_{j(\kappa)}\subseteq N$, and $j(\ell)(\kappa)=\alpha$.

Strongly uplifting version: $\forall\alpha, A\subseteq\kappa$ there are arbitrarily large extensions $\langle V_\kappa,\in,A,\ell\rangle\prec\langle V_\theta,\in,A^*,\ell^*\rangle$ with $\ell^*(\kappa)=\alpha$.

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Proof of Laver functions

Theorem

Every superstrongly unfoldable cardinal has an ordinal-anticipating Laver function.

Proof.

Define $\ell(\delta) = \alpha$, if the number of θ for which $V_\theta \models \delta$ is strongly uplifting is $\langle\beta, \alpha\rangle + 1$ some β. Fix α and $A \subseteq \kappa$, and let η be $(\langle\theta, \alpha\rangle + 2)^{\text{th}}$ with $V_\eta \models \kappa$ is strongly uplifting (by reflection there are many). So get $\langle V_\kappa, \in, A, \ell\rangle \prec \langle V_\gamma, \in, A^*, \ell^*\rangle$ with $\ell^*(\kappa) = \alpha$. □

Can actually anticipate all OD sets. Under $V = \mathrm{HOD}$, all sets.

Open question: must there always be a full Laver function?

Resurrection

Let us turn now to the topic of Resurrection axioms.

These are forcing axioms, involving the idea that truths killed by forcing might be resurrected again.

Existential closure

The resurrrection axioms are inspired by the concept of *existential closure* in model theory.

Definition

A model M is *existentially closed* if whenever M is a submodel of N, then existential witnesses in N exist already in M.

In other words, $M \prec_{\Sigma_1} N$.

Existential closure asserts that objects which could exist in a larger model, already exist in the ground model.

Forcing Axioms as Existential Closure

Many classical forcing axioms can be viewed as expressing to a degree that the universe is existentially closed.

The essence of these axioms is the assertion that certain filters, which could exist in a forcing extension, exist already in V.

$$V \subseteq V[G]$$

Martin's Axiom, the Proper Forcing Axiom, Martin's Maximum are all expressible as instances of existential closure.

Meanwhile, the universe V is never actually existentially closed in all its forcing extensions. But the collection

$$H_{\mathfrak{c}} = \{\text{ sets of hereditary size } < \mathfrak{c}\,\}$$

can be existentially closed in forcing extensions, and this is precisely what the forcing axioms express.

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Forcing Axioms as existential closure

Theorem (Stavi(80s), Bagaria)

Martin's Axiom MA *is equivalent to the assertion that for any c.c.c. forcing extension* $V[g]$

$$H_{\mathfrak{c}} \prec_{\Sigma_1} H_{\mathfrak{c}}^{V[g]}.$$

Theorem (Bagaria)

The Bounded Proper Forcing Axiom is equivalent to the assertion that for any proper forcing extension $V[g]$

$$H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{V[g]}.$$

Thus, $H_{\mathfrak{c}}$ is existentially closed in forcing extensions.

Existential Closure $\iff$ Resurrection

Theorem. The Following are equivalent

- A model M is existentially closed.
- M has *Resurrection*. That is, whenever $M \subseteq N_0$, then there is $M \subseteq N_0 \subseteq N_1$ with
$$M \prec N_1.$$

($\Leftarrow$) Resurrection implies existential closure, since witnesses in N_0 still exist in N_1, which is fully elementary over M.

($\Rightarrow$) If M is existentially closed and $M \subseteq N_0$, then the full elementary diagram of M is consistent with the atomic diagram of N_0. A model of this theory is the desired N_1. QED

The Key Point. Equivalence can break down when the class of models is restricted. But resurrection remains stronger.

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The main idea

This suggests using resurrection to formulate forcing axioms, in place of Σ_1 elementarity.

That is, we can formulate forcing axioms by means of the resurrection concept, considering not just Σ_1 elementarity in the relevant forcing extensions

$$\forall \mathbb{Q} \quad M \prec_{\Sigma_1} M^{V^{\mathbb{Q}}}$$

but instead asking for full elementarity in a further extension

$$\forall \mathbb{Q}\, \exists \dot{\mathbb{R}} \quad M \prec M^{V^{\mathbb{Q} * \dot{\mathbb{R}}}}$$

Resurrection Axiom

This led naturally to the Resurrection Axioms.

Definition (Hamkins, Johnstone)

Suppose Γ is a definable class of forcing notions. The *resurrection axiom* $\mathrm{RA}(\Gamma)$ is the assertion that for every $\mathbb{Q} \in \Gamma$ there is $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$ such that

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[g*h]}$$

whenever $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$ is V-generic.

Equiconsistency Strength of $\mathrm{RA}(\Gamma)$

Theorem (Hamkins, Johnstone)

The following are equiconsistent over ZFC:

1. There is an uplifting cardinal.
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6. and many other instances $\mathrm{RA}(\Gamma) + \neg\mathrm{CH}$.

Boldface resurrection

Now, we extend the analysis to the boldface context. Specifically:

By allowing predicates, we generalize the uplifting cardinals to the strongly uplifting cardinals.

$$\langle V_\kappa, \in, A\rangle \prec \langle V_\theta, \in, A^*\rangle$$

By allowing predicates, we generalize the resurrection axiom RA to the boldface resurrection axiom $\underset{\sim}{\mathrm{RA}}$.

$$\langle H_{\mathfrak{c}}, \in, A\rangle \prec \langle H_{\mathfrak{c}}^{V[g][h]}, \in, A^*\rangle$$

Boldface resurrection

Definition

The boldface resurrection axiom $\underset{\sim}{\mathrm{RA}}(\Gamma)$ asserts that for every $\mathbb{Q} \in \Gamma$ and $A \subseteq \mathfrak{c}$ there is $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$ and A^* with

$$\langle H_{\mathfrak{c}}, \in, A \rangle \prec \langle H_{\mathfrak{c}}^{V[g*h]}, \in, A^* \rangle$$

in the corresponding extension $V[g * h]$.

Embedding characterization

Theorem

If $|H_{\mathfrak{c}}| = \mathfrak{c}$, then the following are equivalent for any class Γ.

1. *The boldface resurrection axiom $\underset{\sim}{\mathrm{RA}}(\Gamma)$.*
2. *For every $\mathbb{Q} \in \Gamma$ and every transitive set $M \models \mathrm{ZFC}^-$ with $|M| = \mathfrak{c} \in M$ and $H_{\mathfrak{c}} \subseteq M$, there is $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$, such that in the corresponding extension $V[g * h]$, there is elementary*

$$j : M \to N$$

with $j \upharpoonright H_{\mathfrak{c}} = id$ and $j(\mathfrak{c}) = \mathfrak{c}^{V[g*h]}$ and $H_{\mathfrak{c}}^{V[g*h]} \subseteq N$.

Similarly, the weak boldface axiom $\mathrm{w}\underset{\sim}{\mathrm{RA}}(\Gamma)$ is equivalent to the embedding characterization obtained by omitting the requirement that $\dot{\mathbb{R}} \in \Gamma^{V^{\mathbb{Q}}}$.

Strength of Boldface Resurrection

Main Theorem

The following theories are equiconsistent over ZFC.

1. There is a strongly uplifting cardinal.
2. The boldface Resurrection Axiom $\underset{\sim}{\mathrm{RA}}(\text{all})$.
3. The boldface Reserrection Axiom $\underset{\sim}{\mathrm{RA}}(\mathrm{ccc})$ for c.c.c. forcing.
4. The boldface Resurrection Axiom for proper forcing.
5. The boldface Resurrection Axiom for semi-proper forcing.

strongly uplifting $\implies \underset{\sim}{\mathrm{RA}}$

Assume κ is strongly uplifting. We produce a forcing extension with $\underset{\sim}{\mathrm{RA}}$(proper). Let $f:\kappa\to\kappa$ be a strongly uplifting Menas function. Let $\mathbb{P}$ be the PFA lottery preparation, which forces at stage $\gamma<\kappa$ with the lottery sum

$$\mathbb{Q}_\gamma = \oplus\{\,\mathbb{Q}\in H_{f(\gamma)^+}^{V[G_\gamma]} \mid \mathbb{Q}\text{ is proper}\,\}.$$

Let $G\subseteq\mathbb{P}$ be V-generic. Since the generic will often opt to add a Cohen real, $\kappa=\mathfrak{c}^{V[G]}$. If $\mathbb{Q}$ is proper in $V[G]$ and $A\subseteq\kappa$, find extension $\langle V_\kappa,\in,A,f,\mathbb{P}\rangle\prec\langle V_\theta,\in,A^*,f^*,\mathbb{P}^*\rangle$ with $f^*(\kappa)>|\mathbb{Q}|$. Opt for $\mathbb{Q}$ in the stage κ lottery of $\mathbb{P}^*$, which becomes $\mathbb{P}*\mathbb{Q}*\dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is the rest of the forcing. So $\langle V_\kappa[G],\in,A\rangle\prec\langle V_\theta[G][g][h],\in,A^*\rangle$. In other words,

$$\langle H_{\mathfrak{c}}^{V[G]},\in,A\rangle\prec\langle H_{\mathfrak{c}}^{V[G][g][h]},\in,A^*\rangle,$$

which witnesses $\underset{\sim}{\mathrm{RA}}$ in $V[G]$.

$\underset{\sim}{\mathrm{RA}} \implies$ strongly uplifting

For this direction, the basic fact is that if $\underset{\sim}{\mathrm{RA}}(\Gamma)$, then $\kappa = \mathfrak{c}$ is strongly uplifting in L.

In the case of $\underset{\sim}{\mathrm{RA}}(\text{proper})$, for example, fix any $A \subseteq \kappa$ with $A \in L$. Let $\mathbb{Q} = \mathrm{Coll}(\omega_1, \theta)$. By resurrection, there is $\dot{\mathbb{R}}$, such that $\langle H_{\mathfrak{c}}, \in, A\rangle \prec \langle H_{\mathfrak{c}}^{V[g][h]}, \in, A^*\rangle$ in the corresponding extension $V[g][h]$. It follows that $\langle L_\kappa, \in, A\rangle \prec \langle L_{\mathfrak{c}^{V[g][h]}}, \in, A^*\rangle$, and all initial segments of A^* are in L, which is enough to conclude κ is strongly uplifting in L.

For $\underset{\sim}{\mathrm{RA}}(\mathrm{ccc})$, use $\mathrm{Add}(\omega, \theta)$.

Thank you.

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