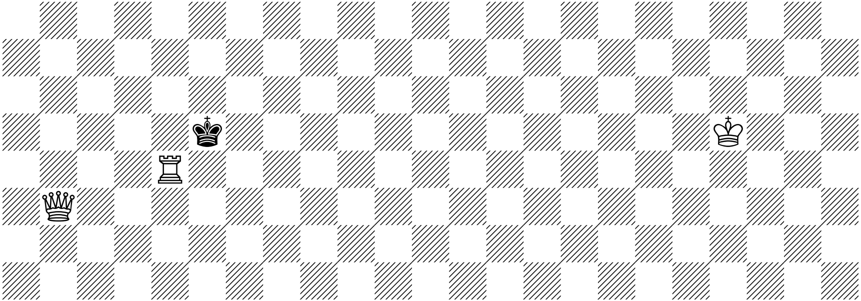
A mate-in-12 problem

White to move on an infinite, edgeless board.



Can white force mate in 12 moves?

The theory of infinite games
How to play infinite chess and win!

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Mathematics, Philosophy, Computer Science

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Dan Brumleve

Philip Schlicht

Links to articles available at http://jdh.hamkins.org.

What is an infinite game?

Consider two-player infinite games of perfect information.

The two players take turns making moves:

White:	a_0		a_2		a_4		a_6		$\cdots$
Black:		a_1		a_3		a_5		$\cdots$	

Together they build a particular play $\vec{a} = \langle a_0, a_1, a_2, \ldots \rangle$ of the game.

White wins this instance of the game if the play $\vec{a} \in A$, a fixed set specifying the winning conditions. Otherwise, Black wins.

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Winning strategies

A *strategy* for a player is a function that tells the player what move to make next, given the finite sequence of preceding moves.

$$\sigma : \langle a_0, a_1, \ldots, a_{k-1}\rangle \quad \mapsto \quad a_k$$

A *winning* strategy for a player is a strategy that, when followed, results in a winning play, no matter how the opponent plays.

Obviously, at most one of the players can have a winning strategy.

Example infinite games

Consider the following winning conditions for White, where the players play 0 or 1 at each step.

- $A =$ the set of eventually periodic sequences.
- $B =$ set of universal sequences (all finite strings occur)
- $C =$ set of strings containing 00 or 111 but not 0001.
- $D =$ a countable set.

If white has a winning strategy for X, does black have a winning strategy for the complement?

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Fundamental question

Question

In any game, must one of the players have a winning strategy?

Definition

A game is *determined* if one player has a winning strategy.

Question

Is every game determined?

Determinacy

Axiom of Determinacy (AD)

The axiom of determinacy is the assertion that every game has a strategy for one of the players.

Do we have any reason to believe this axiom?

In what sense is this an “axiom”?

- Axioms are self-evident principles
  (Euclidean axioms, Peano axioms, set theory axioms)
- Axioms define a collection of structures, a domain of discourse
  (group axioms)
- Axioms express a succinct prescient vision of possibility
  (CH, MA, $V = L$, $\Diamond$)

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Evidence for AD

White has a winning strategy: $\exists a_0 \, \forall a_1 \, \exists a_2 \cdots A(\vec{a})$

Black has a winning strategy: $\forall a_0 \, \exists a_1 \, \forall a_2 \cdots \neg A(\vec{a})$

AD asserts: White has no ws $\iff$ Black has a ws.

In other words, AD is the assertion:

$$\neg \left(\exists a_0 \, \forall a_1 \, \exists a_2 \cdots A(\vec{a}) \right) \iff \left(\forall a_0 \, \exists a_1 \, \forall a_2 \cdots \neg A(\vec{a}) \right).$$

Thus, AD is an infinitary de Morgan law.

$$\neg \exists x \, A(x) \iff \forall x \, \neg A(x)$$

$$\neg \exists x \, \forall y \, A(x, y) \iff \forall x \, \exists y \, \neg A(x, y)$$

Conclusion: all finite-length games are determined.

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Further evidence: AD has nice consequences

The axiom of determinacy (without using AC) has amazingly attractive consequences:

- Every set of reals is Lebesgue measurable.
- Every set of reals differs from an open set by a meager set.
- Every uncountable set of reals contains a perfect set.

On consequentialist grounds, this is strong evidence for AD.

AD leverages innate human capacity for strategic reasoning.

AD is deeply connected with large cardinals, equiconsistent with infinitely many Woodin cardinals.

Evidence against AD

Nevertheless,

Theorem

AD contradicts the axiom of choice.

Proof idea

To prove this, construct the counterexample set in a transfinite recursion of length continuum. Well-order the binary-play strategies in order type continuum, and at stage α, we've made fewer than continuum many promises, and so we can add a play to the set and to the complement so as to prevent a given strategy from winning for either side.

Set theorists often study fragments of AD in ZFC.

Open games

Open games generalize the concept of finite games.

A game is *open* for a player, if every winning instance of the game for that player is won already at a finite stage. That is, if $\vec{a}$ is a winning play, then there is some finite play $s = \vec{a} \upharpoonright n$ such that all extensions of s are winning, no matter how play continues from s.

This is equivalent to saying that A is an open set in the product topology on the space of all plays.

Similarly, a game is *closed* for a player, if it is open for the opposing player.

To win an open game, you must win at some finite stage. To win a closed game, it suffices never to lose at any finite stage.

Open Determinacy

Theorem (Gale,Stewart 1953)

Every open game is determined. One of the players has a winning strategy.

We give a proof using the concept of ordinal game *values*.

Recall the ordinals...

$0 \quad 1 \quad 2 \quad \cdots \quad \omega \quad \omega{+}1 \quad \cdots \quad \omega{\cdot}2 \quad \omega{\cdot}2{+}1 \quad \cdots \quad \omega{\cdot}n{+}k \quad \cdots \quad \omega^2$

$\omega^2{+}1 \quad \cdots \quad \omega^n \quad \cdots \quad \omega^\omega \quad \cdots \quad \epsilon_0 \quad \cdots \quad \omega_1^{\mathfrak{Ch}} \overset{?}{=} \omega_1^{CK} \quad \cdots \quad \omega_1 \quad \cdots$

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Game value

In any open game, we define the ordinal *game value* of positions:

- A position that is already won has value 0.
- A position one move away from being won has value 1.

The values continue transfinitely:

- value is $\alpha + 1$, if white may play to a position with value α.
- if black to play, value is supremum of the values of the positions to which black may play.

Some positions may be left without value.

White can play to strictly decrease value. Black can maintain value, but not increase it.

Game values provide strategies

White wants to reduce value. From a valued position, white can do so.

Black wants to maintain value. He cannot increase it.

Conclusion

A position is winning for white if and only if it has an ordinal value.

White wins via the value-reducing strategy.

Black wins via the value-maintaining strategy.

Illustrating game value ω^2

ω^2	The initial value, with black to play.
$\omega \cdot 3 + 17$	Black must play to some ordinal below ω^2.
$\omega \cdot 3 + 16$	White plays to reduce 1 each step, leading in 17 moves to
$\omega \cdot 3$	Black to play.
$\omega \cdot 2 + 2014$	Black must play below $\omega \cdot 3$
$\vdots$	Black survives another 2014 moves until...
$\omega \cdot 2$	Black to play.
$\omega + 10^{100}$	Black can now survive another googol many moves...
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ω	Black to play.
G	With Graham's number, Black can survive a bit longer.
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0	Eventually, the value hits 0, at which time white wins.

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$\omega \cdot 2 + 2014$	Black must play below $\omega \cdot 3$
$\vdots$	Black survives another 2014 moves until...
$\omega \cdot 2$	Black to play.
$\omega + 10^{100}$	Black can now survive another googol many moves...
$\vdots$	
ω	Black to play.
G	With Graham's number, Black can survive a bit longer.
$\vdots$	
0	Eventually, the value hits 0, at which time white wins.

Illustrating game value ω^2

ω^2	The initial value, with black to play.
$\omega \cdot 3 + 17$	Black must play to some ordinal below ω^2.
$\omega \cdot 3 + 16$	White plays to reduce 1 each step, leading in 17 moves to
$\omega \cdot 3$	Black to play.
$\omega \cdot 2 + 2014$	Black must play below $\omega \cdot 3$
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$\vdots$	
0	Eventually, the value hits 0, at which time white wins.

Concrete meaning of the game values

Finite value n. . . . . . white can force a win in n moves, but not faster.

Value ω. . . . . . black to move, but every move leads to some (arbitrarily large) finite value n. Black will lose, but he can cause an arbitrarily long delay.

Value $\omega + 10$. . . . . . white can force value ω in 10 moves.

Value ω^2. . . . . . black to play, but every move creates value $\omega \cdot n + k$. Think: n is the number of future announcements he will make, for each of which play will take that long before the next announcement.

Value ω^3. . . . . . black can announce an arbitrary number, which will be the number of large announcements. Each large announcement is the number of small announcements to be made; each small announcement is the number of moves before the next announcement.

Infinite chess

I'd like to illustrate various transfinite game values as they arise in the case of infinite chess.

However much I'd like to sit down in a café for a game of infinite chess, over espresso, nevertheless it remains a game purely of the mind.

Infinite chess

Infinite chess is chess played on an infinite edgeless chess board, arranged like the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

The familiar chess pieces—kings, queens, bishops, knights, rooks and pawns—move about according to their usual chess rules, with bishops on diagonals, rooks on ranks and files and so on, with each player striving to place the opposing king into checkmate.

There is no standard starting configuration in infinite chess, but rather a game proceeds by setting up a particular position on the board and then playing from that position.

Clarifying the rules

Let me clarify a few of the rules as they relate to infinite chess.

- At most one king of each color
- There is no boundary, hence no pawn promotion
- There is no castling and no *en passant*
- Abandon the 50 move rule as limiting
- Infinite play is a draw
- We may abandon the three-fold repetition rule
- For arbitrary starting positions, one must clarify some weird boundary cases, e.g. White to move, black in check, white in checkmate

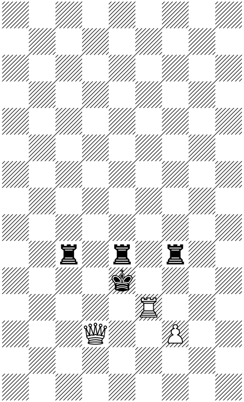
Infinite chess is an open game

Checkmate, when it occurs, does so after finitely many moves. Infinite chess is therefore an open game, and thus subject to the theory of transfinite game values.

So we can imagine chess positions with various large ordinal game values.

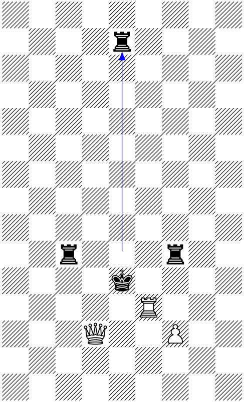
(The possibility of draw does not significantly upset the transfinite game value analysis.)

A finite position with value ω



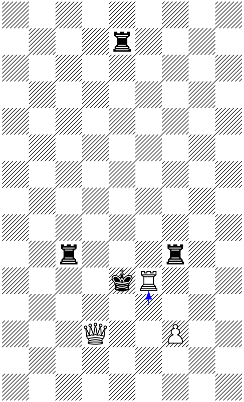
Black to move.

A finite position with value ω



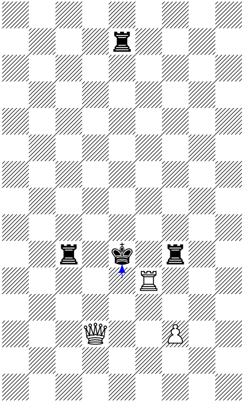
Black moves up arbitrary height

A finite position with value ω

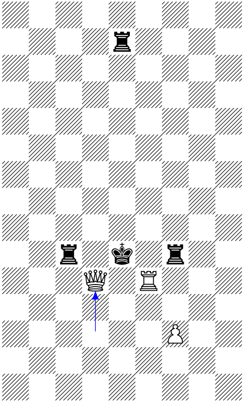


Check

A finite position with value ω

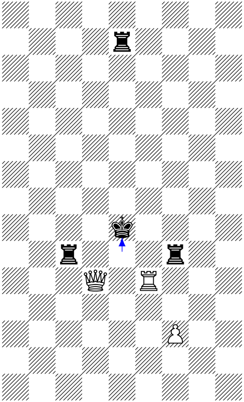


A finite position with value ω

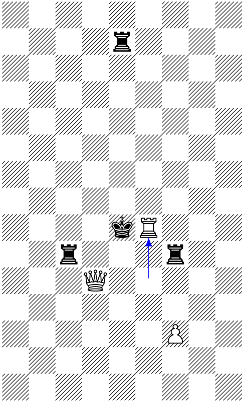


Check

A finite position with value ω

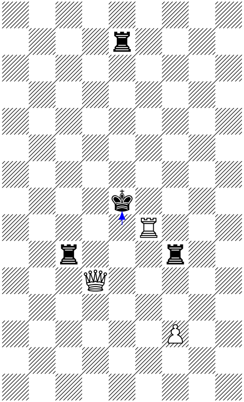


A finite position with value ω

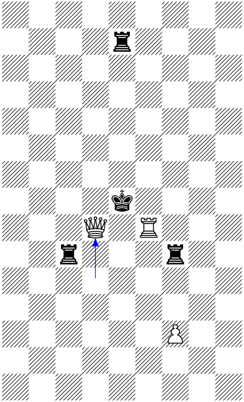


Check

A finite position with value ω

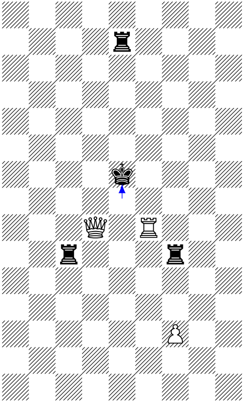


A finite position with value ω

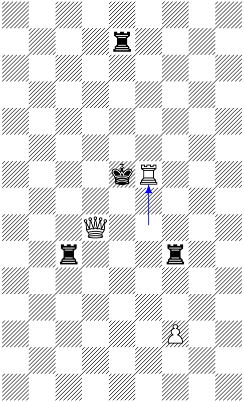


Check

A finite position with value ω

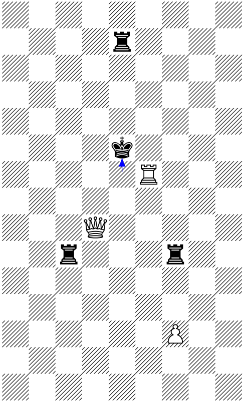


A finite position with value ω

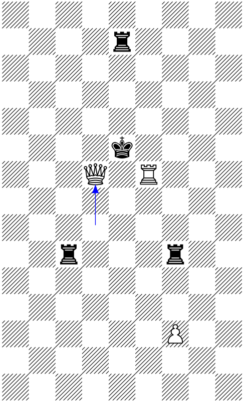


Check

A finite position with value ω

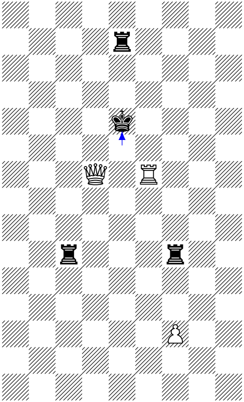


A finite position with value ω

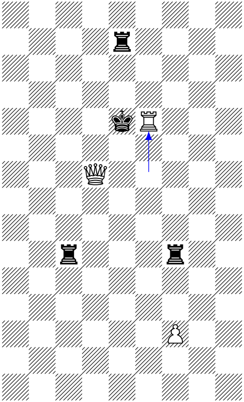


Check

A finite position with value ω

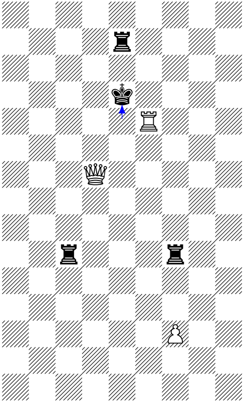


A finite position with value ω

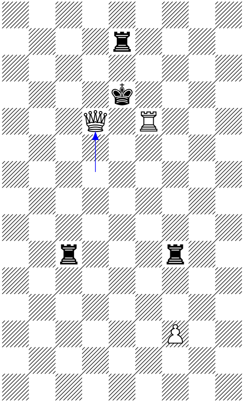


Check

A finite position with value ω

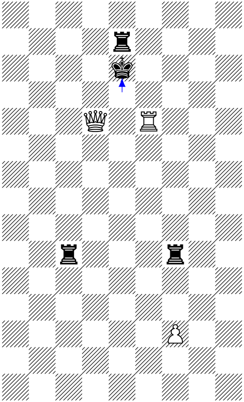


A finite position with value ω

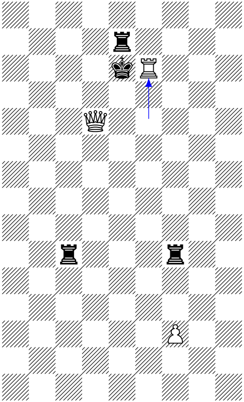


Check

A finite position with value ω

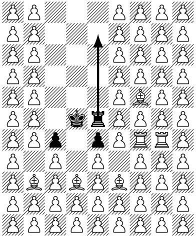


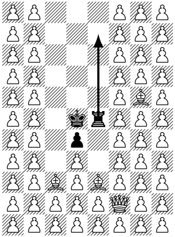
A finite position with value ω

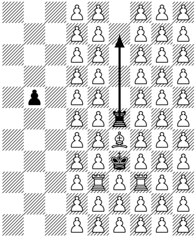


Checkmate. Black can cause arbitrary delay, but the only choice is on first move, so the initial value is ω.

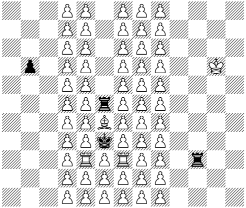
Several infinite positions with value ω

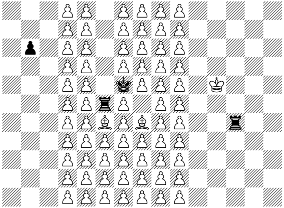


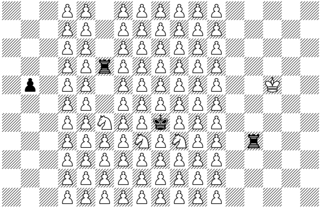




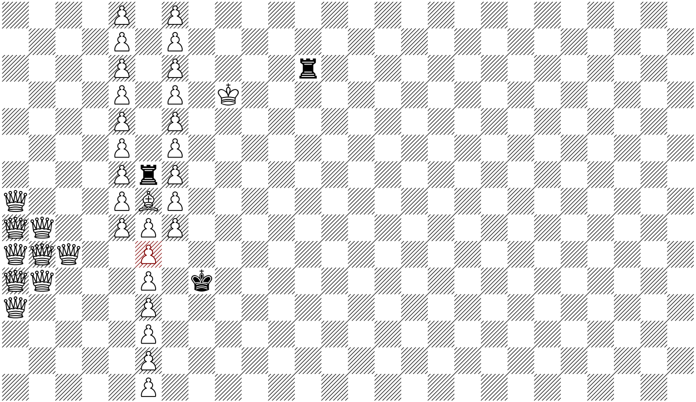
Positions with value ω^2





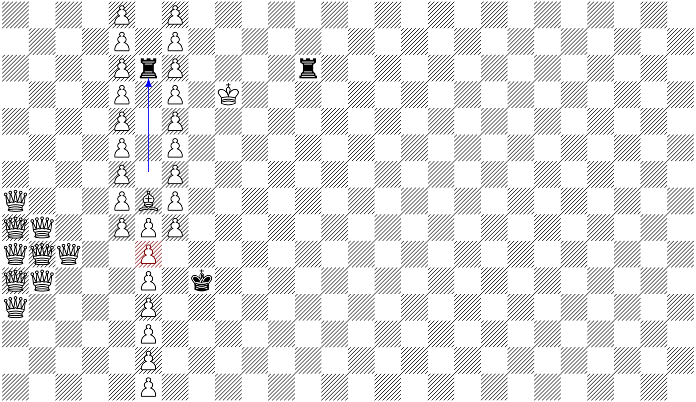


Releasing the Hordes, with value ω^2



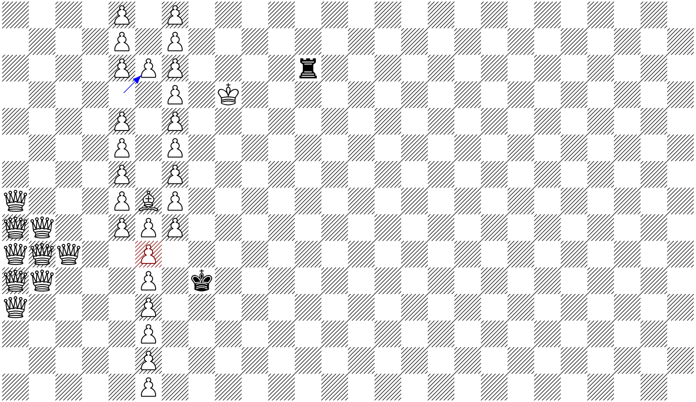
Black to move.

Releasing the Hordes, with value ω^2



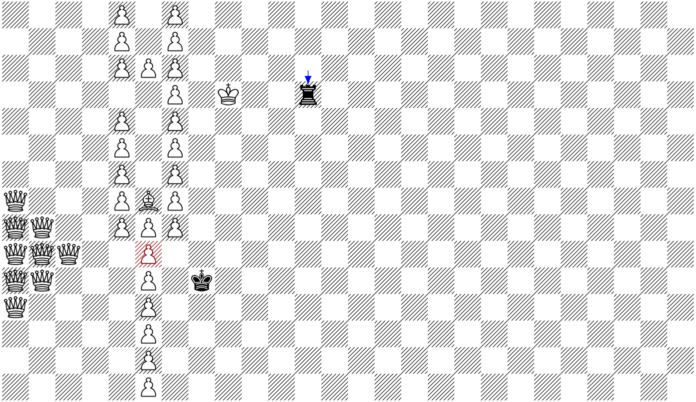
He moves trapped rook up arbitrary height.

Releasing the Hordes, with value ω^2



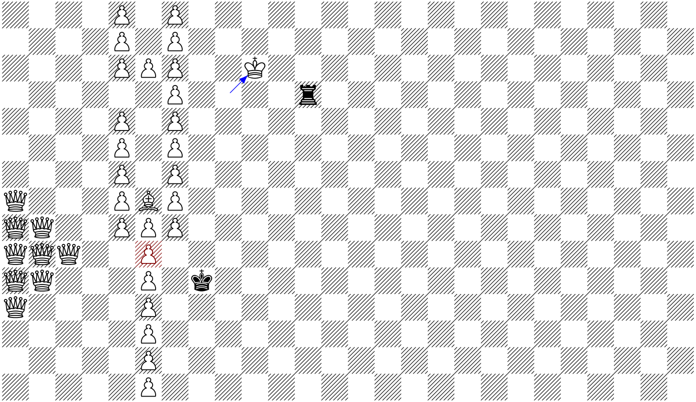
White should capture from left side.

Releasing the Hordes, with value ω^2



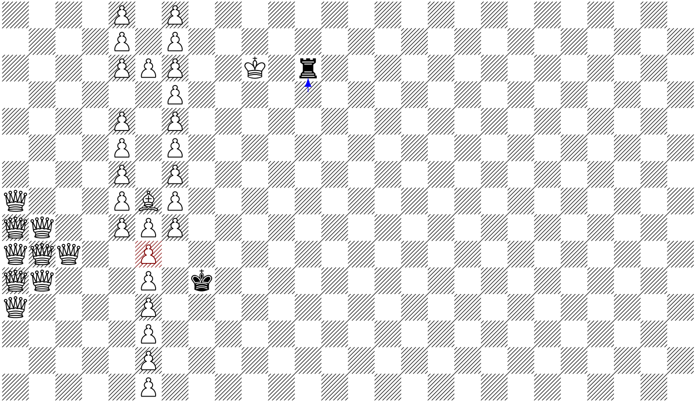
Now black begins to harass white king.

Releasing the Hordes, with value ω^2



White must chase down the rook to avoid perpetual check.

Releasing the Hordes, with value ω^2



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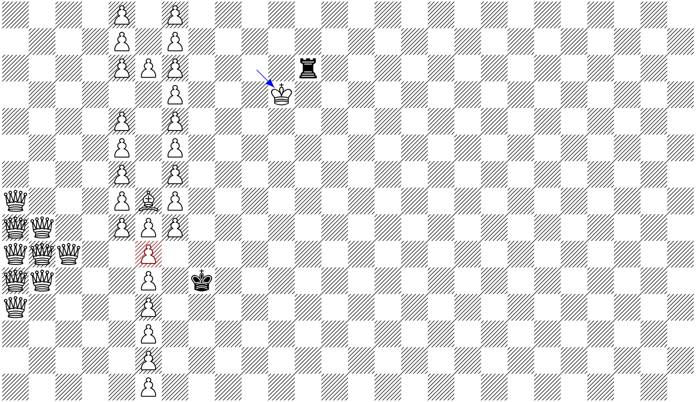

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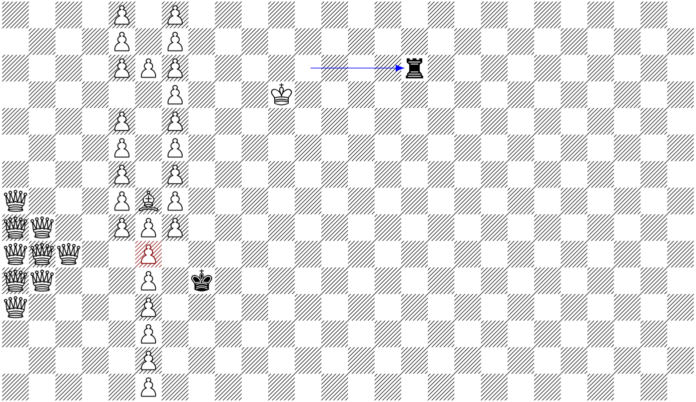
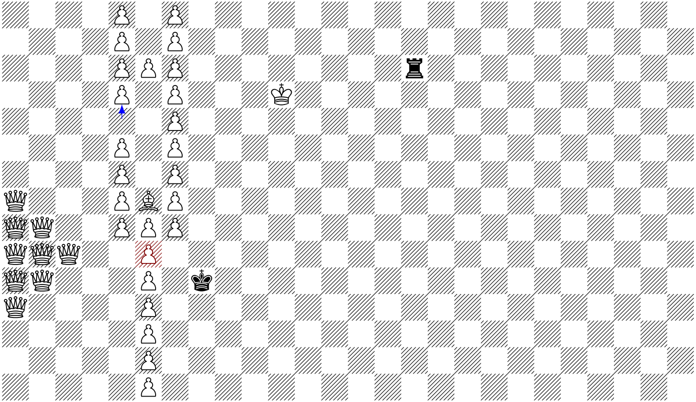


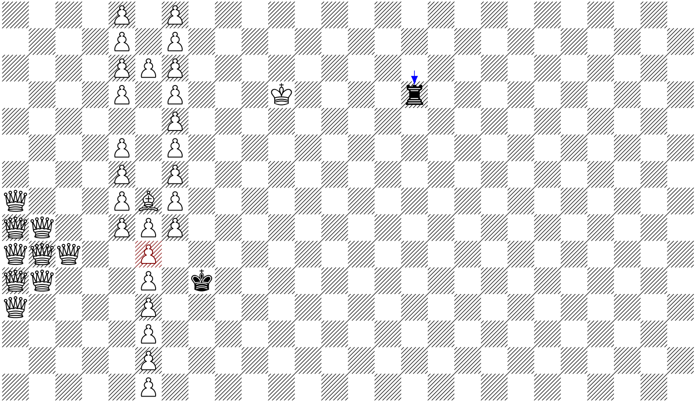
Black must move away to save rook.

Releasing the Hordes, with value ω^2



Now is white's chance to advance a pawn.

Releasing the Hordes, with value ω^2



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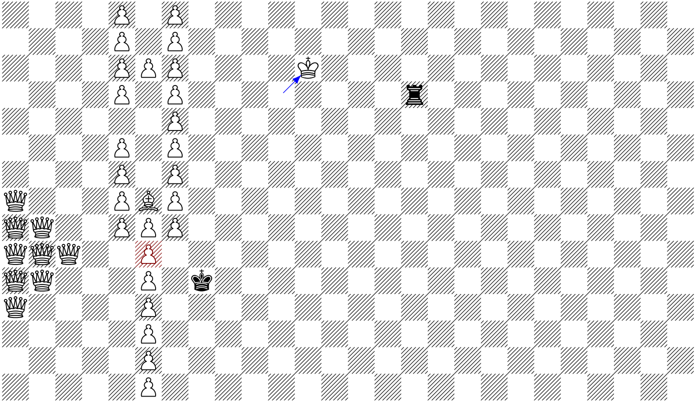

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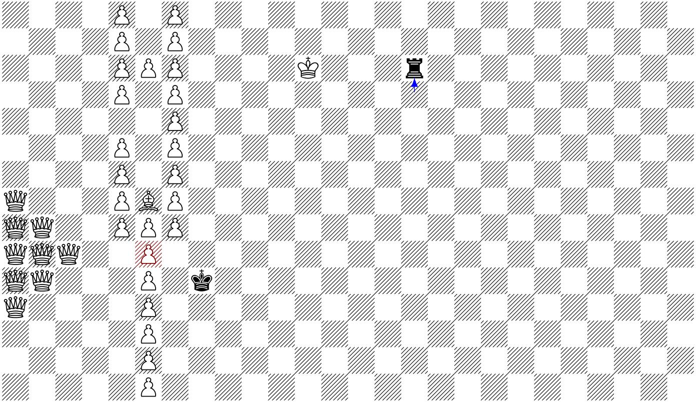

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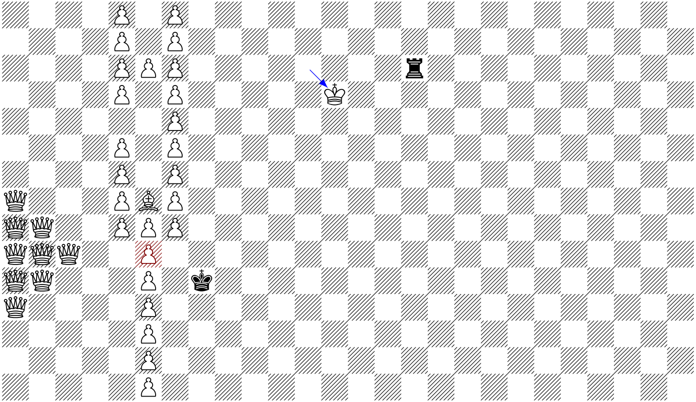

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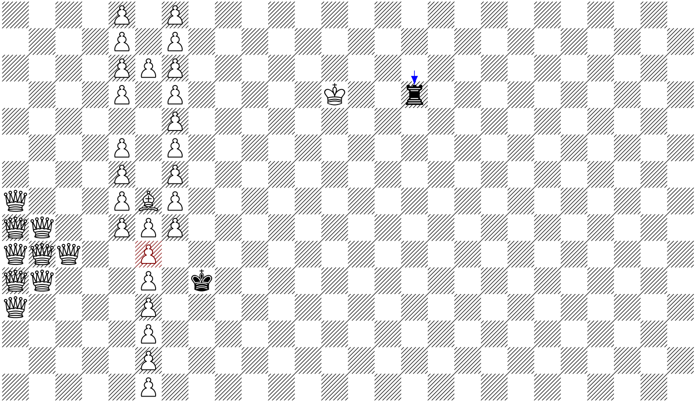

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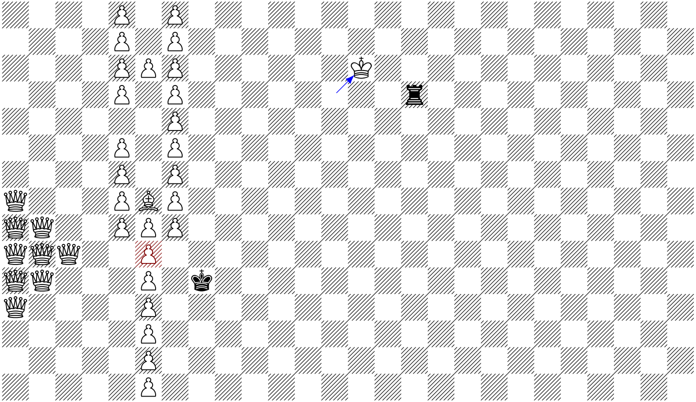

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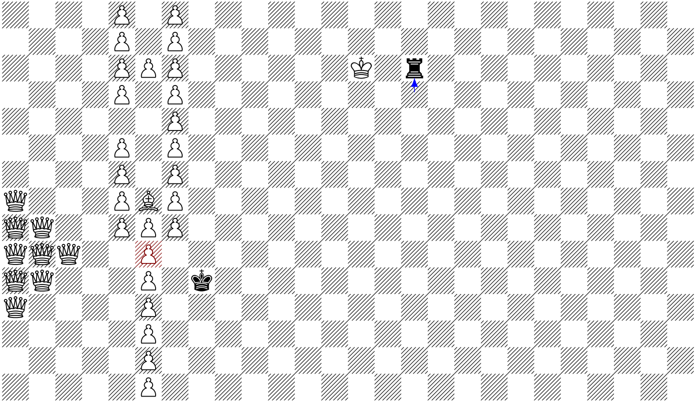
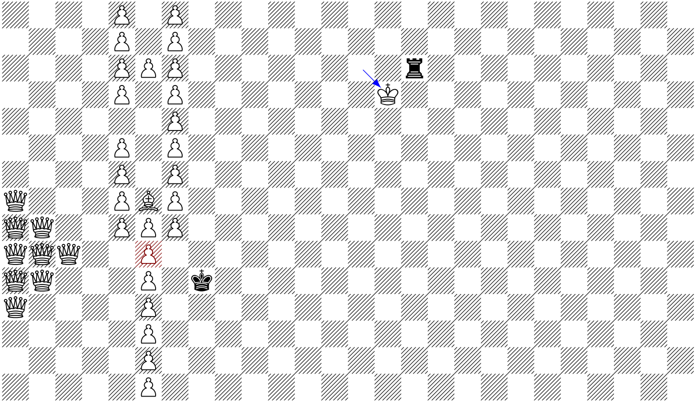


Releasing the Hordes, with value ω^2



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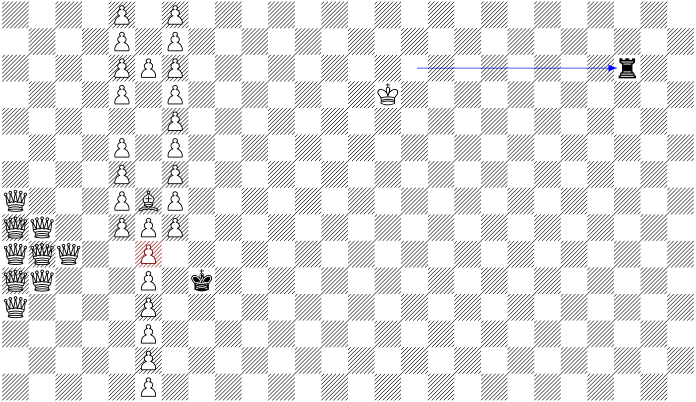
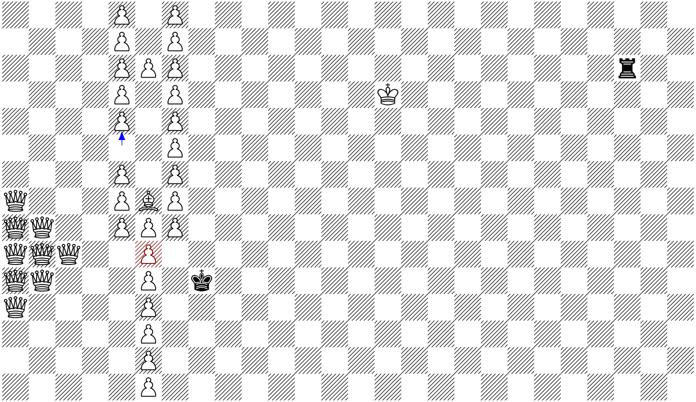


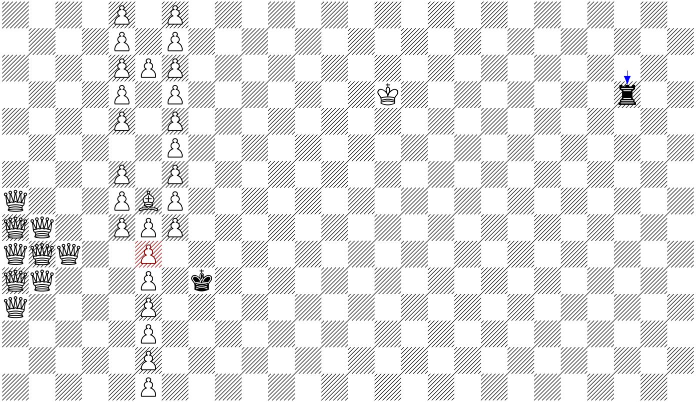
Black moves arbitrary distance out.

Releasing the Hordes, with value ω^2



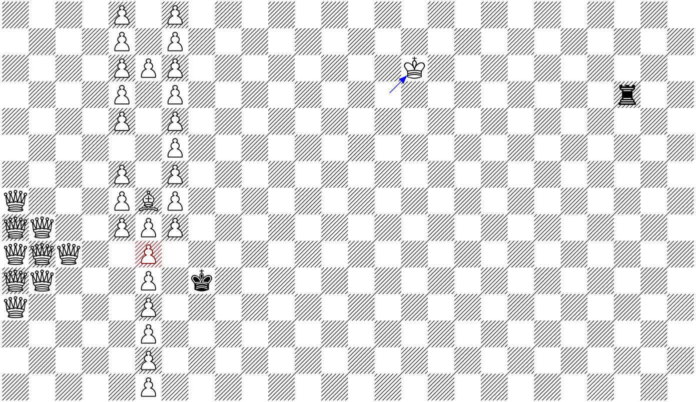
Another chance to advance a pawn.

Releasing the Hordes, with value ω^2



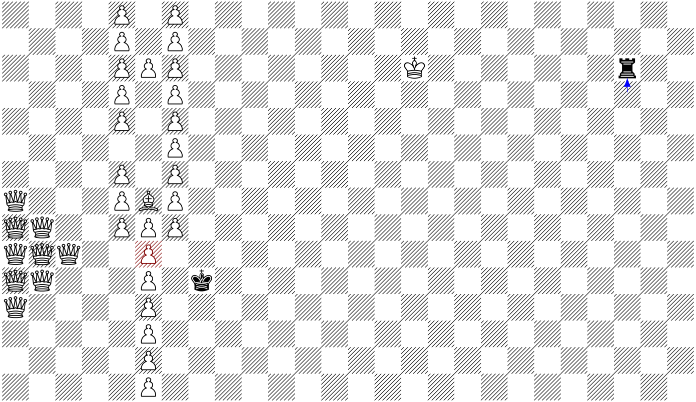
Black harasses the white king.

Releasing the Hordes, with value ω^2

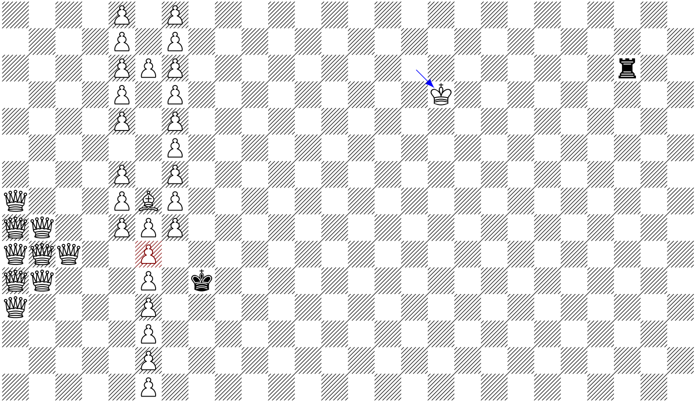


White must chase him down.

Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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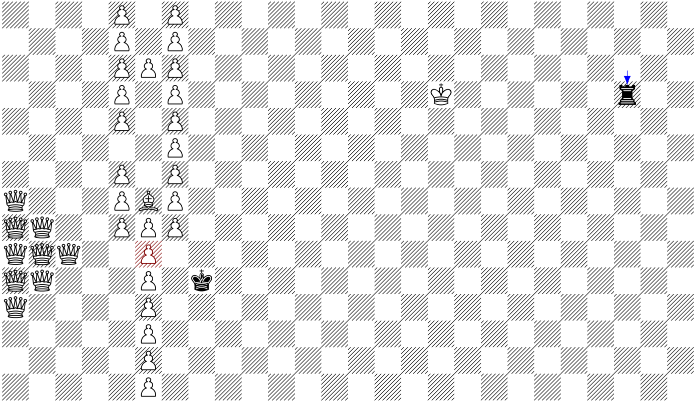

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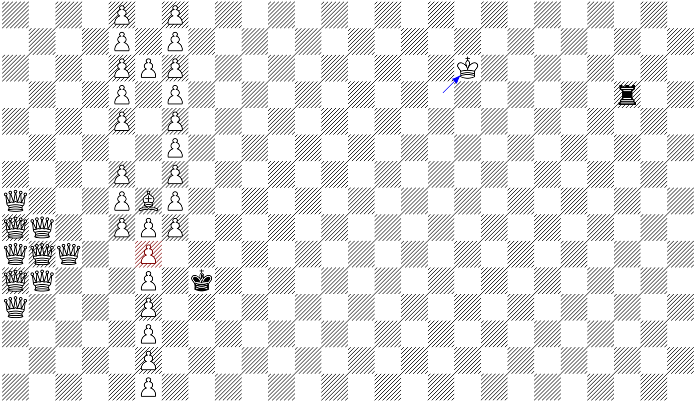

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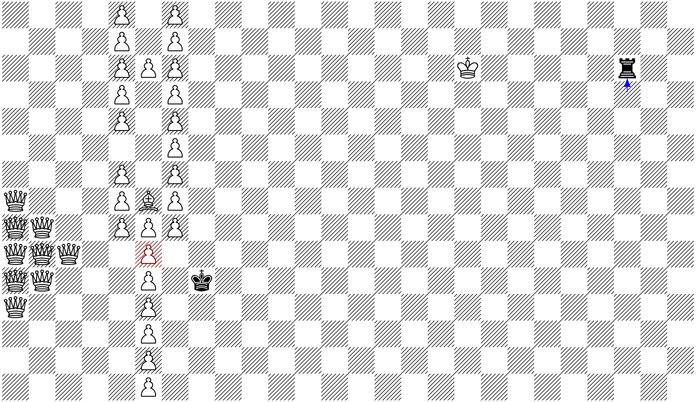

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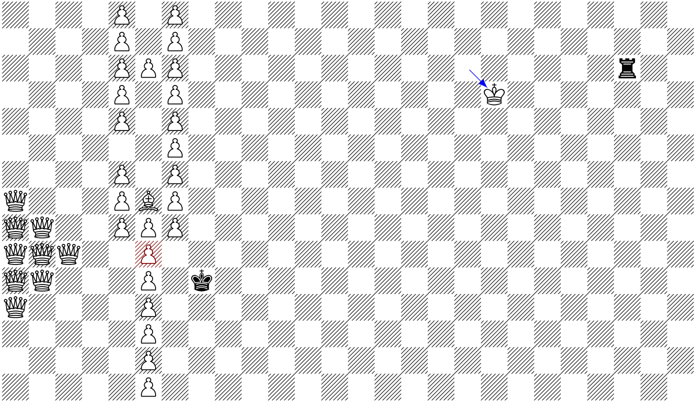

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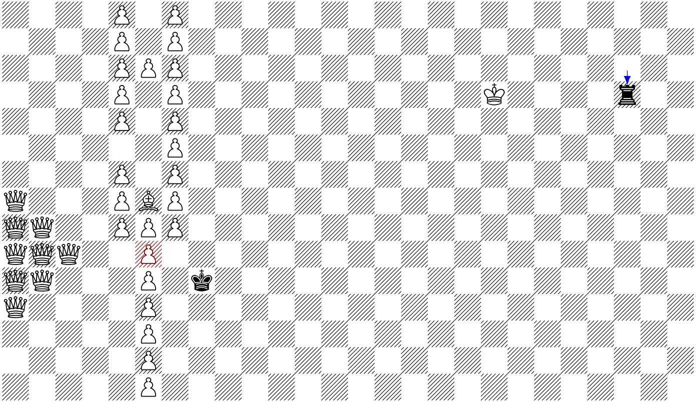

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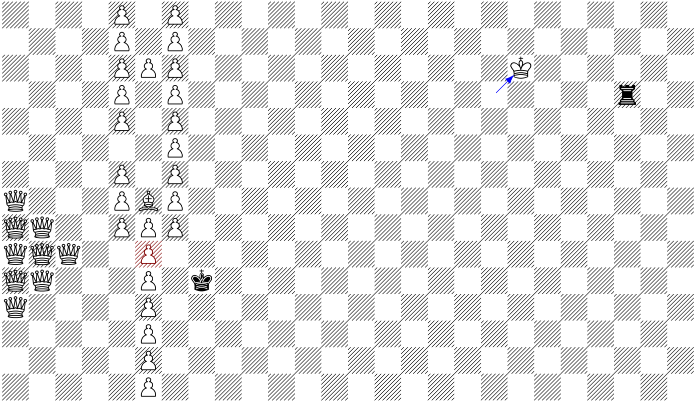

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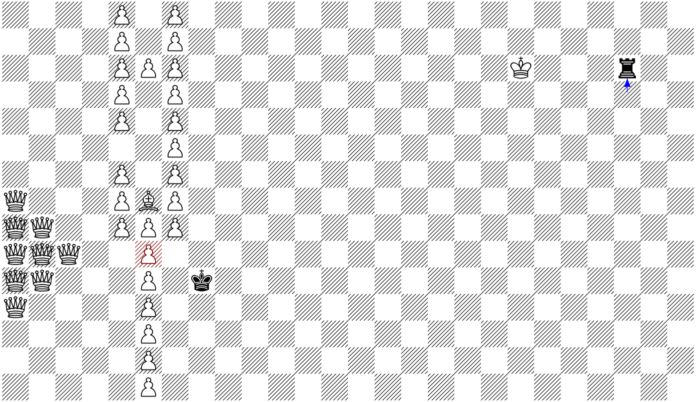

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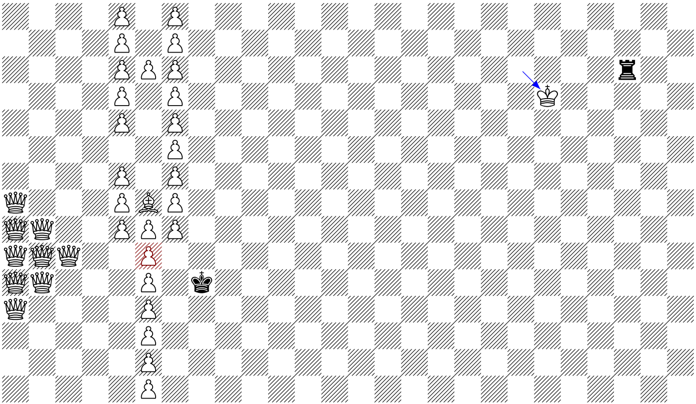

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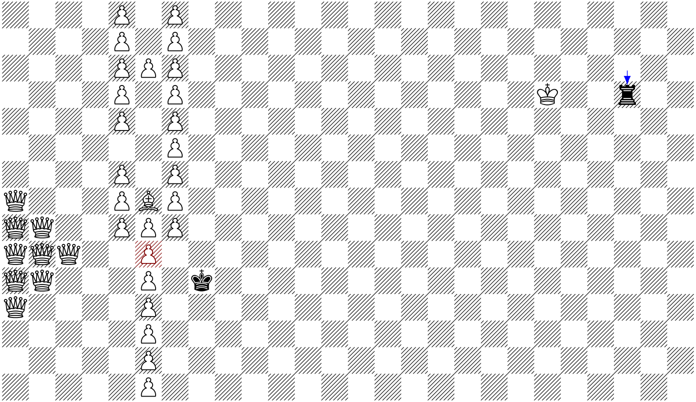
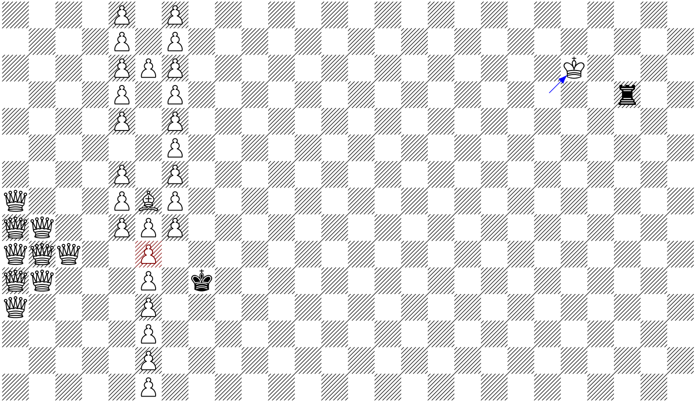
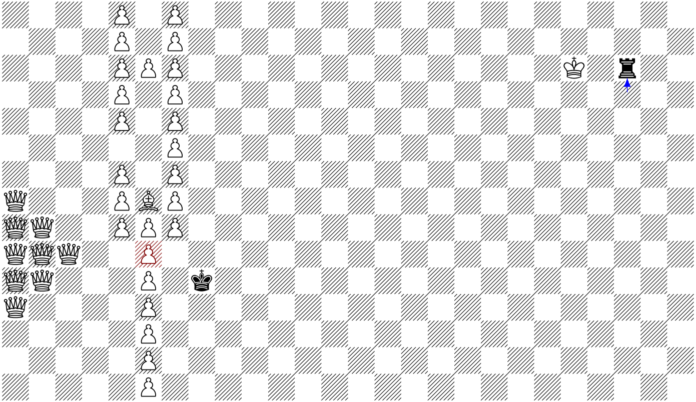


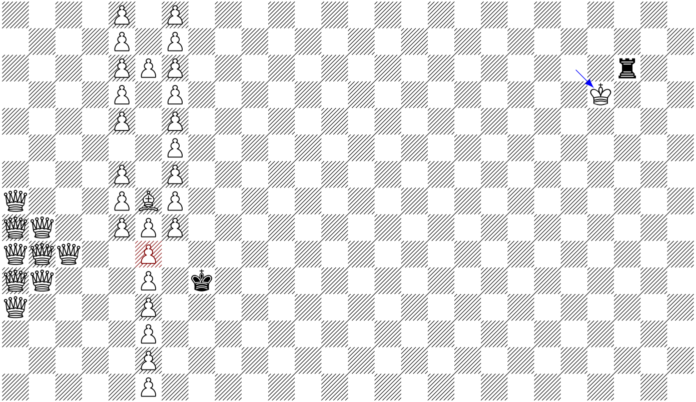
Releasing the Hordes, with value ω^2



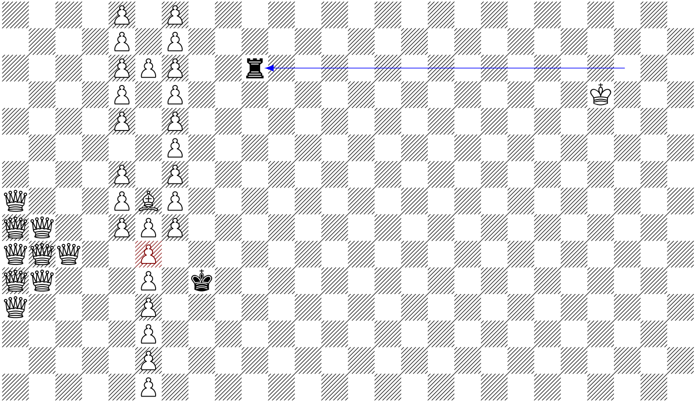
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

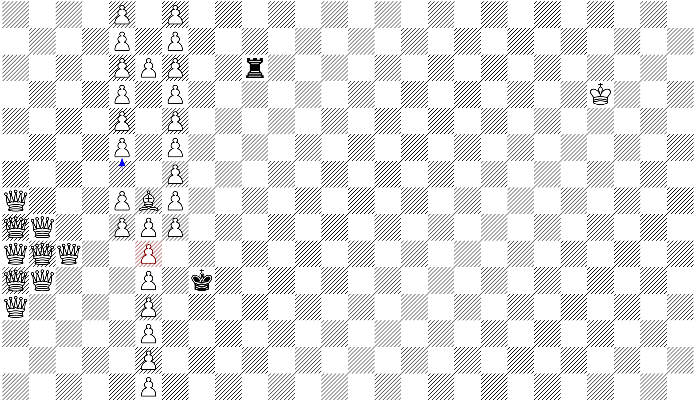


Releasing the Hordes, with value ω^2

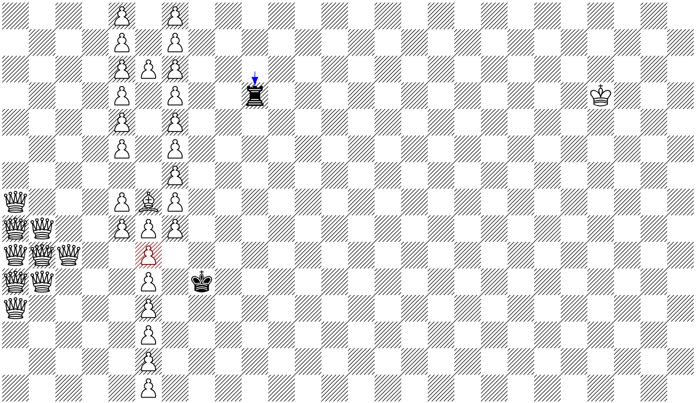


(Black should actually move arbitrary distance to the right.)

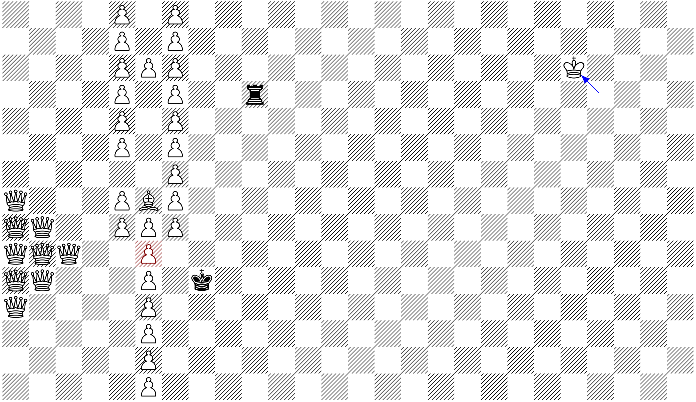
Releasing the Hordes, with value ω^2



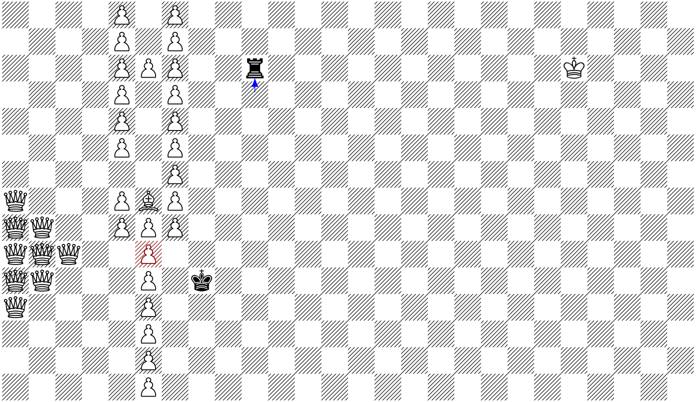
Releasing the Hordes, with value ω^2



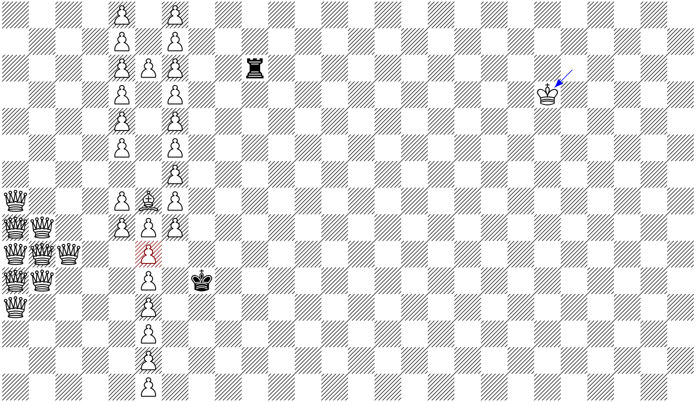
Releasing the Hordes, with value ω^2



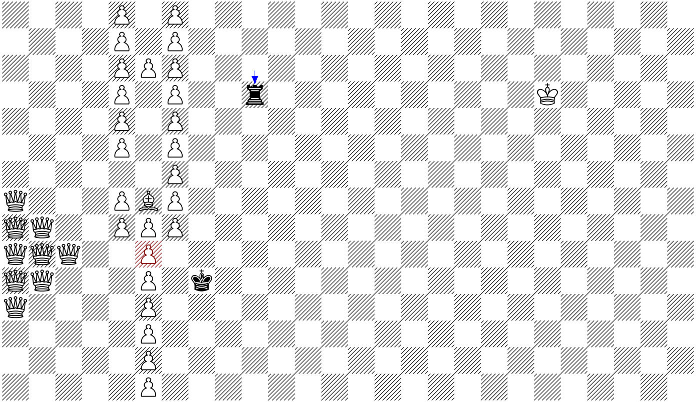
Releasing the Hordes, with value ω^2



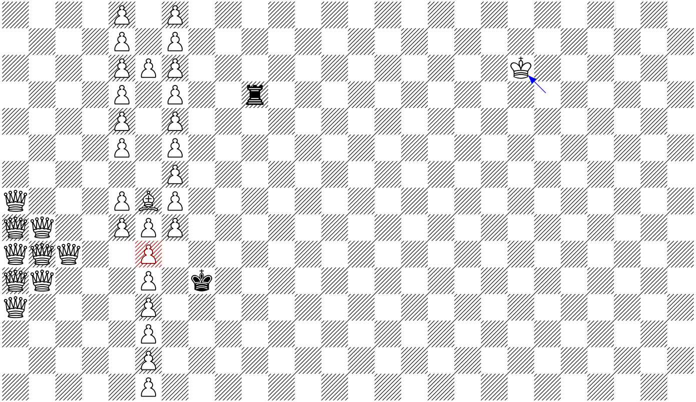
Releasing the Hordes, with value ω^2



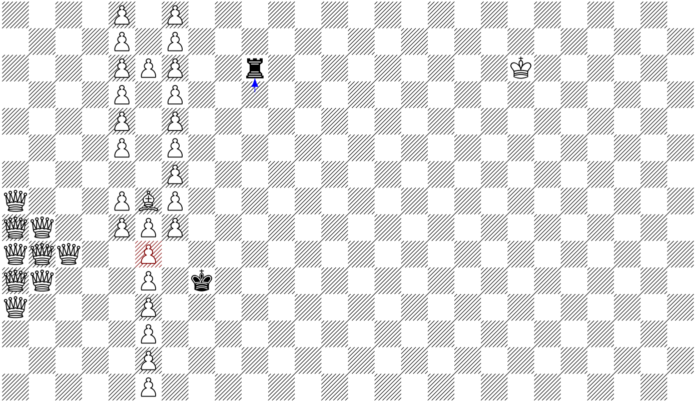
Releasing the Hordes, with value ω^2



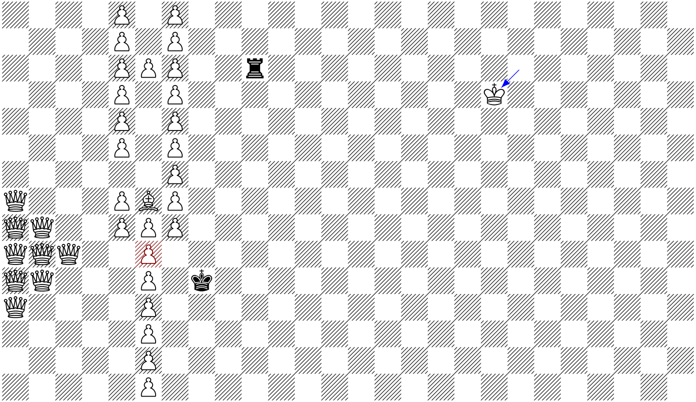
Releasing the Hordes, with value ω^2



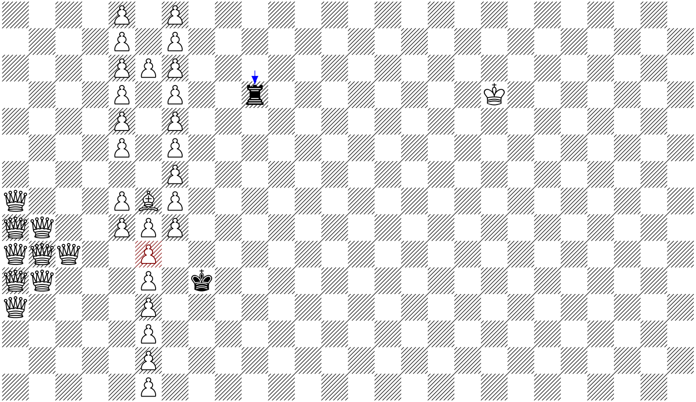
Releasing the Hordes, with value ω^2



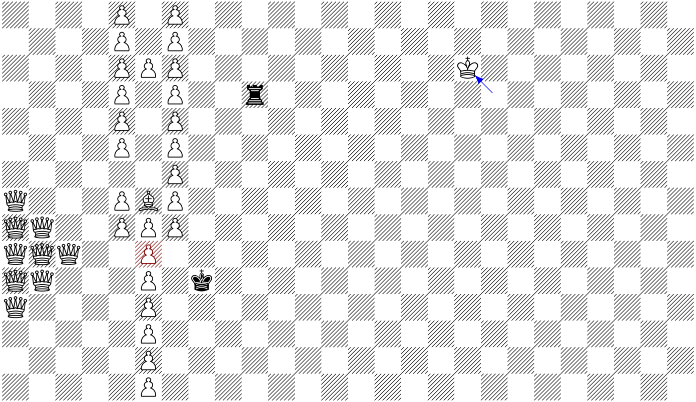
Releasing the Hordes, with value ω^2



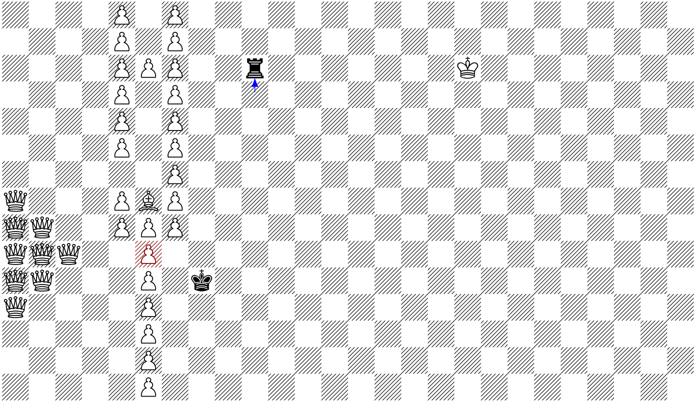
Releasing the Hordes, with value ω^2



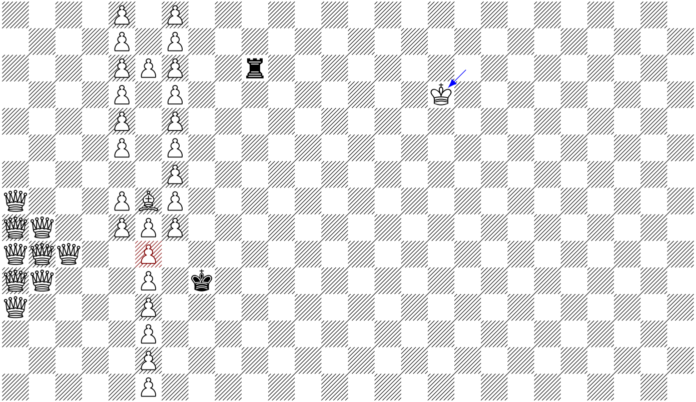
Releasing the Hordes, with value ω^2



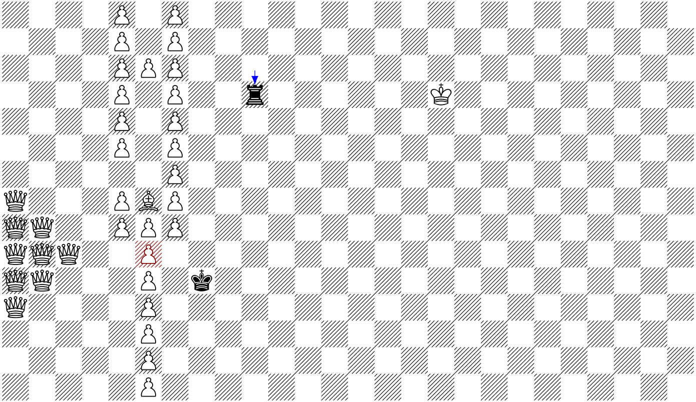
Releasing the Hordes, with value ω^2



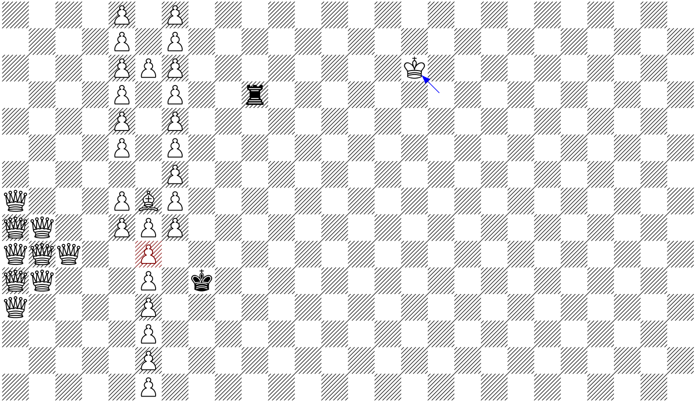
Releasing the Hordes, with value ω^2



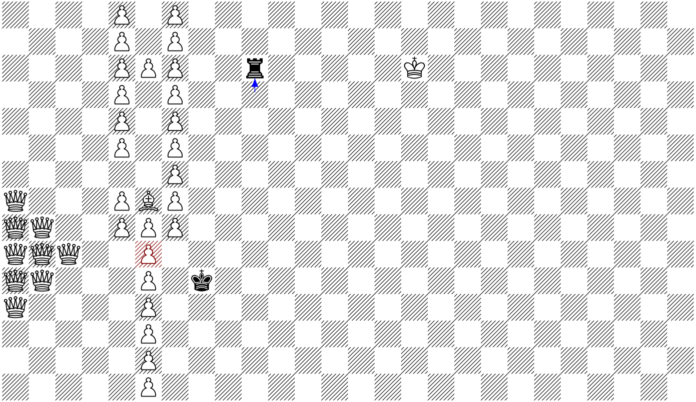
Releasing the Hordes, with value ω^2



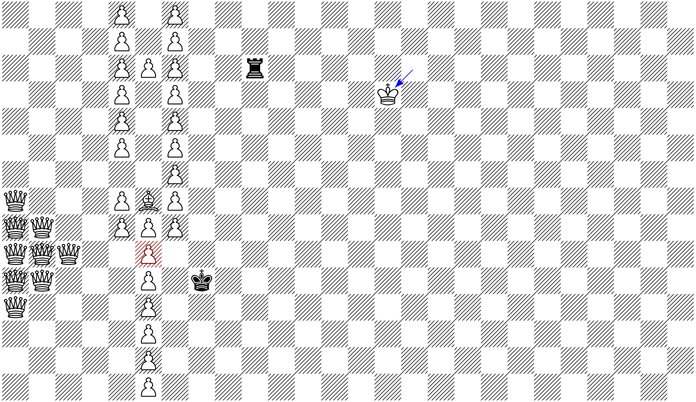
Releasing the Hordes, with value ω^2



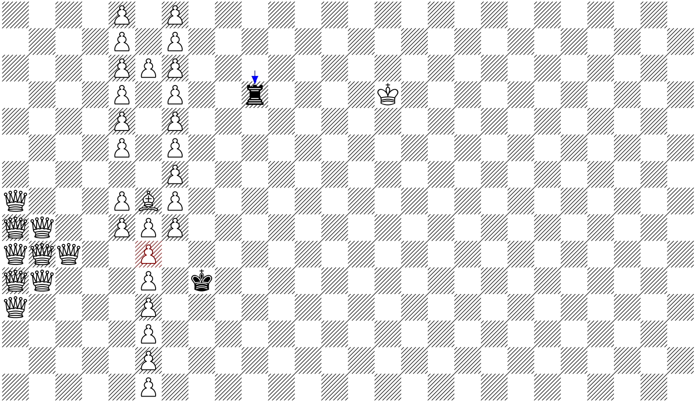
Releasing the Hordes, with value ω^2



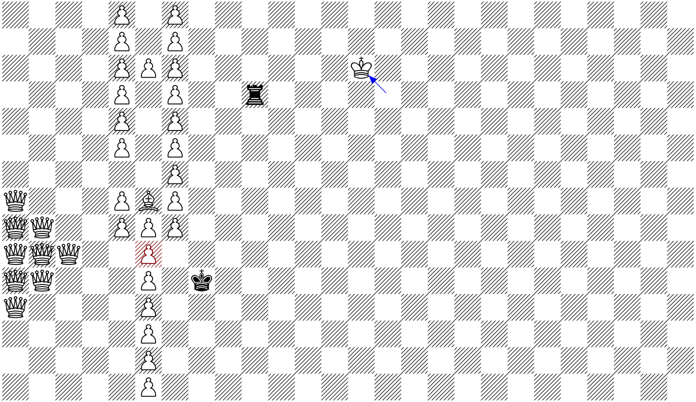
Releasing the Hordes, with value ω^2



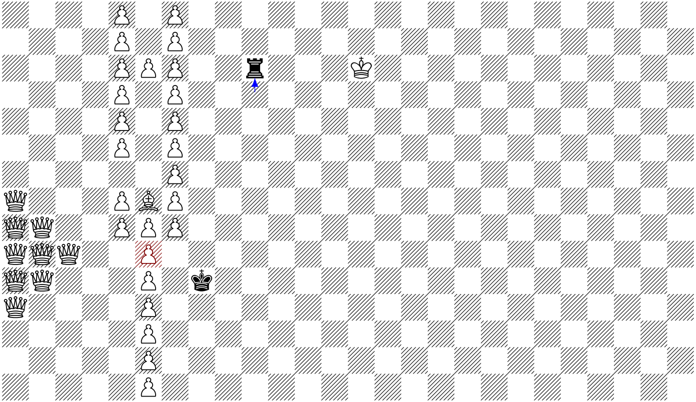
Releasing the Hordes, with value ω^2



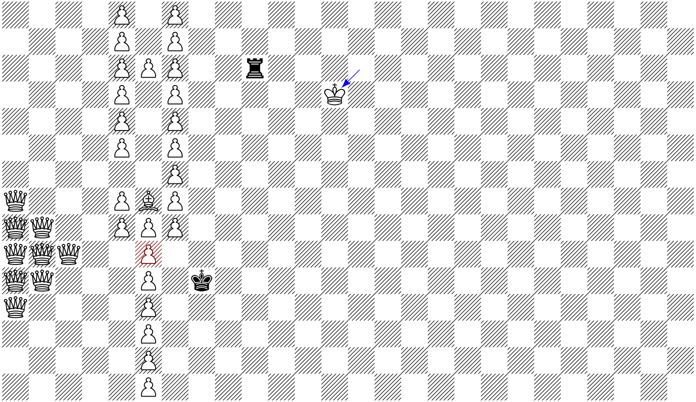
Releasing the Hordes, with value ω^2



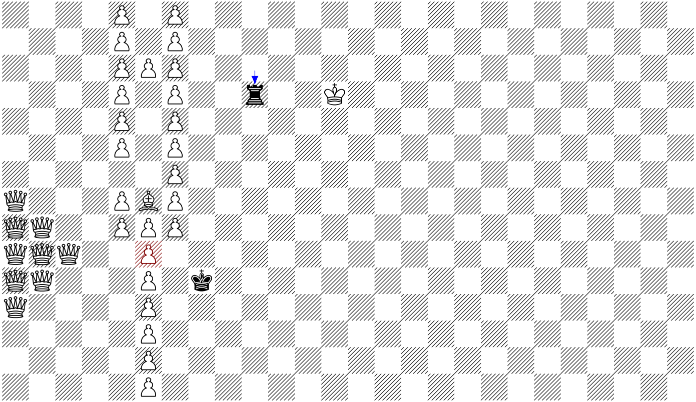
Releasing the Hordes, with value ω^2



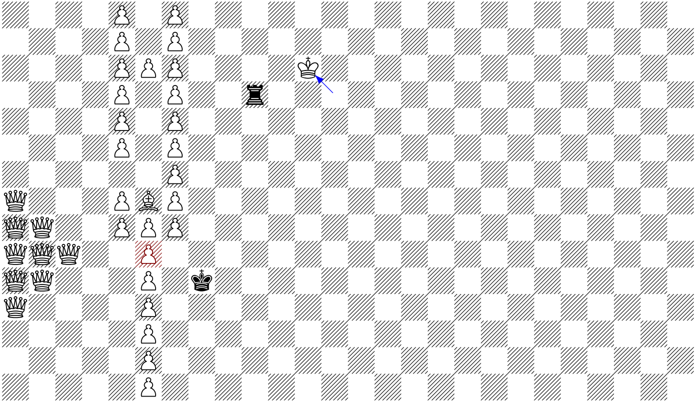
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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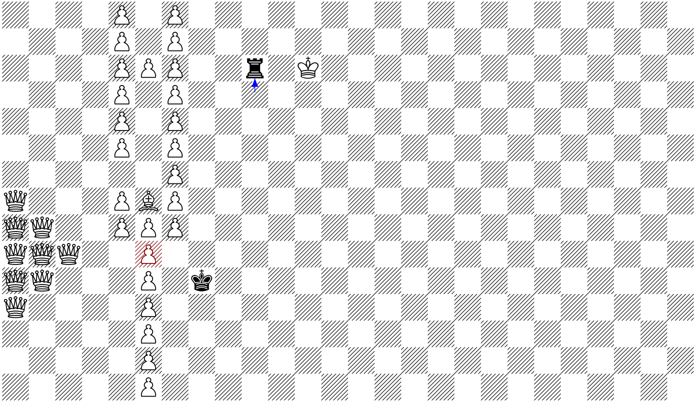
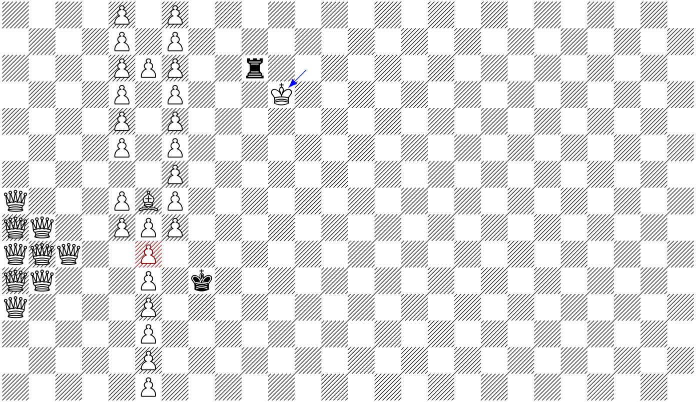
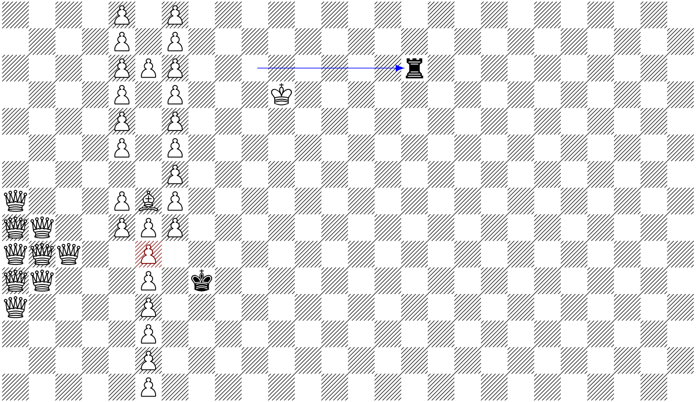


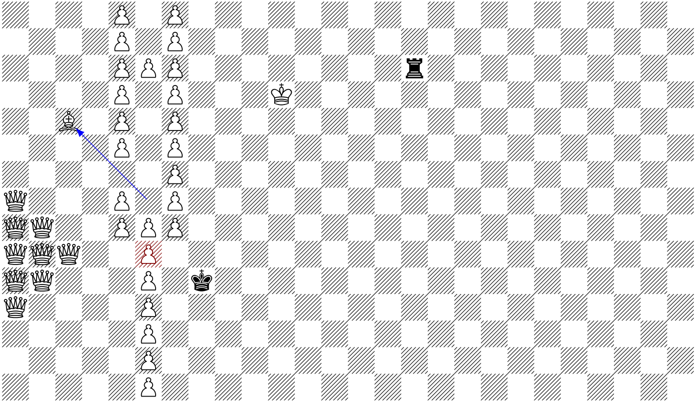
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

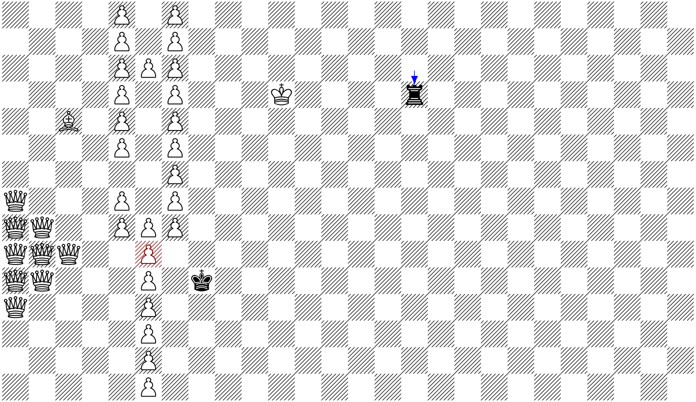


Releasing the Hordes, with value ω^2

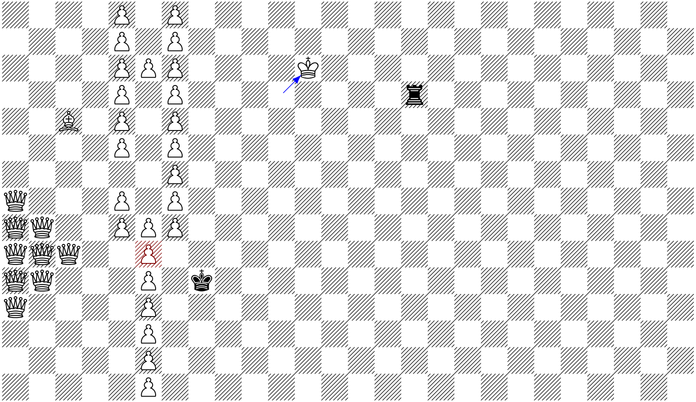


The bishop unlocks the door.

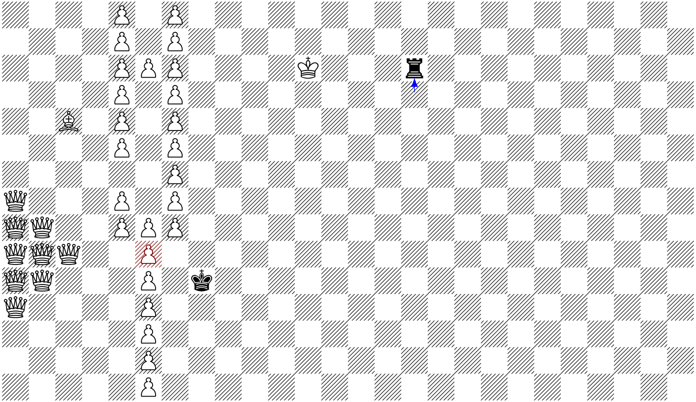
Releasing the Hordes, with value ω^2



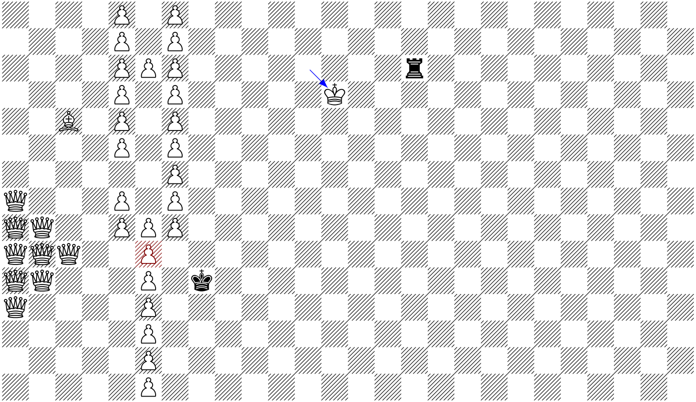
Releasing the Hordes, with value ω^2



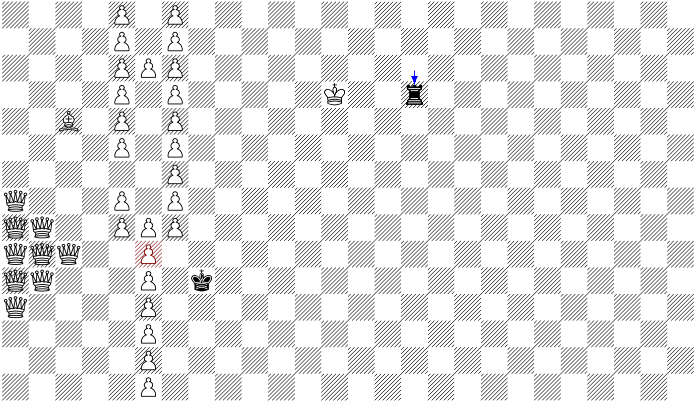
Releasing the Hordes, with value ω^2



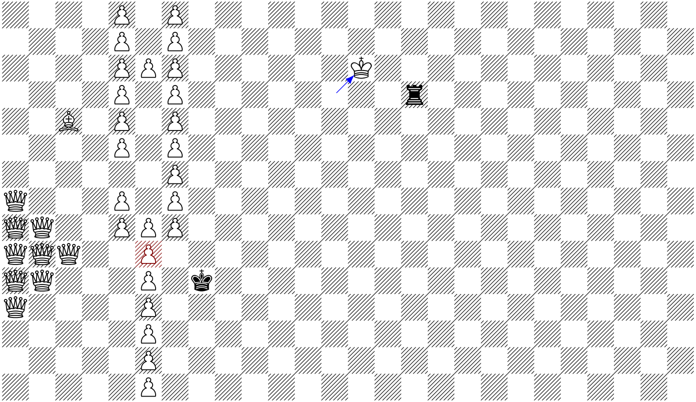
Releasing the Hordes, with value ω^2



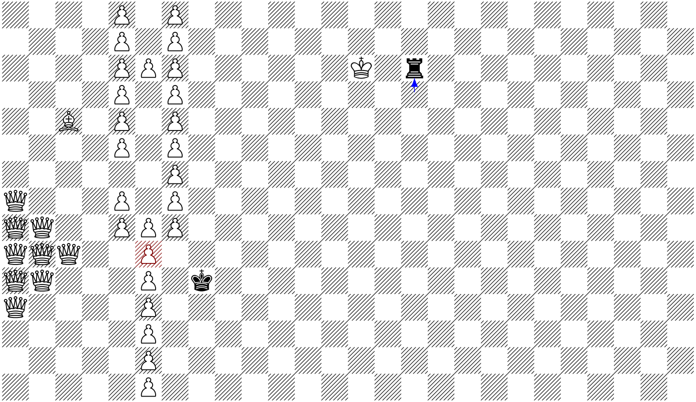
Releasing the Hordes, with value ω^2



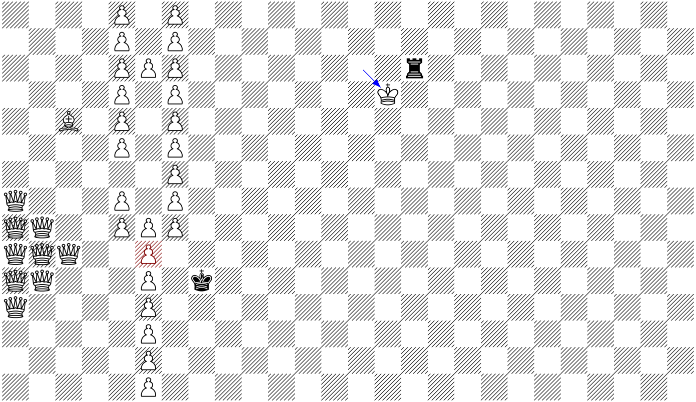
Releasing the Hordes, with value ω^2



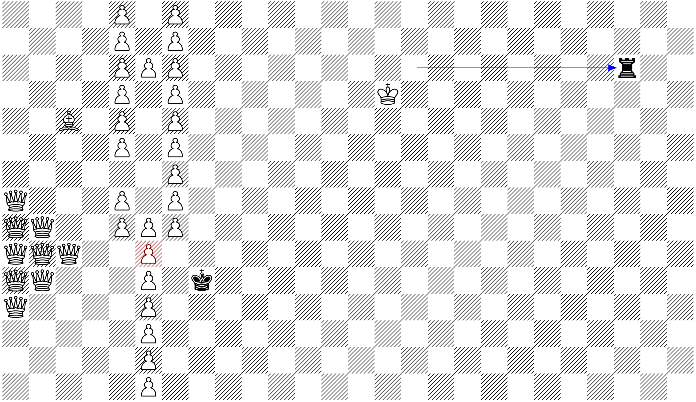
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

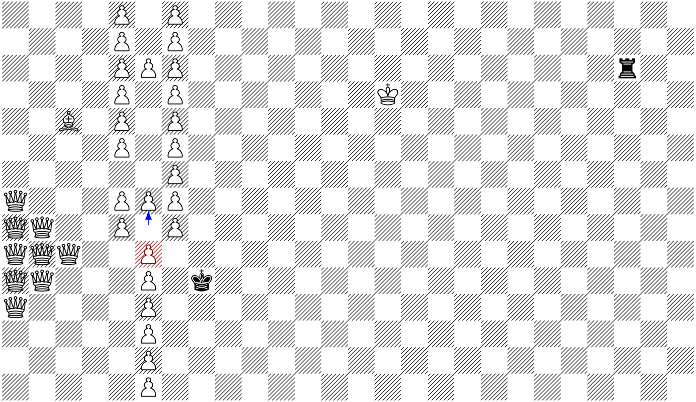


Releasing the Hordes, with value ω^2

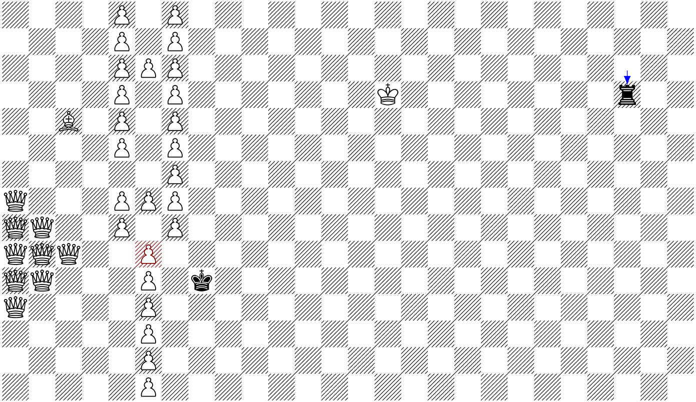


Black can move rook arbitrary distance.

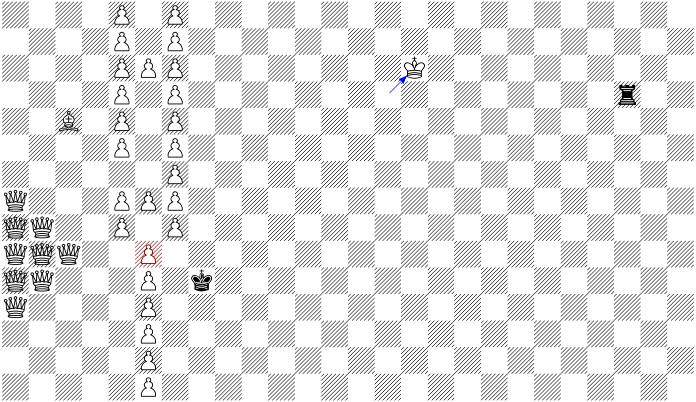
Releasing the Hordes, with value ω^2



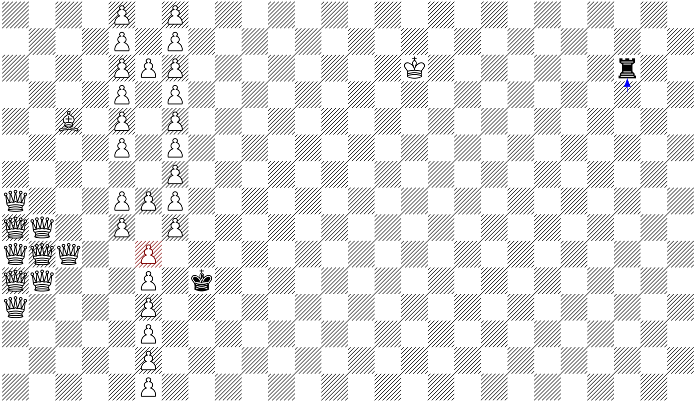
Releasing the Hordes, with value ω^2



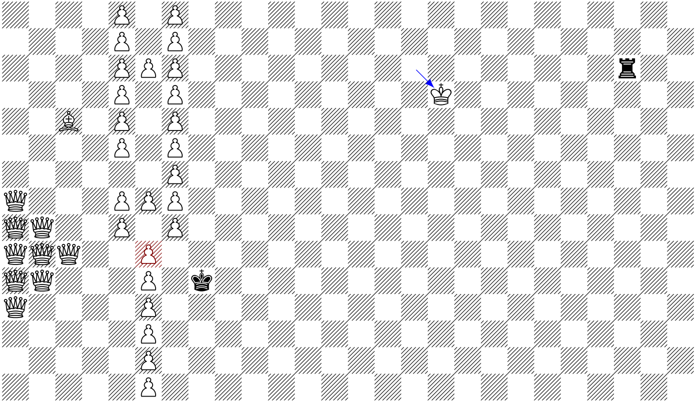
Releasing the Hordes, with value ω^2



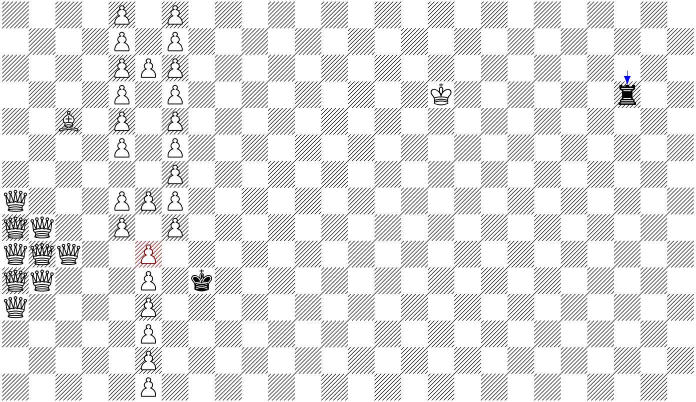
Releasing the Hordes, with value ω^2



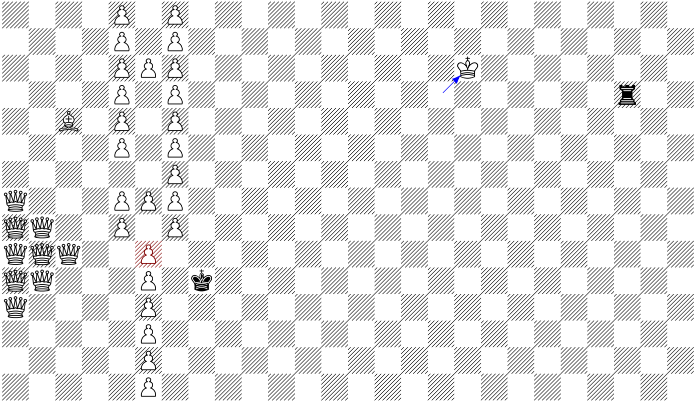
Releasing the Hordes, with value ω^2



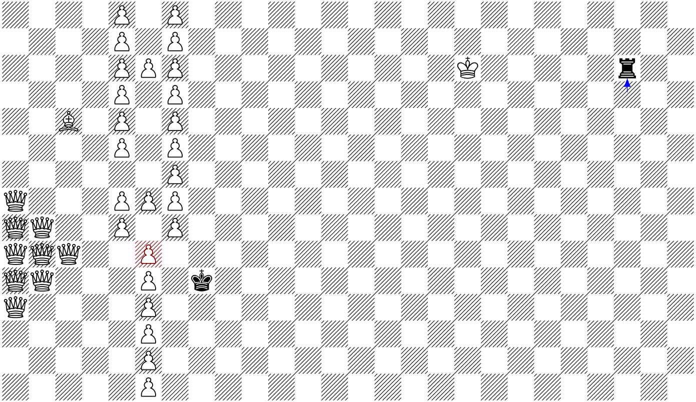
Releasing the Hordes, with value ω^2



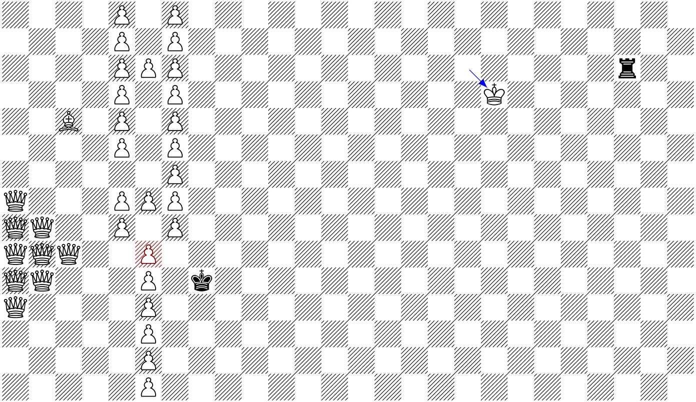
Releasing the Hordes, with value ω^2



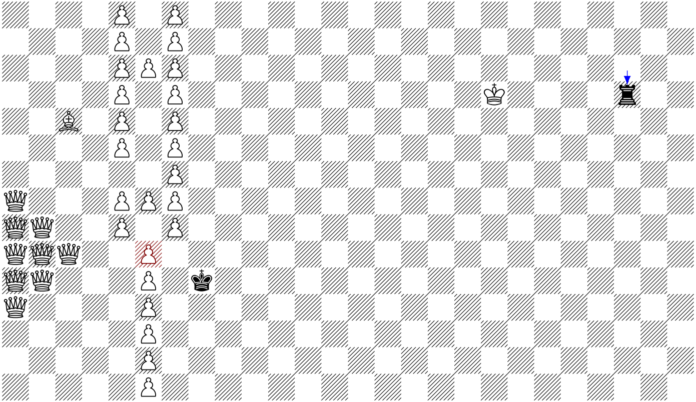
Releasing the Hordes, with value ω^2



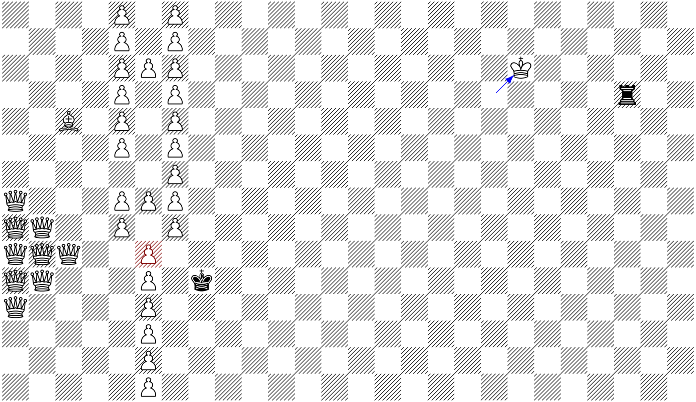
Releasing the Hordes, with value ω^2



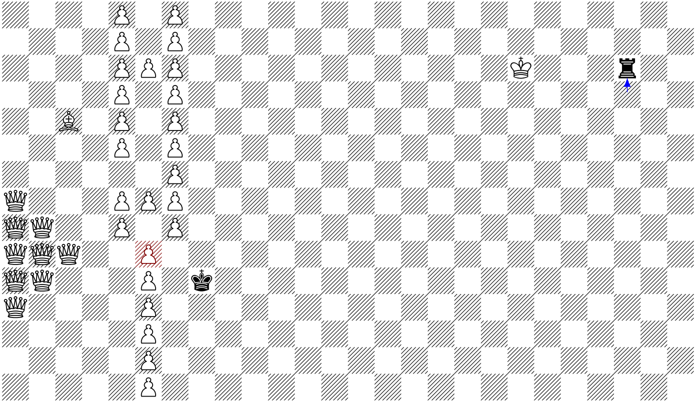
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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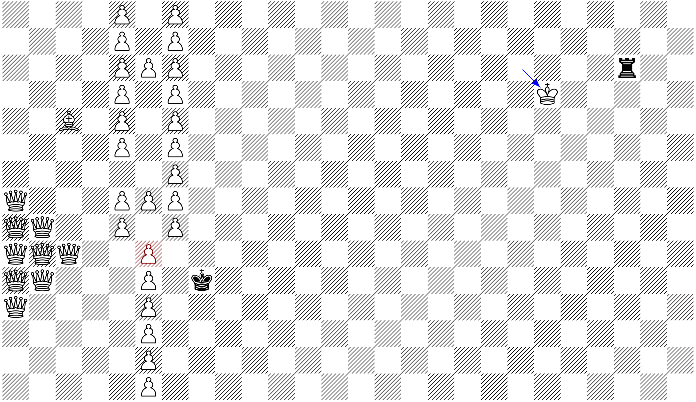
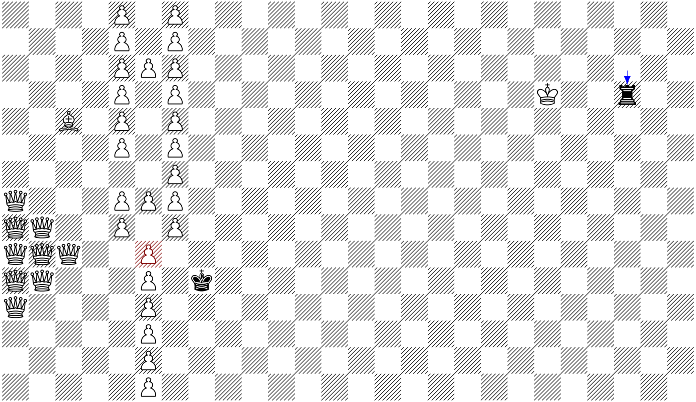
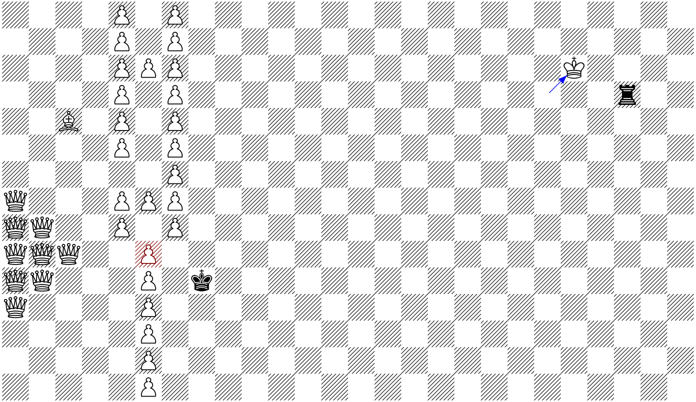
Releasing the Hordes, with value ω^2



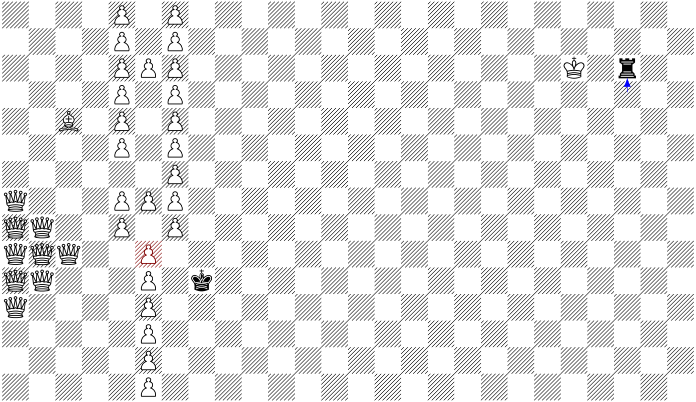
Releasing the Hordes, with value ω^2



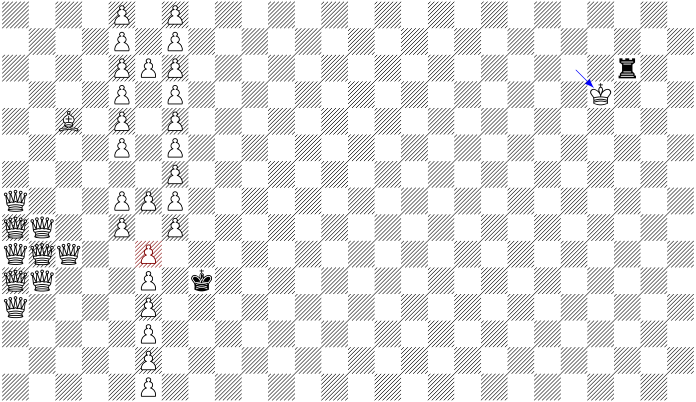
Releasing the Hordes, with value ω^2



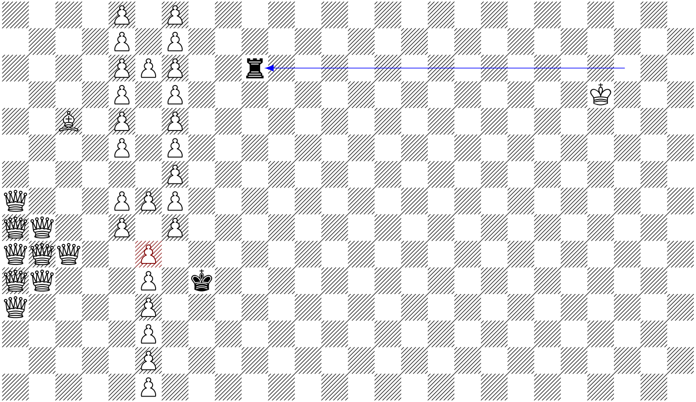
Releasing the Hordes, with value ω^2



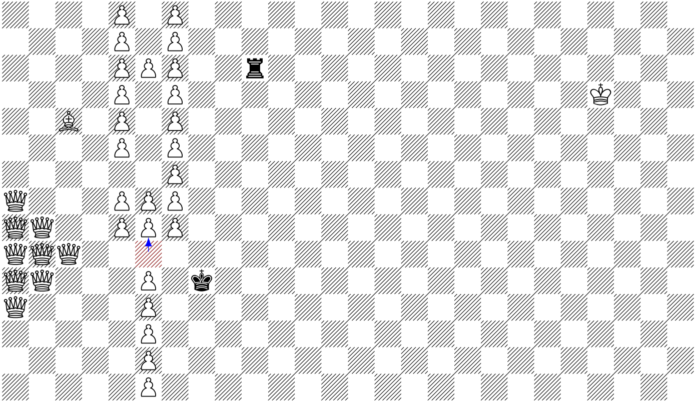
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

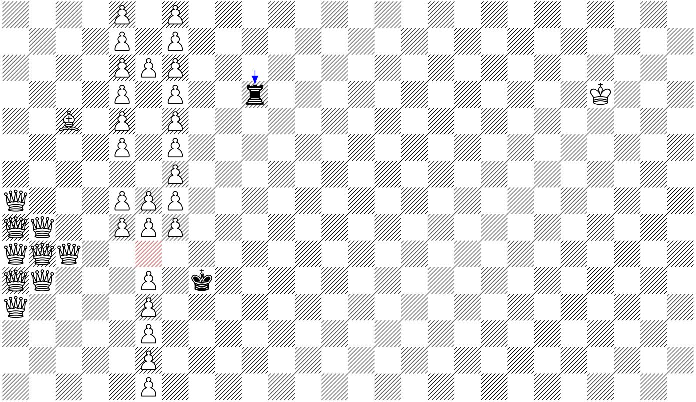


Releasing the Hordes, with value ω^2

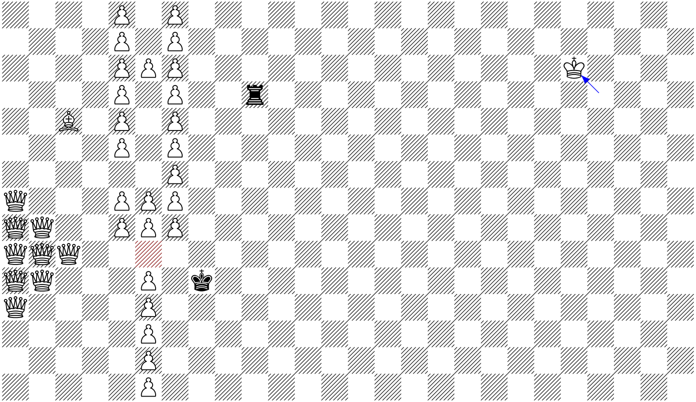


The portcullis opens...

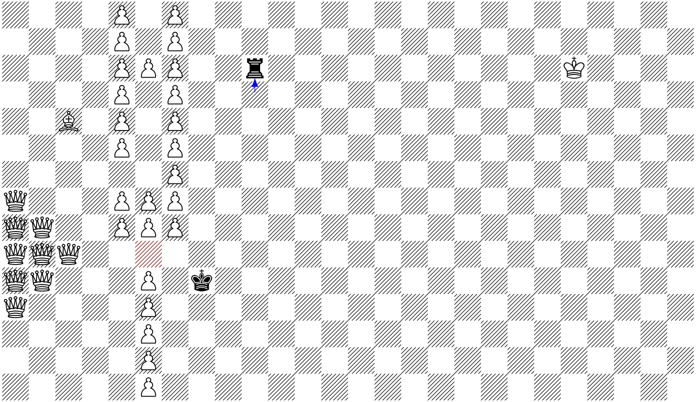
Releasing the Hordes, with value ω^2



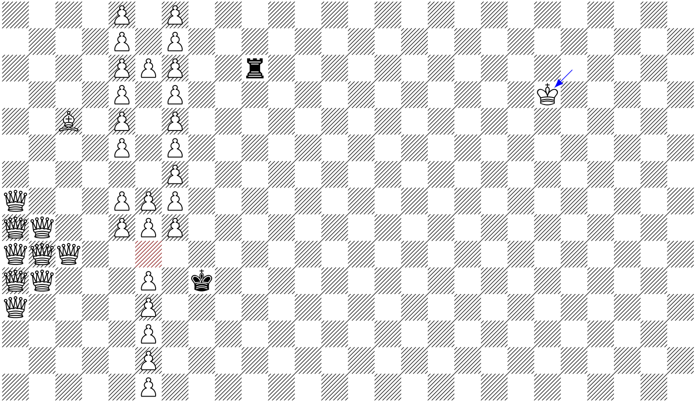
Releasing the Hordes, with value ω^2



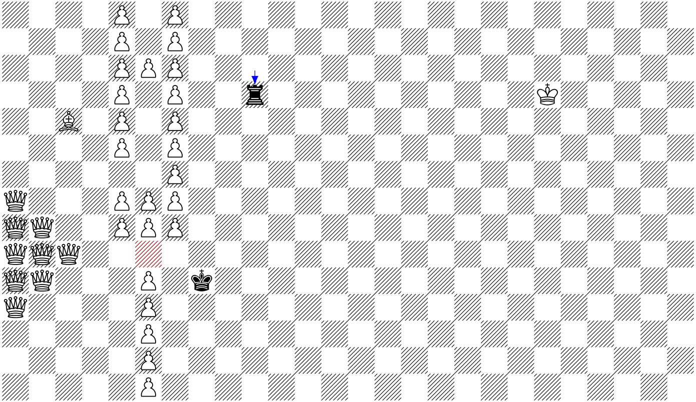
Releasing the Hordes, with value ω^2



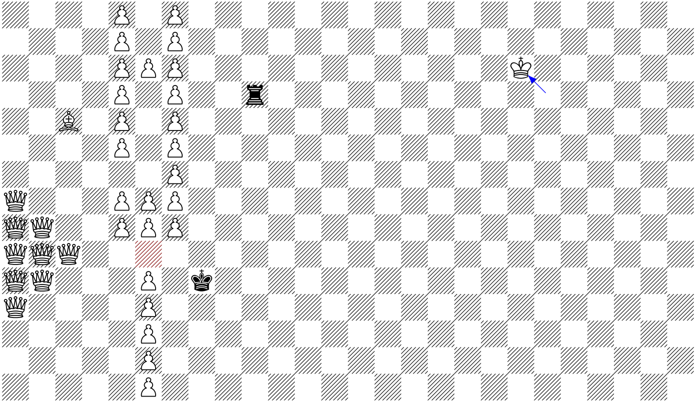
Releasing the Hordes, with value ω^2



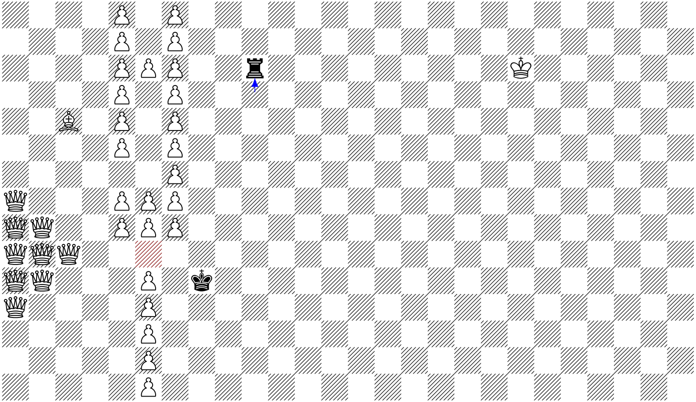
Releasing the Hordes, with value ω^2



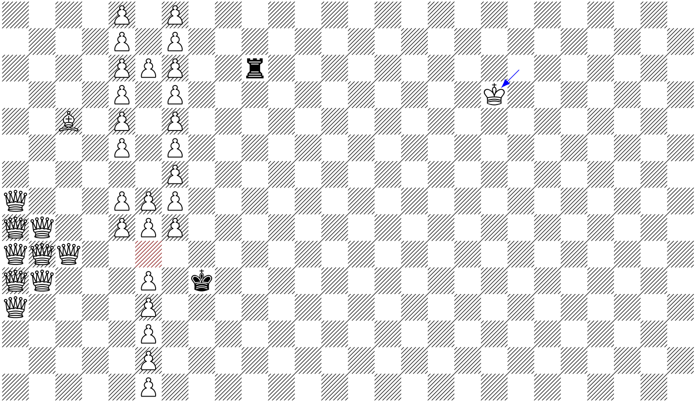
Releasing the Hordes, with value ω^2



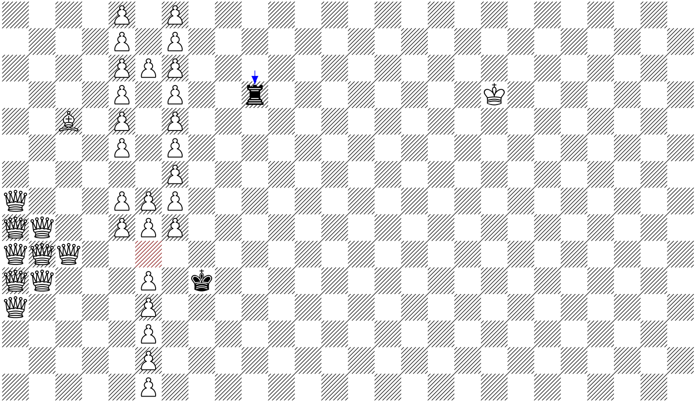
Releasing the Hordes, with value ω^2



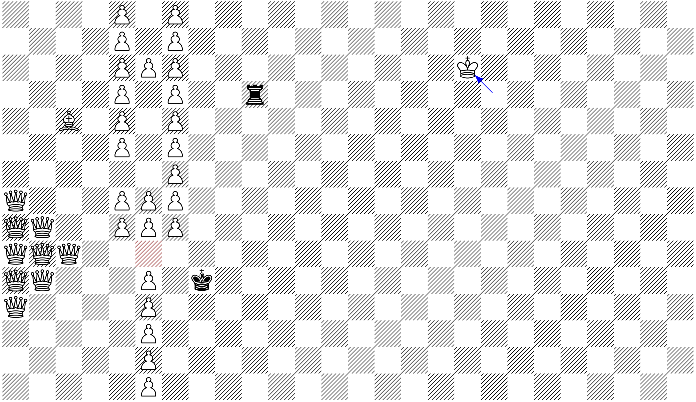
Releasing the Hordes, with value ω^2



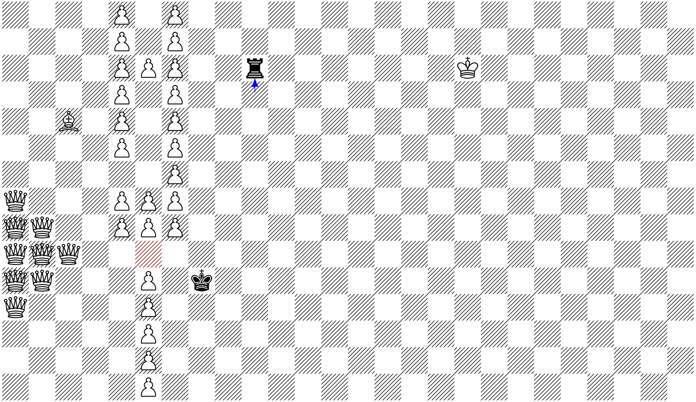
Releasing the Hordes, with value ω^2



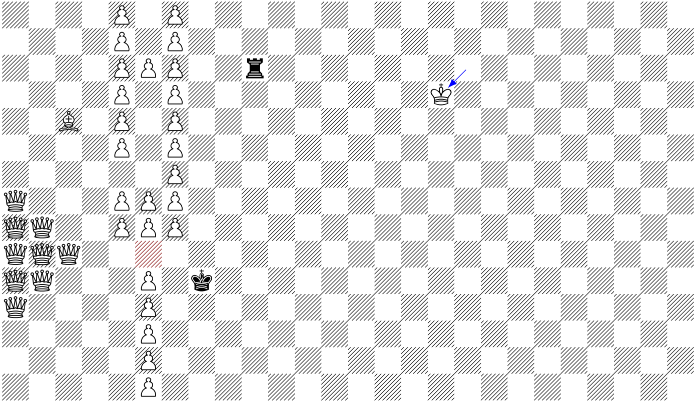
Releasing the Hordes, with value ω^2



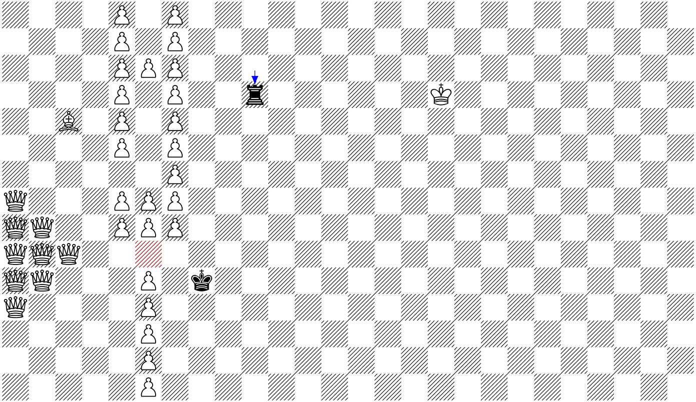
Releasing the Hordes, with value ω^2



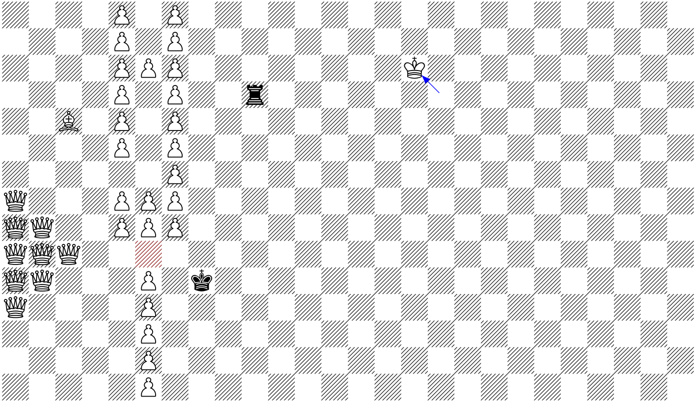
Releasing the Hordes, with value ω^2



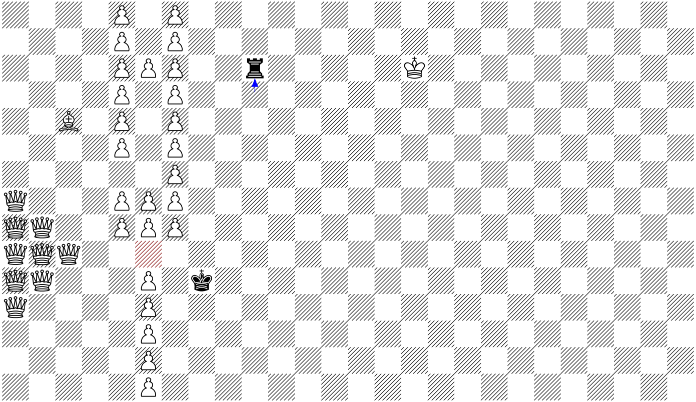
Releasing the Hordes, with value ω^2



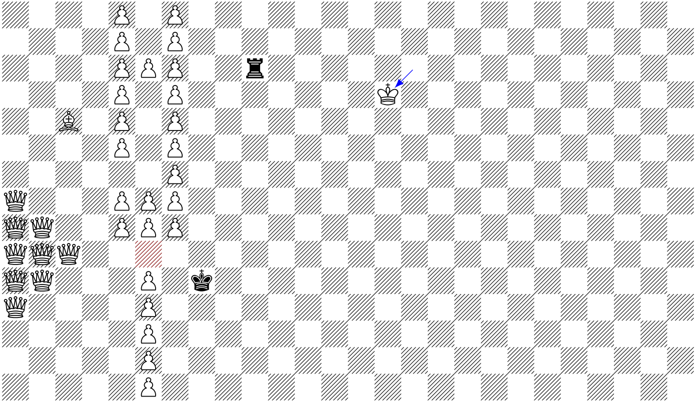
Releasing the Hordes, with value ω^2



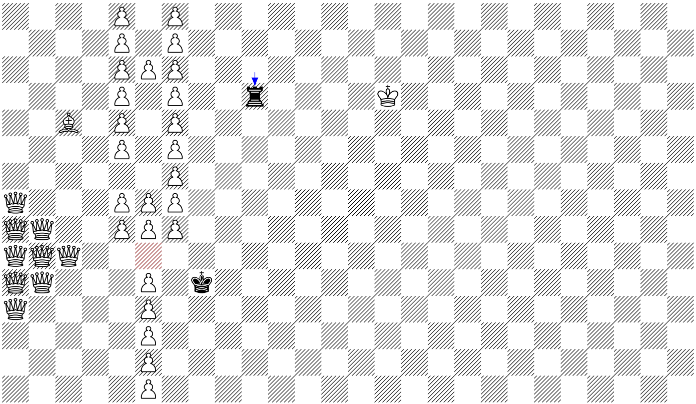
Releasing the Hordes, with value ω^2



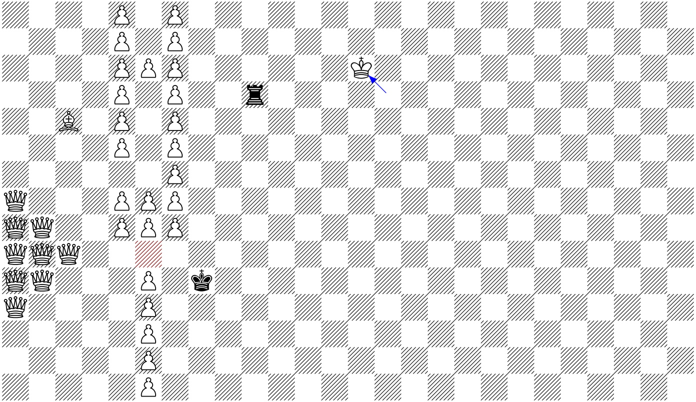
Releasing the Hordes, with value ω^2



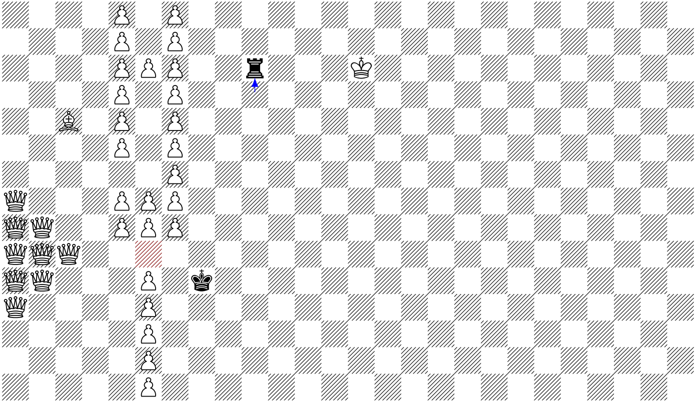
Releasing the Hordes, with value ω^2



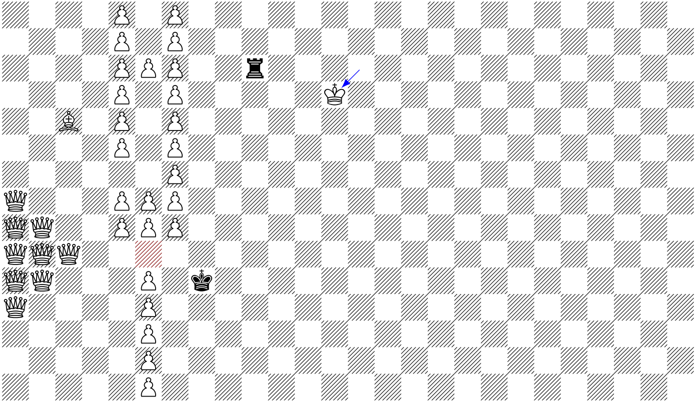
Releasing the Hordes, with value ω^2



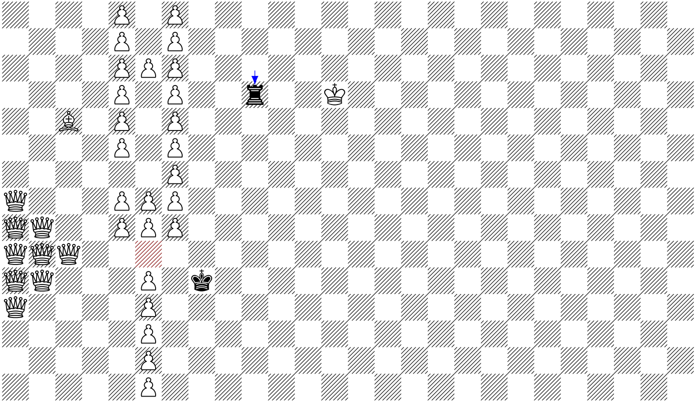
Releasing the Hordes, with value ω^2



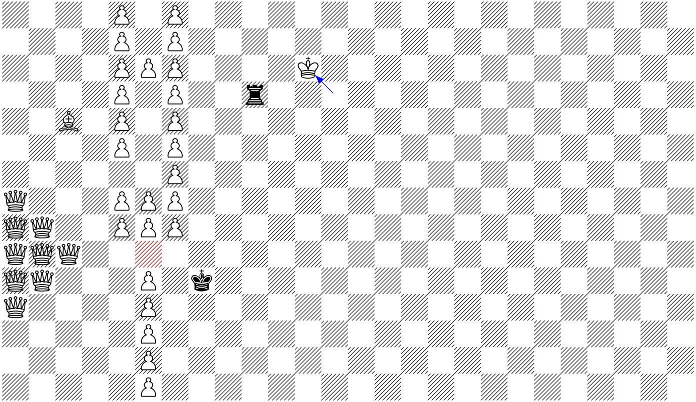
Releasing the Hordes, with value ω^2



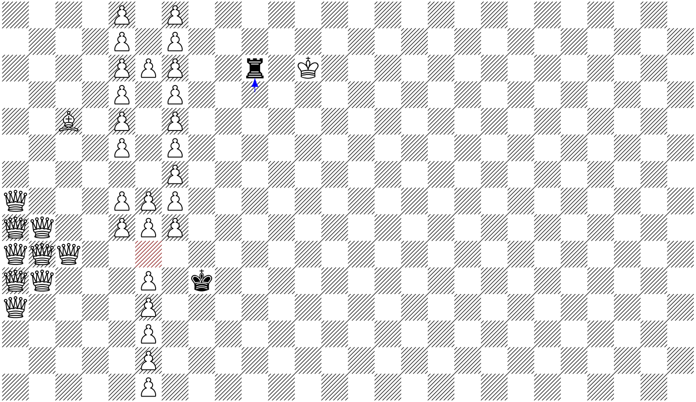
Releasing the Hordes, with value ω^2



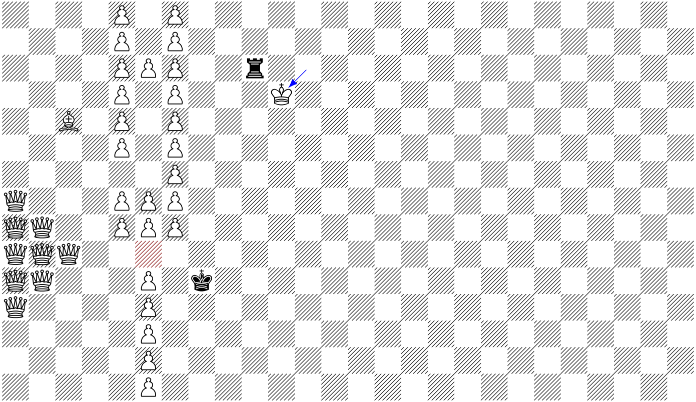
Releasing the Hordes, with value ω^2



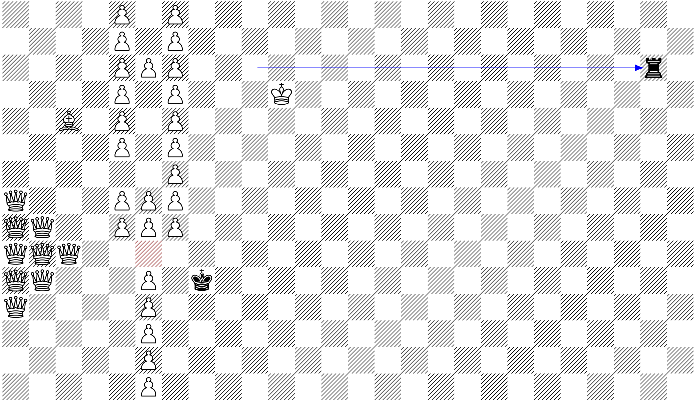
Releasing the Hordes, with value ω^2



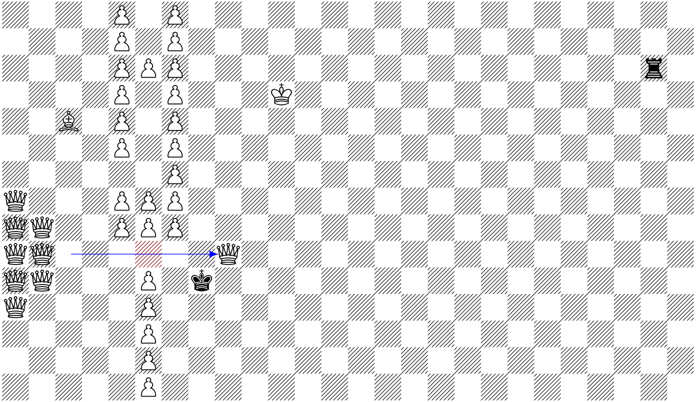
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

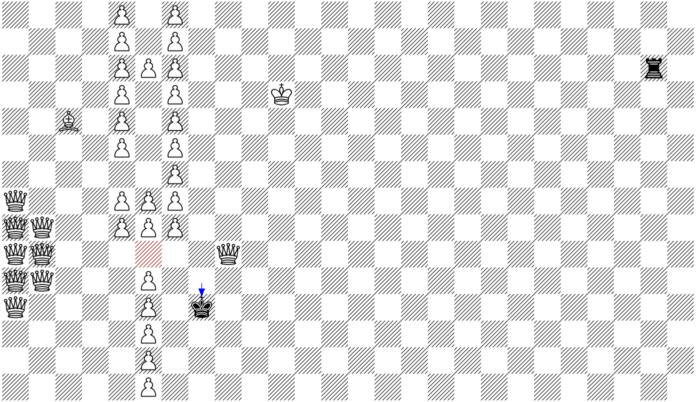


Releasing the Hordes, with value ω^2

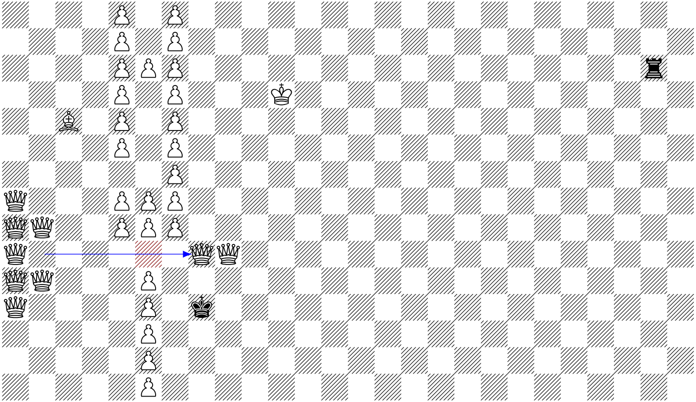


Queens enter the mating chamber.

Releasing the Hordes, with value ω^2

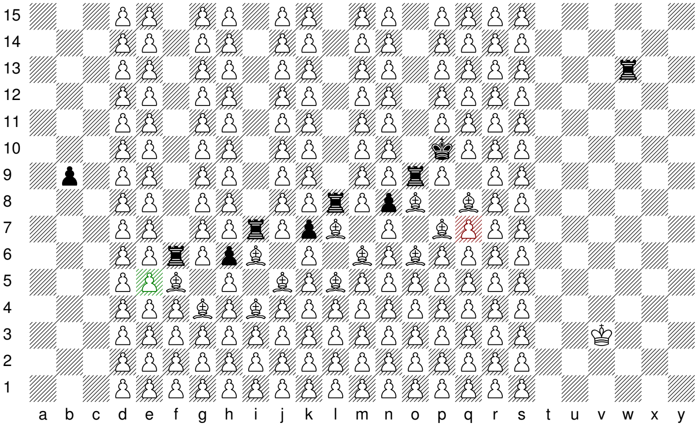


Releasing the Hordes, with value ω^2



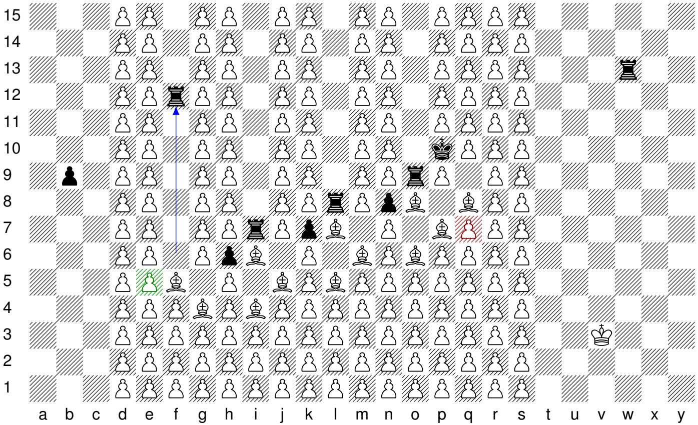
Checkmate

Iterated lock and key, value $\omega^2 \cdot 4$



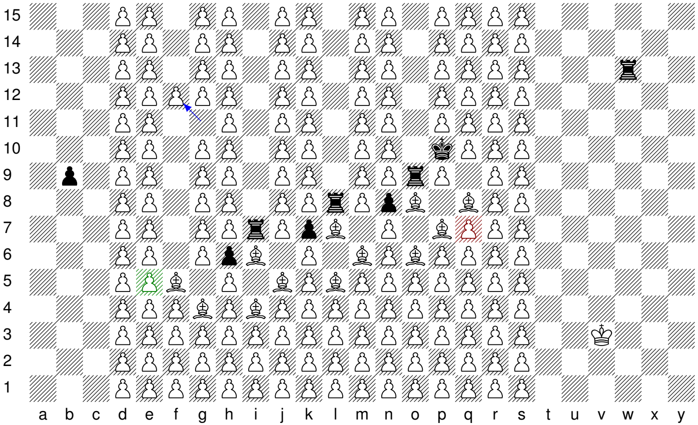
Black to move.

Iterated lock and key, value $\omega^2 \cdot 4$

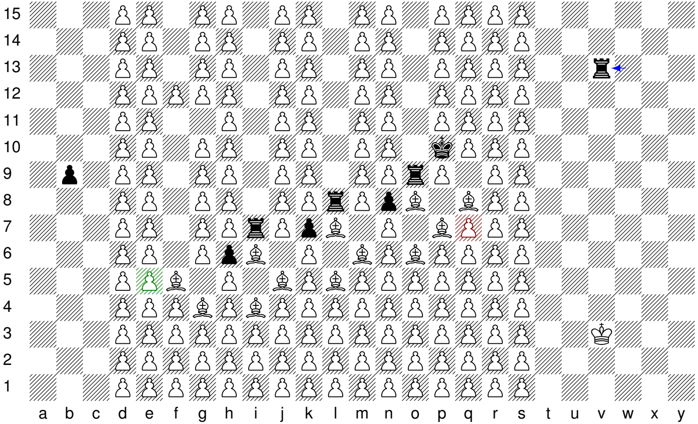


The first black tower ascends.

Iterated lock and key, value $\omega^2 \cdot 4$

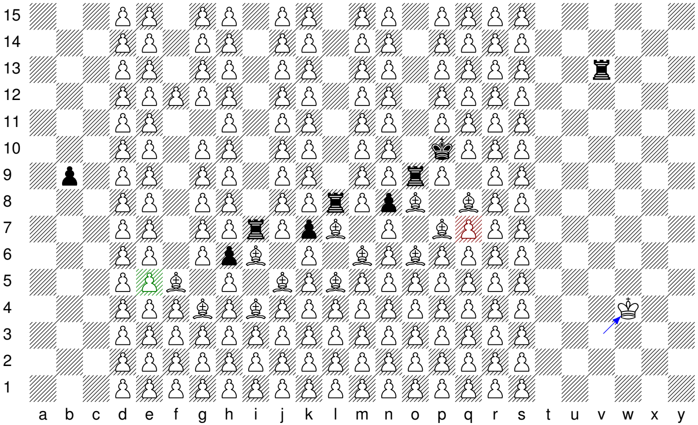


Iterated lock and key, value $\omega^2 \cdot 4$

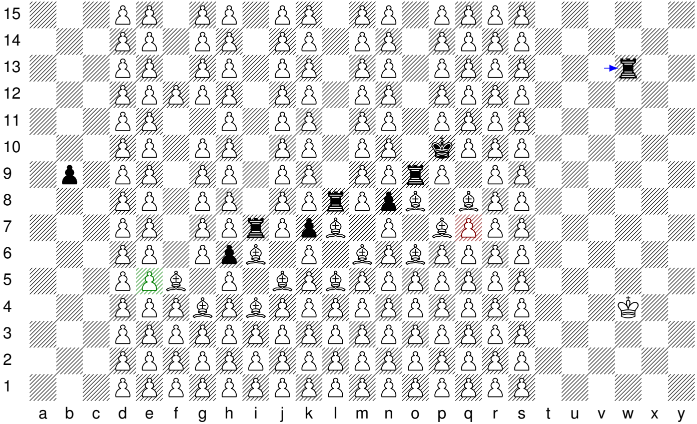


Black harasses white king.

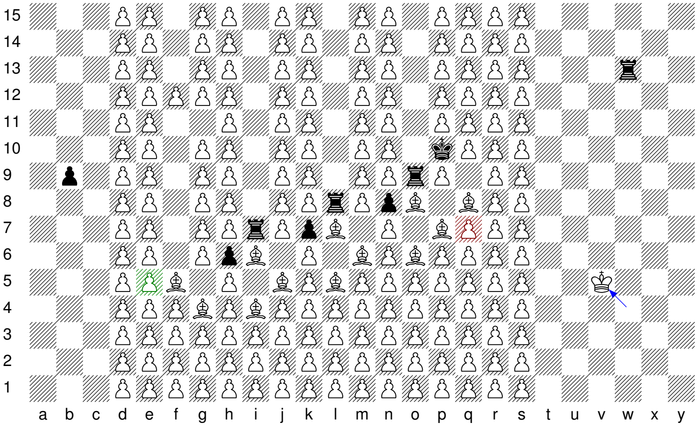
Iterated lock and key, value $\omega^2 \cdot 4$



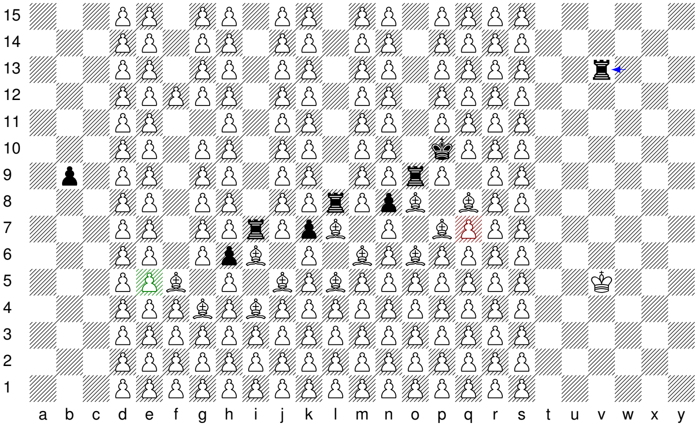
Iterated lock and key, value $\omega^2 \cdot 4$



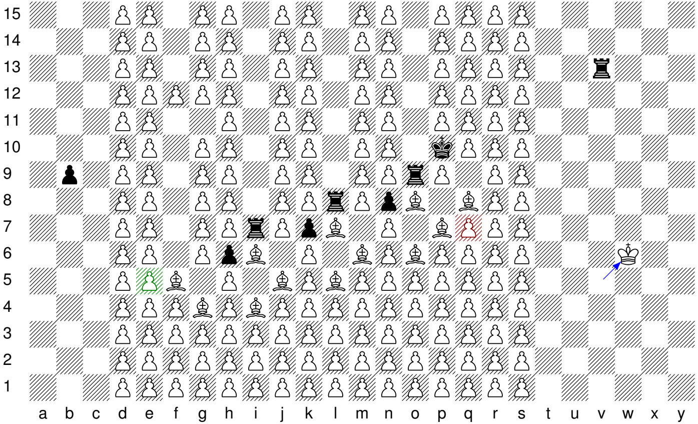
Iterated lock and key, value $\omega^2 \cdot 4$



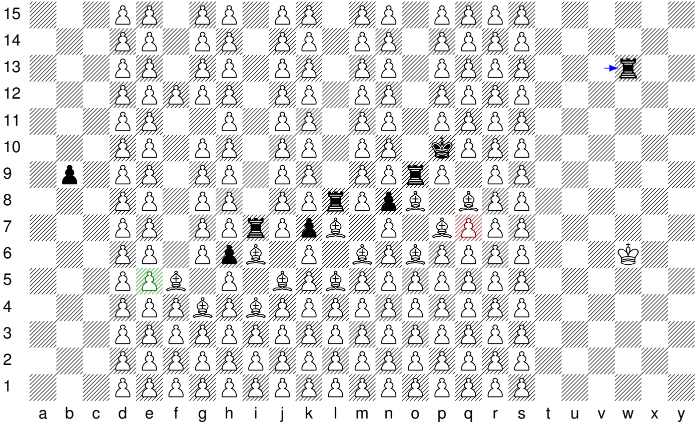
Iterated lock and key, value $\omega^2 \cdot 4$



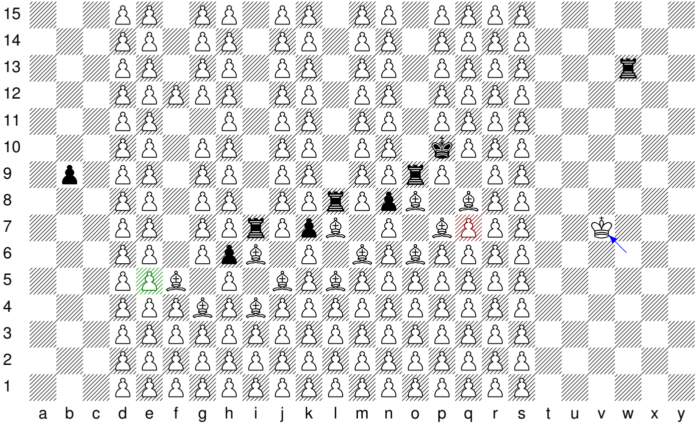
Iterated lock and key, value $\omega^2 \cdot 4$



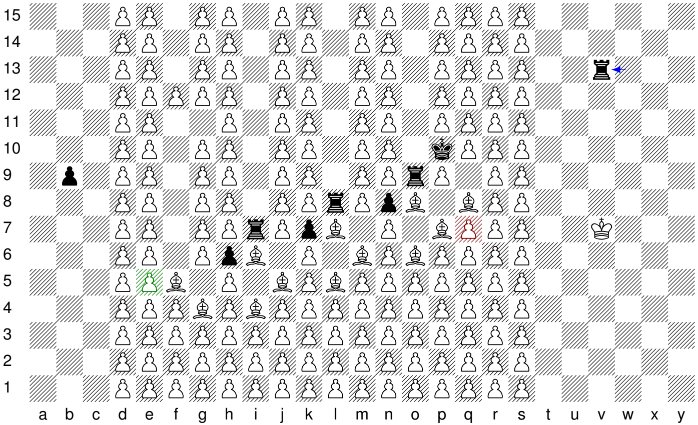
Iterated lock and key, value $\omega^2 \cdot 4$



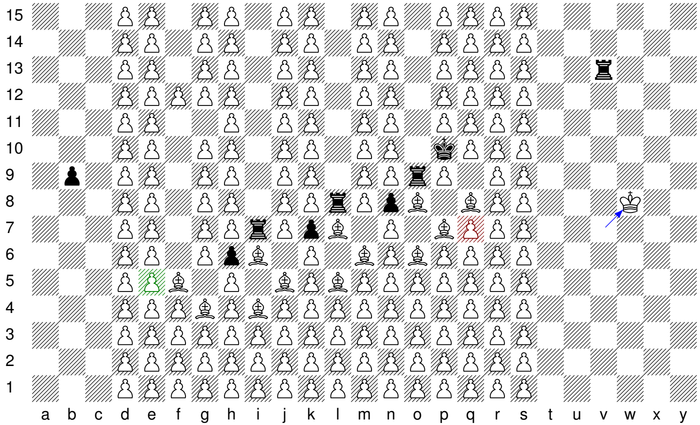
Iterated lock and key, value $\omega^2 \cdot 4$



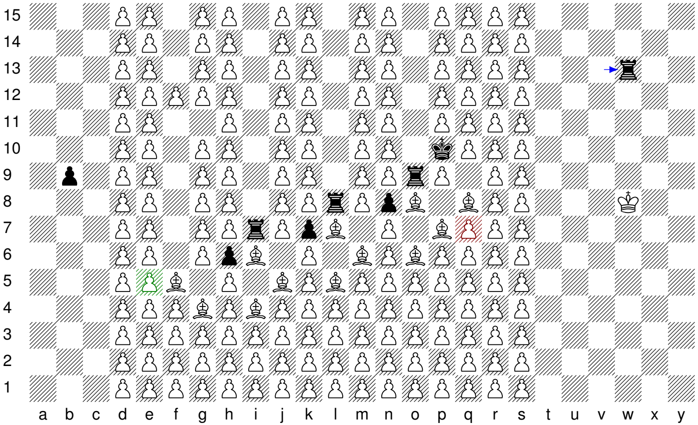
Iterated lock and key, value $\omega^2 \cdot 4$



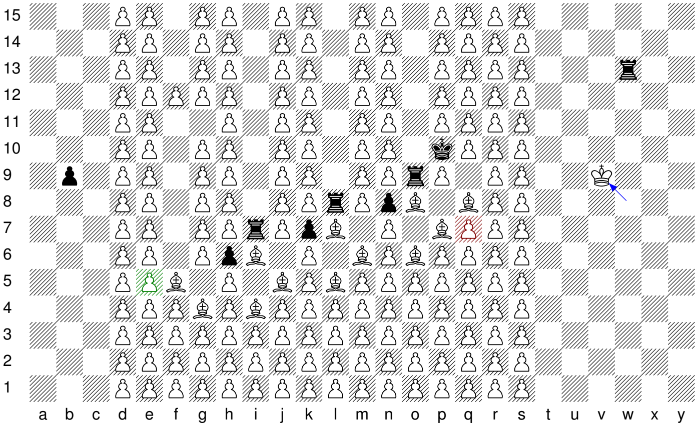
Iterated lock and key, value $\omega^2 \cdot 4$



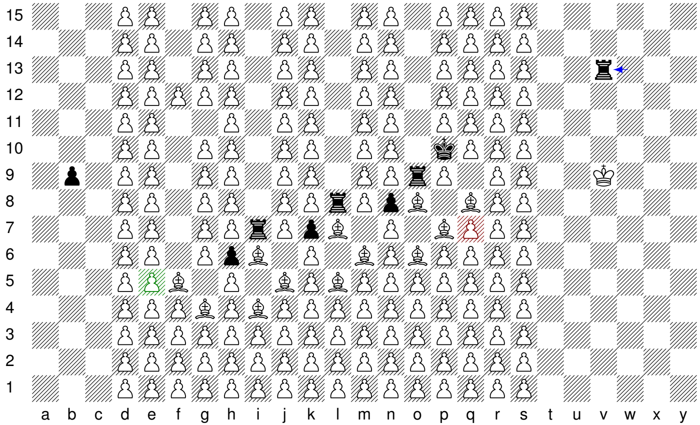
Iterated lock and key, value $\omega^2 \cdot 4$



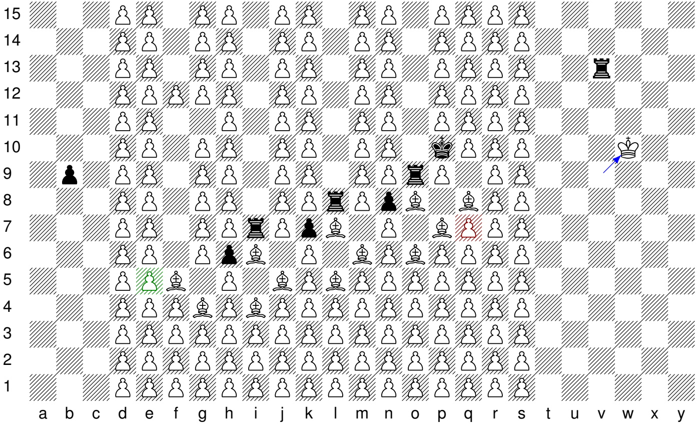
Iterated lock and key, value $\omega^2 \cdot 4$



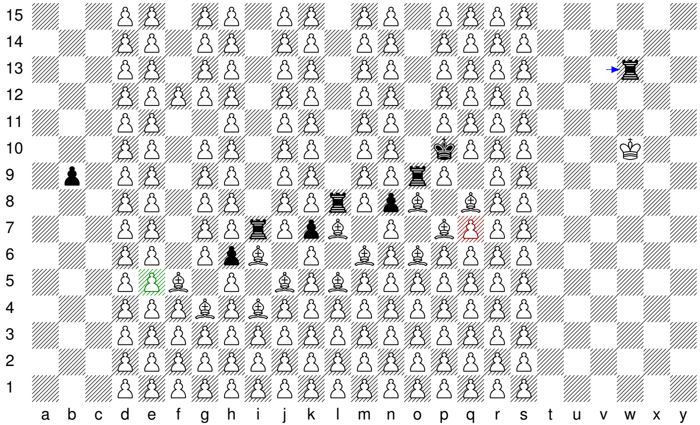
Iterated lock and key, value $\omega^2 \cdot 4$



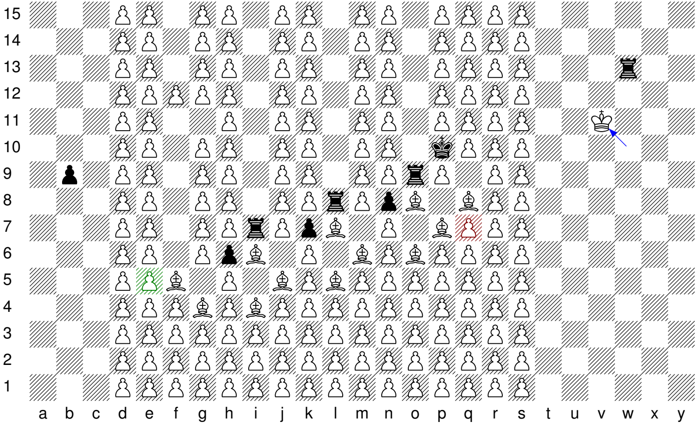
Iterated lock and key, value $\omega^2 \cdot 4$



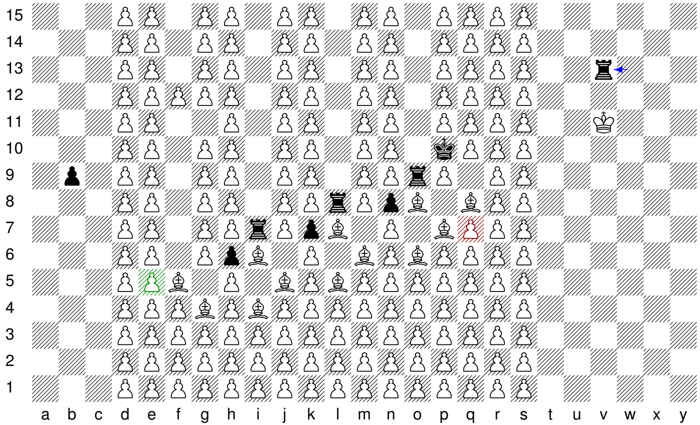
Iterated lock and key, value $\omega^2 \cdot 4$



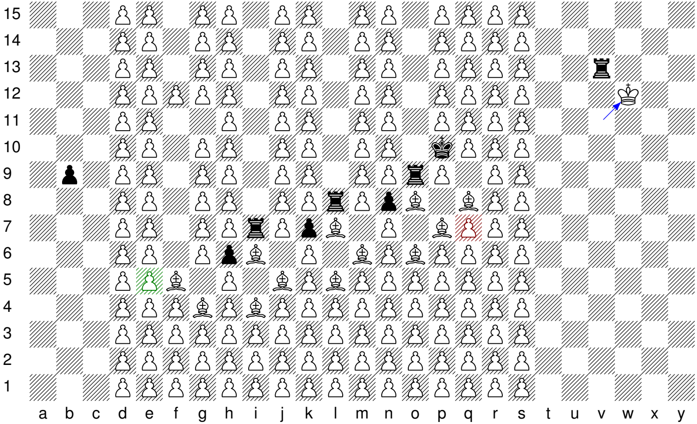
Iterated lock and key, value $\omega^2 \cdot 4$



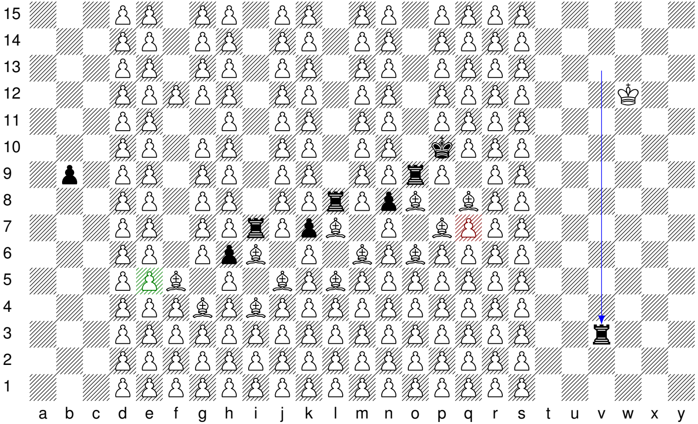
Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$

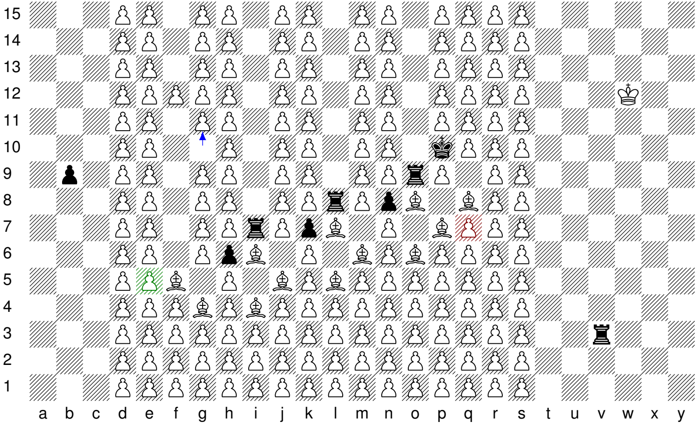


Iterated lock and key, value $\omega^2 \cdot 4$



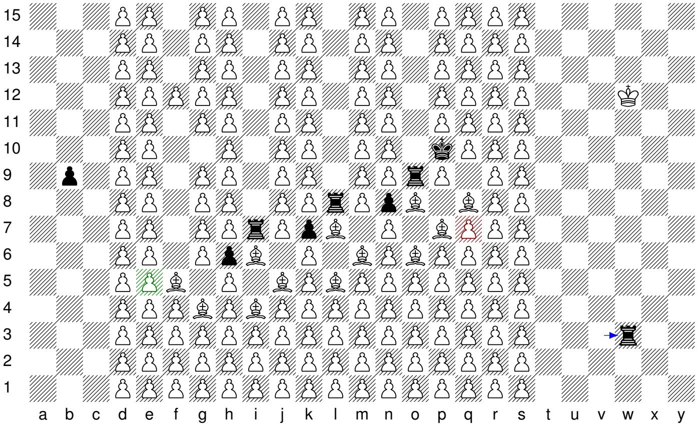
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$

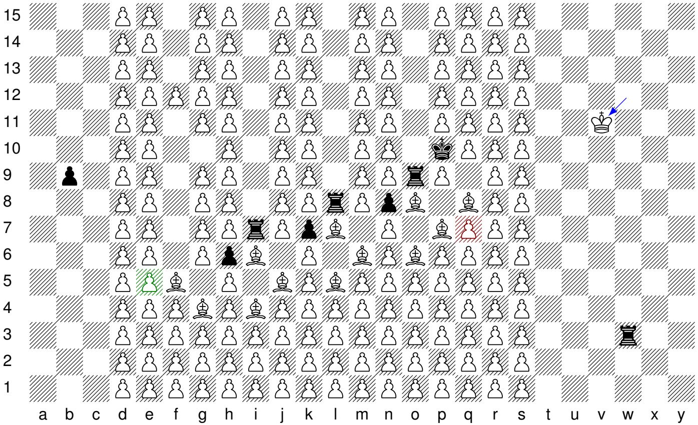


White advances a pawn.

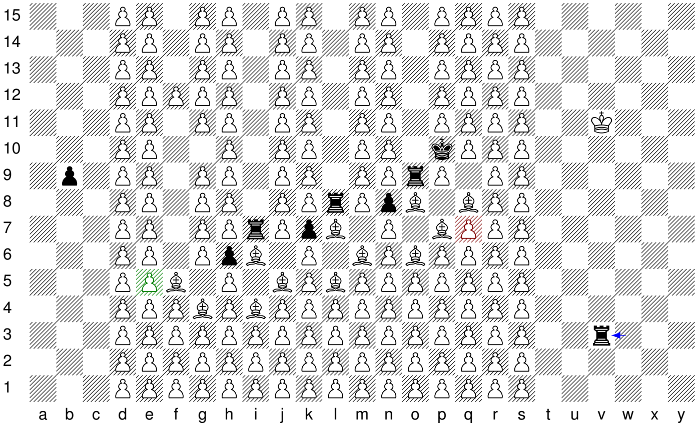
Iterated lock and key, value $\omega^2 \cdot 4$



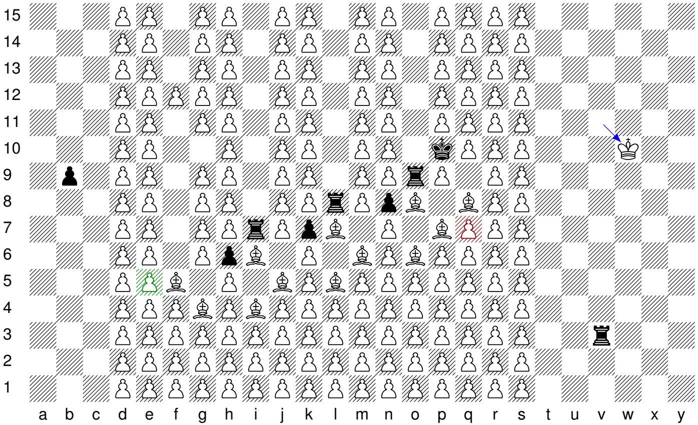
Iterated lock and key, value $\omega^2 \cdot 4$



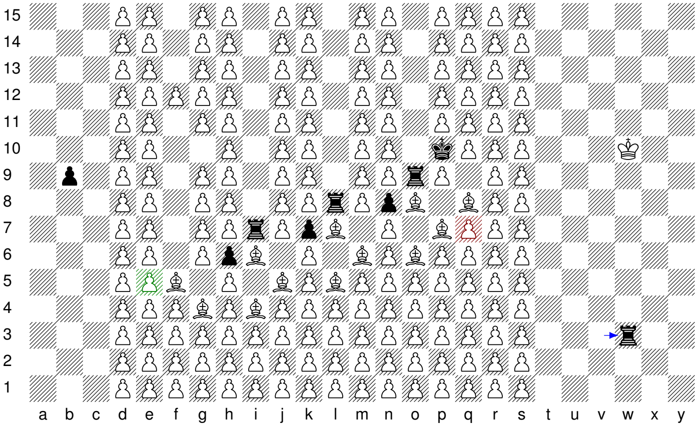
Iterated lock and key, value $\omega^2 \cdot 4$



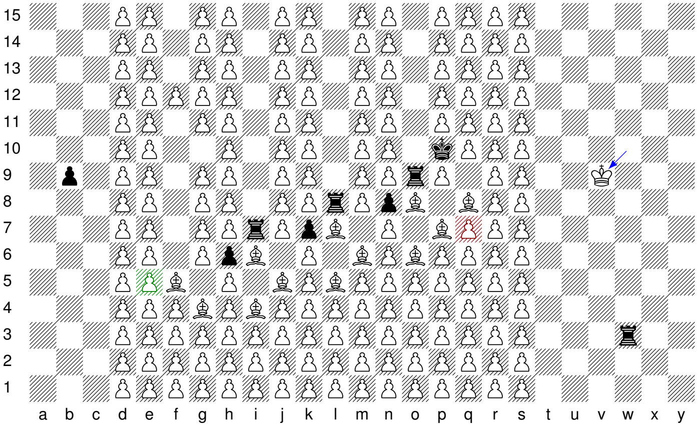
Iterated lock and key, value $\omega^2 \cdot 4$



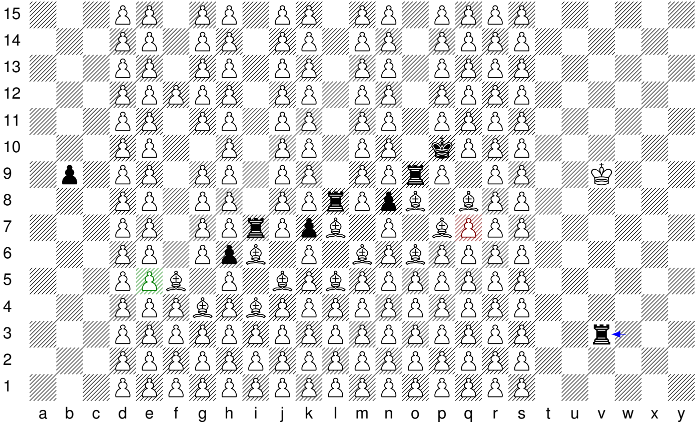
Iterated lock and key, value $\omega^2 \cdot 4$



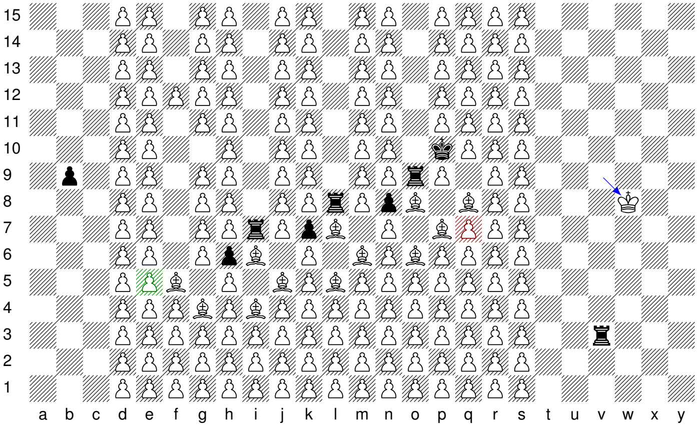
Iterated lock and key, value $\omega^2 \cdot 4$



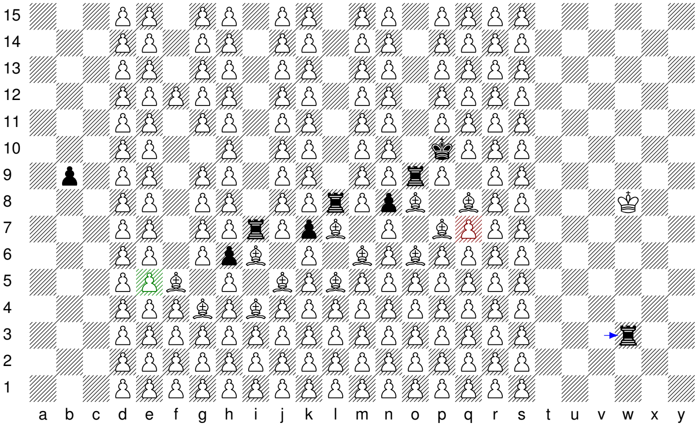
Iterated lock and key, value $\omega^2 \cdot 4$



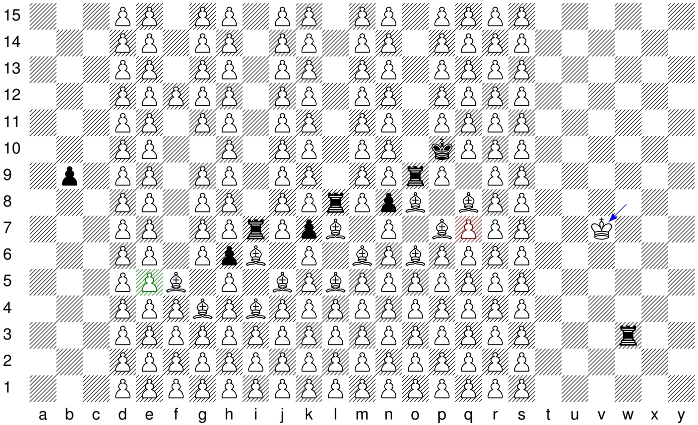
Iterated lock and key, value $\omega^2 \cdot 4$



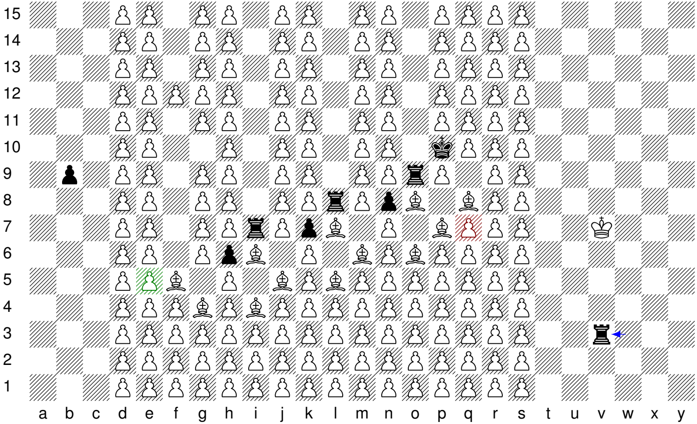
Iterated lock and key, value $\omega^2 \cdot 4$



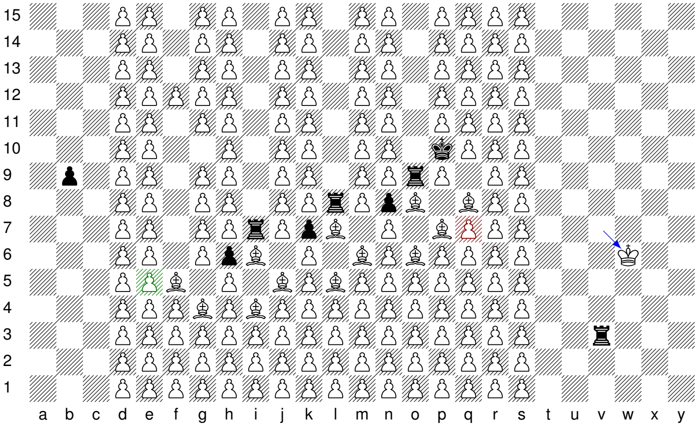
Iterated lock and key, value $\omega^2 \cdot 4$



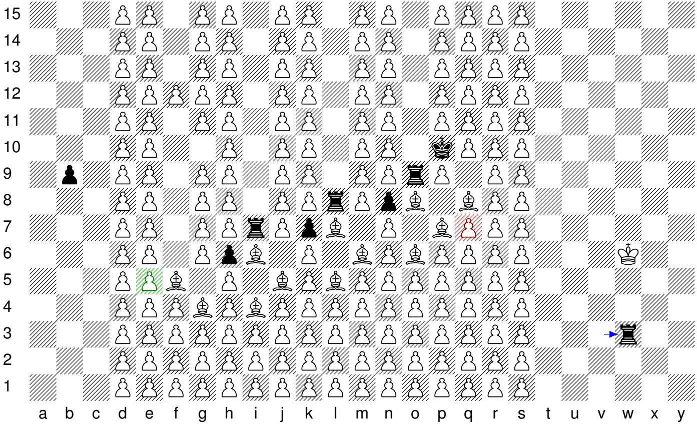
Iterated lock and key, value $\omega^2 \cdot 4$



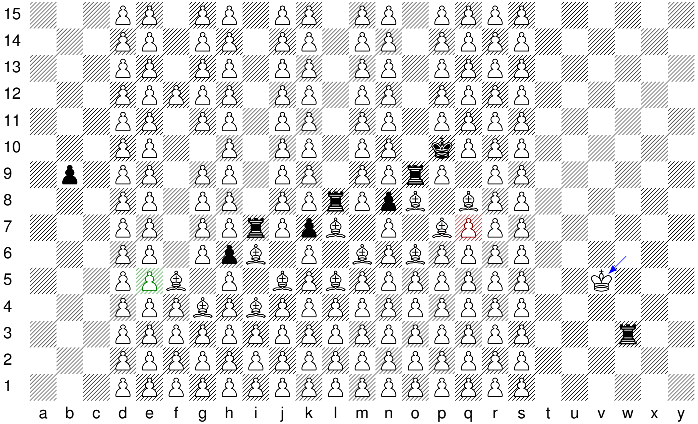
Iterated lock and key, value $\omega^2 \cdot 4$



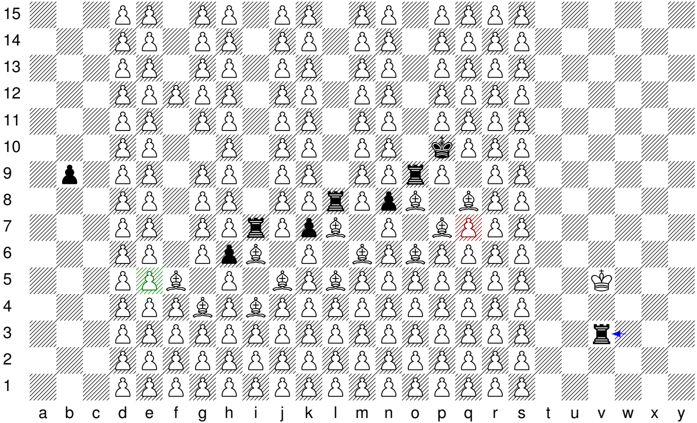
Iterated lock and key, value $\omega^2 \cdot 4$



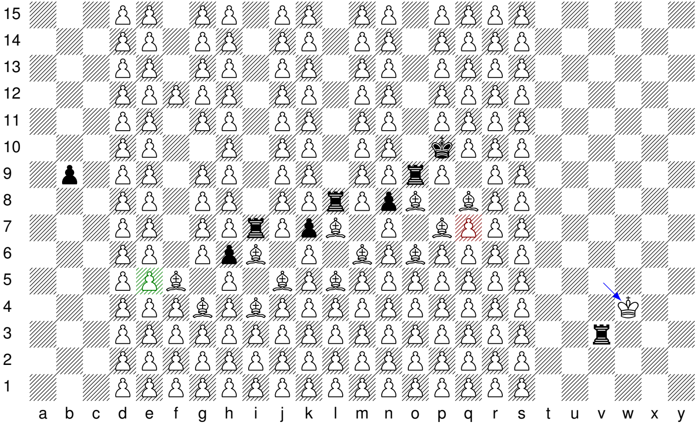
Iterated lock and key, value $\omega^2 \cdot 4$



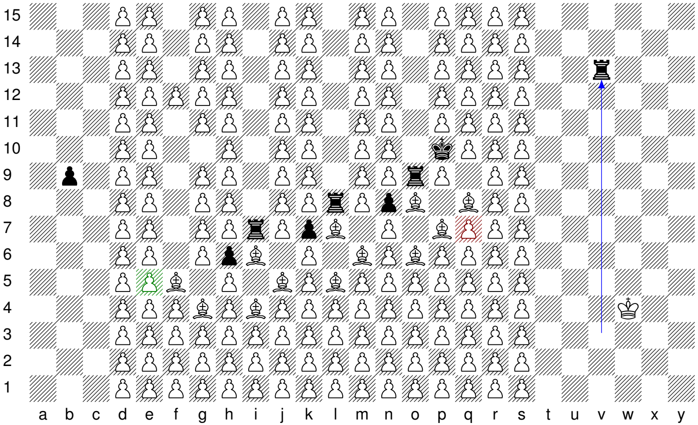
Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$

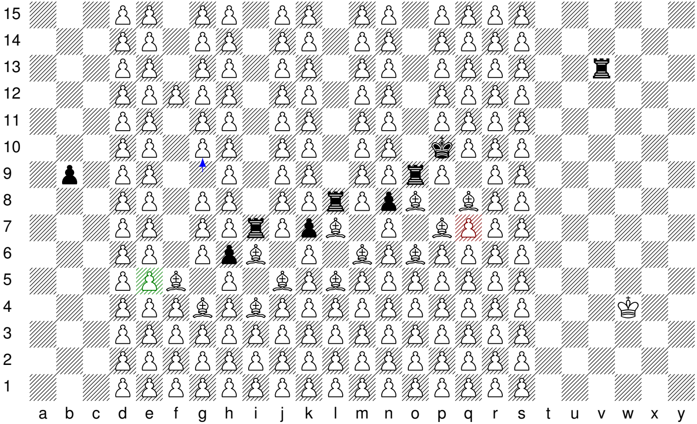


Iterated lock and key, value $\omega^2 \cdot 4$



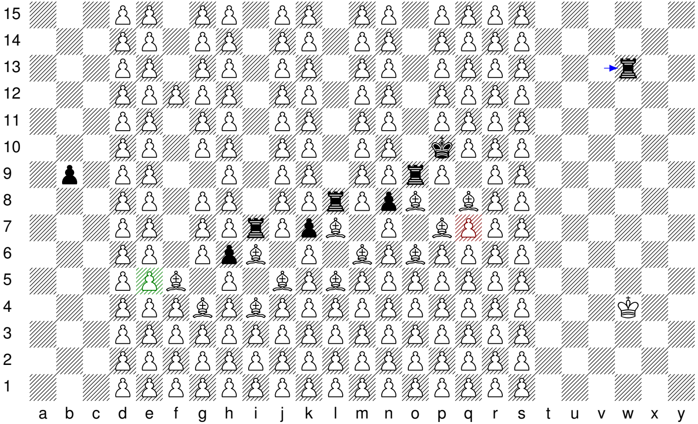
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$

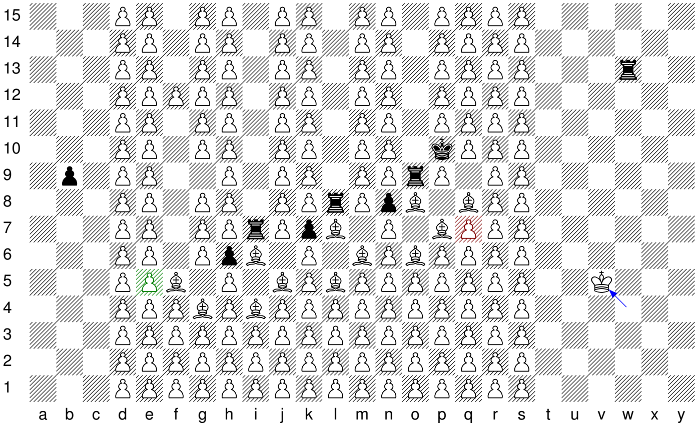


White advances pawn.

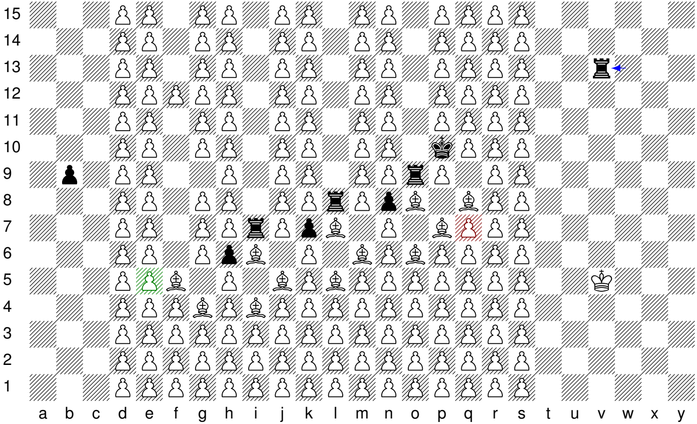
Iterated lock and key, value $\omega^2 \cdot 4$



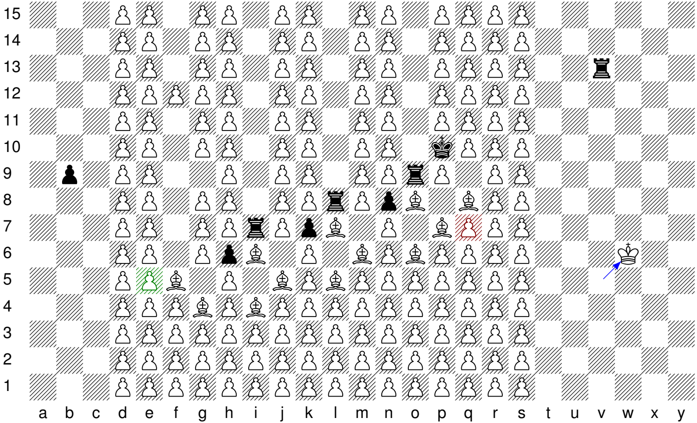
Iterated lock and key, value $\omega^2 \cdot 4$



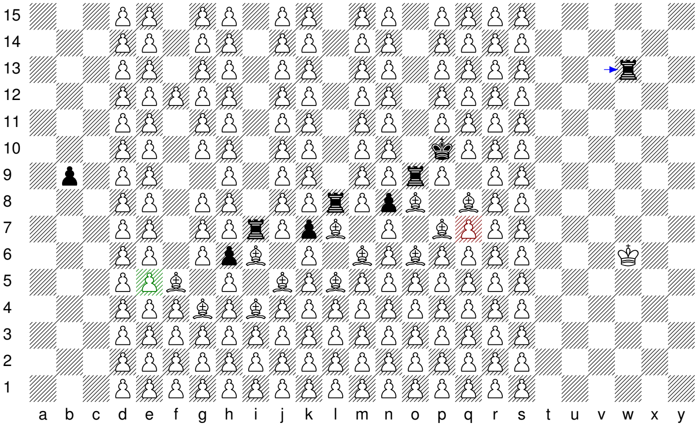
Iterated lock and key, value $\omega^2 \cdot 4$



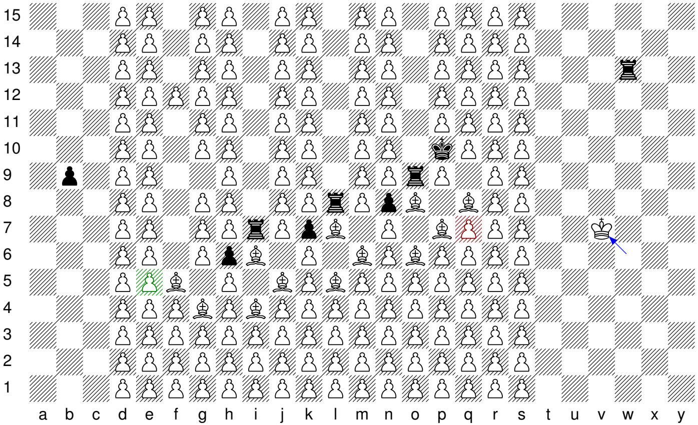
Iterated lock and key, value $\omega^2 \cdot 4$



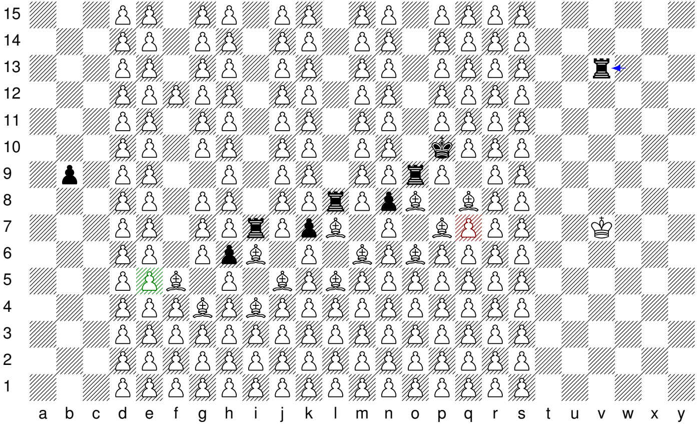
Iterated lock and key, value $\omega^2 \cdot 4$



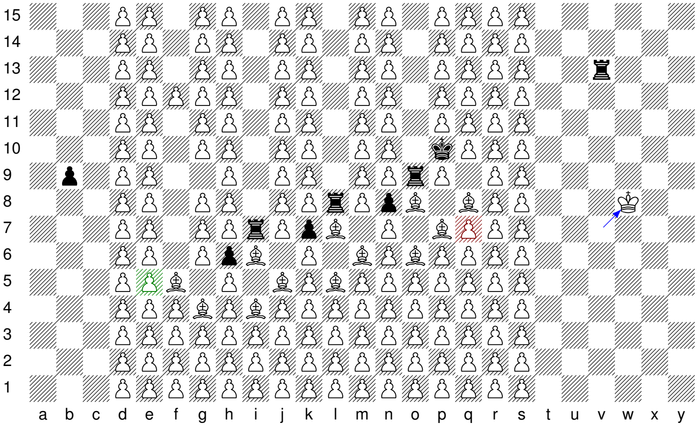
Iterated lock and key, value $\omega^2 \cdot 4$



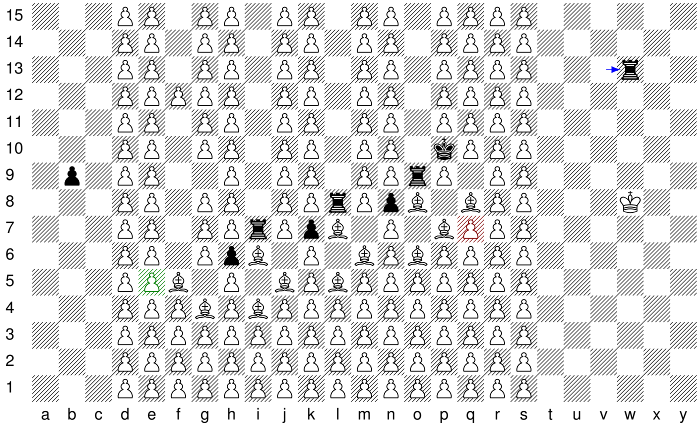
Iterated lock and key, value $\omega^2 \cdot 4$



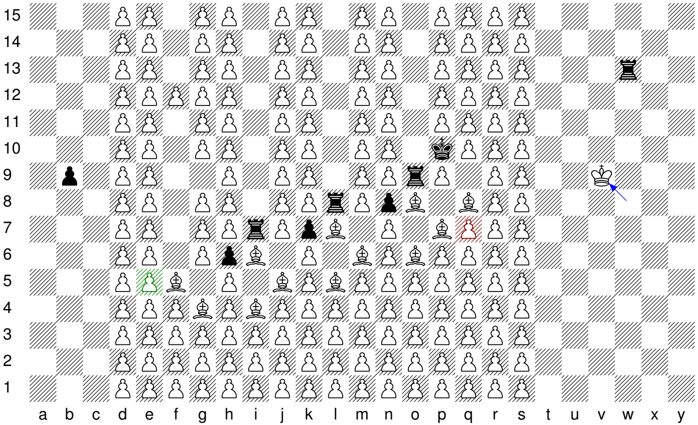
Iterated lock and key, value $\omega^2 \cdot 4$



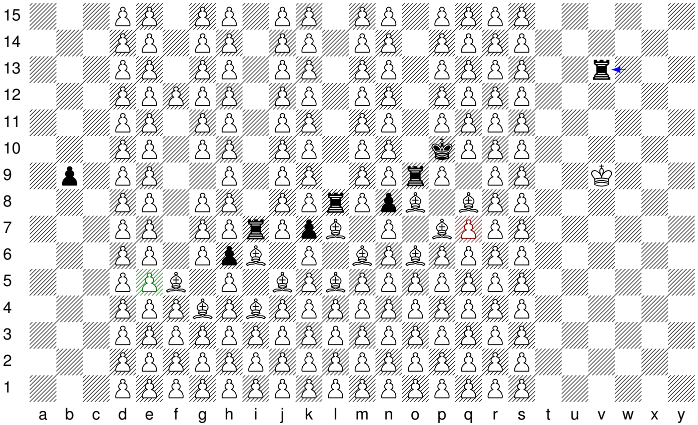
Iterated lock and key, value $\omega^2 \cdot 4$



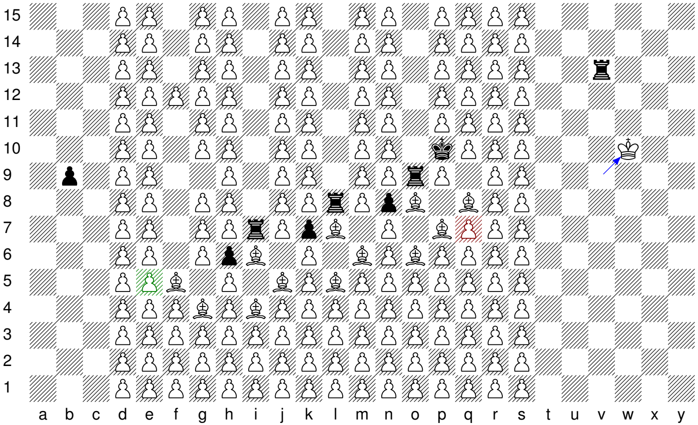
Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$



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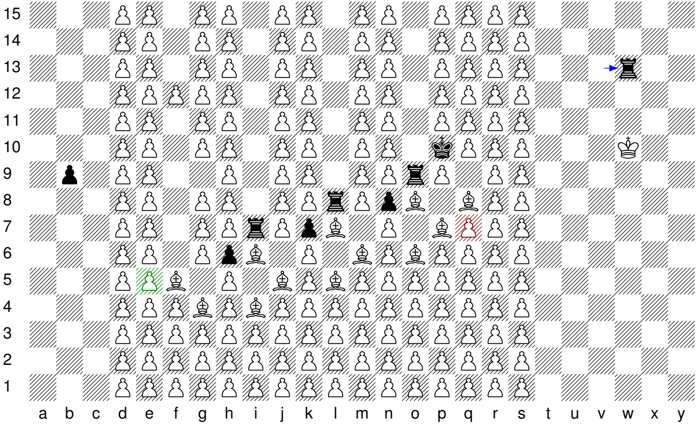

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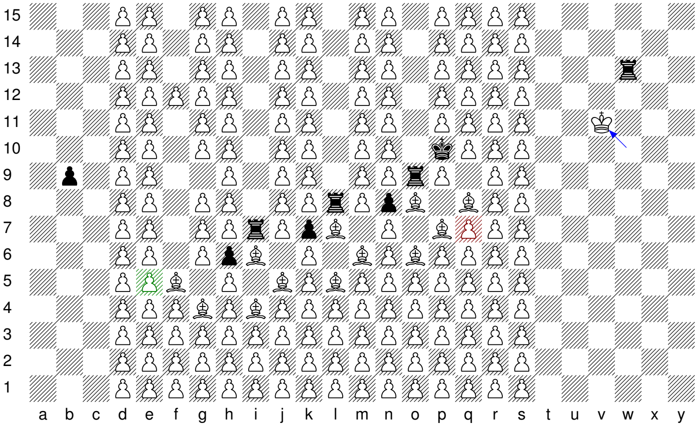
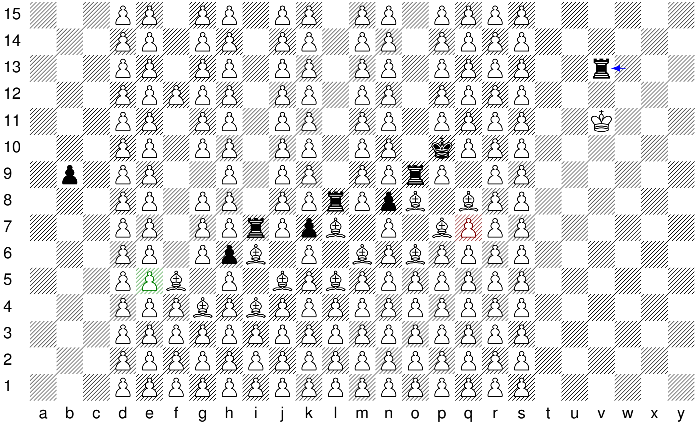


Iterated lock and key, value $\omega^2 \cdot 4$



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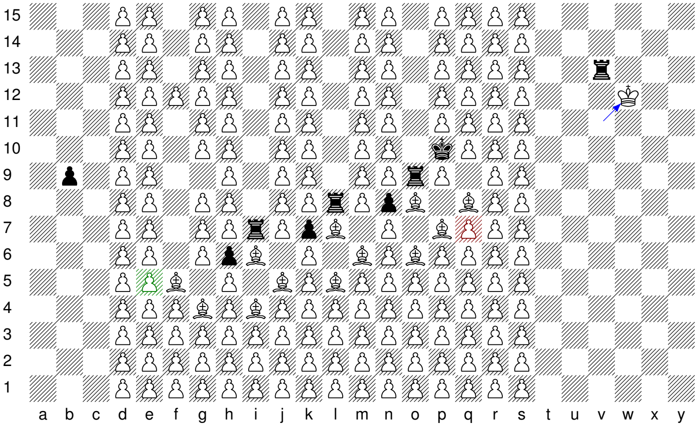
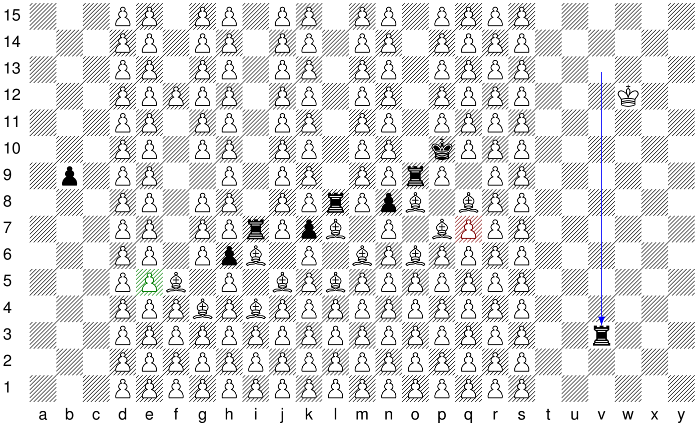
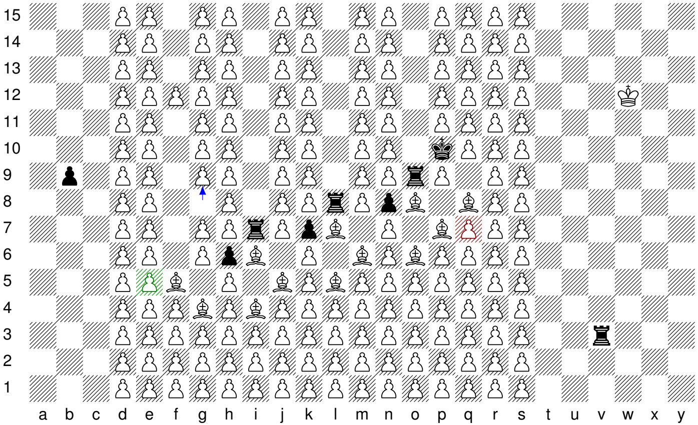


Iterated lock and key, value $\omega^2 \cdot 4$



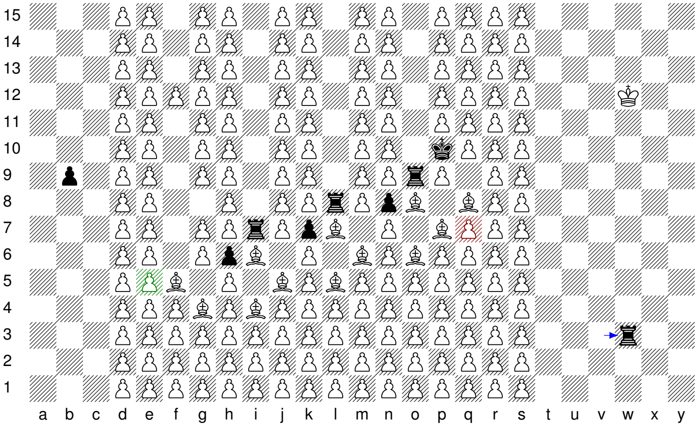
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$



White advances pawn.

Iterated lock and key, value $\omega^2 \cdot 4$



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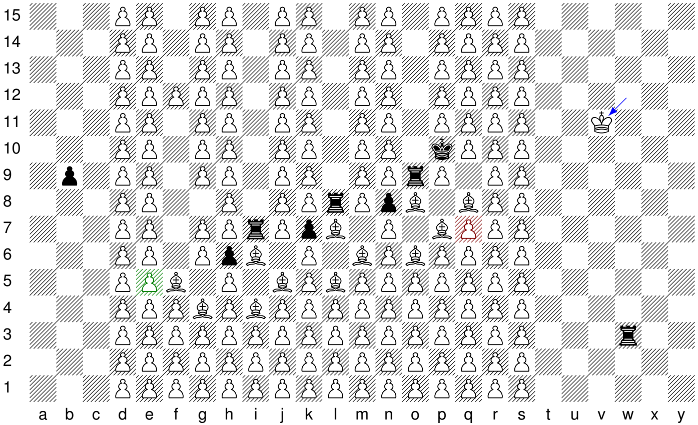

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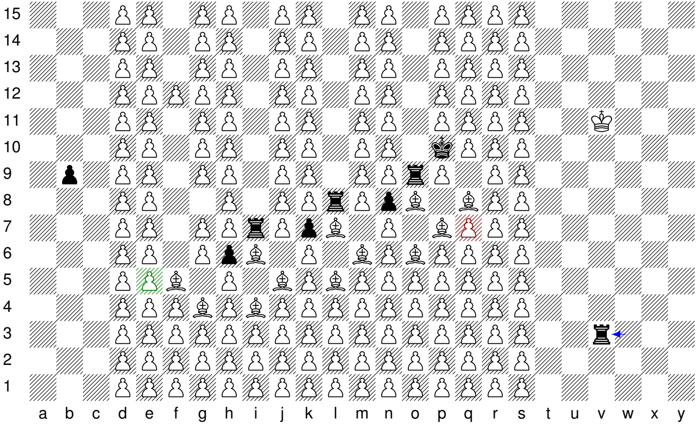

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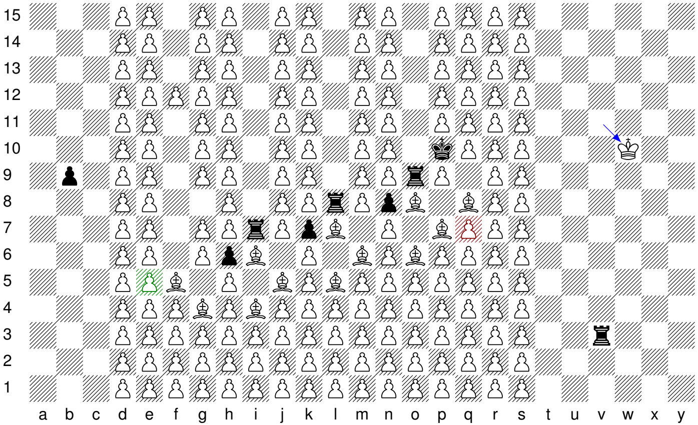

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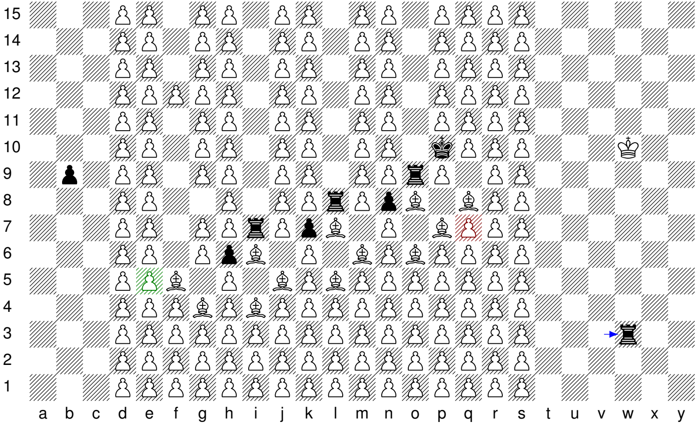

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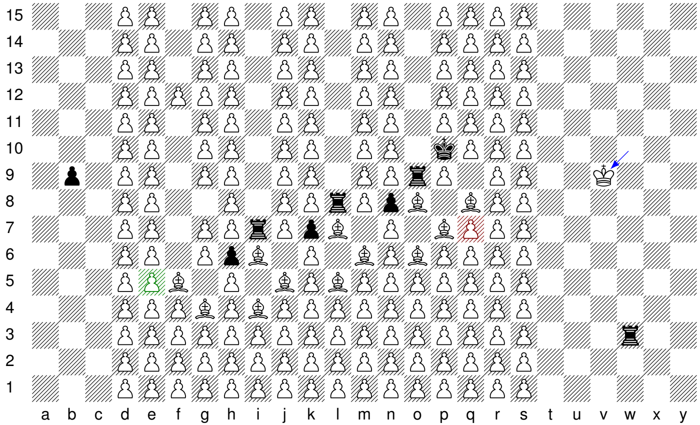

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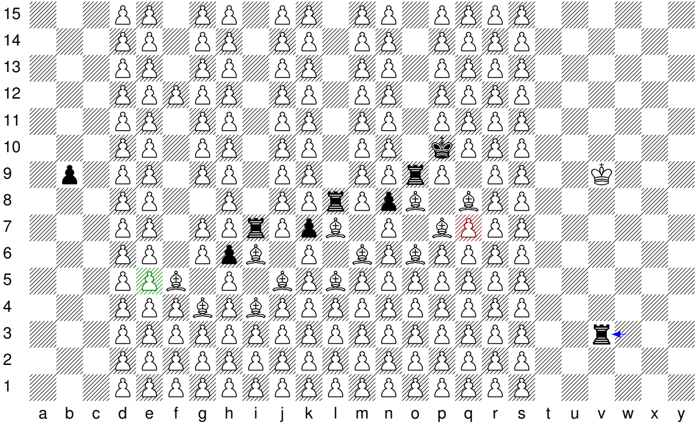

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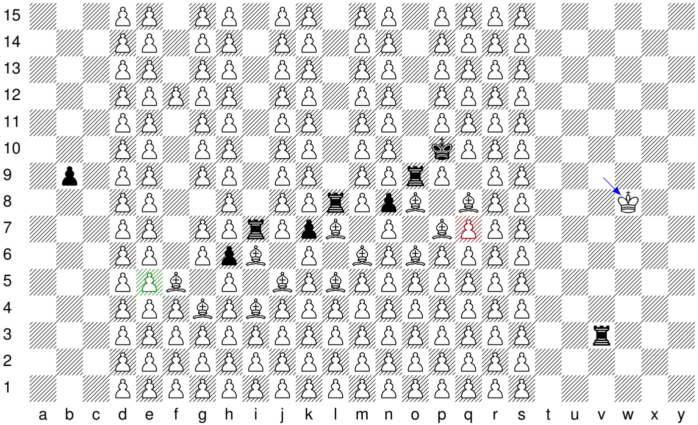

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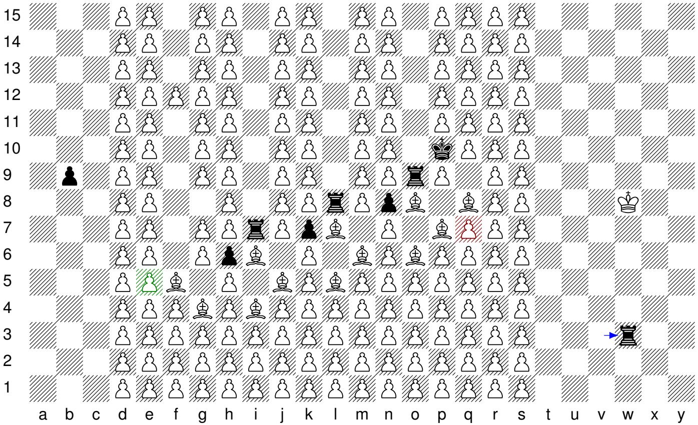

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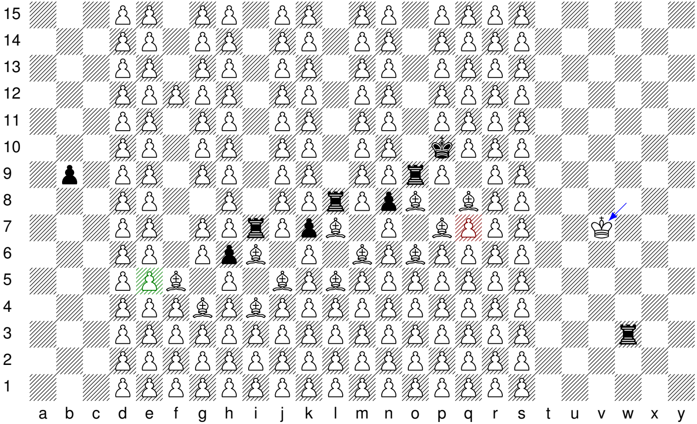

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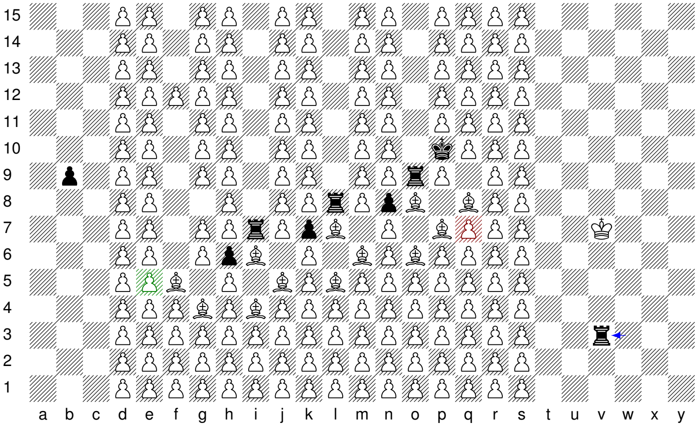

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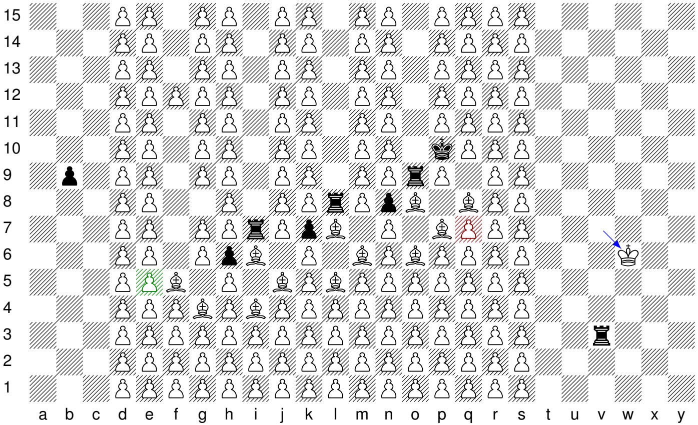

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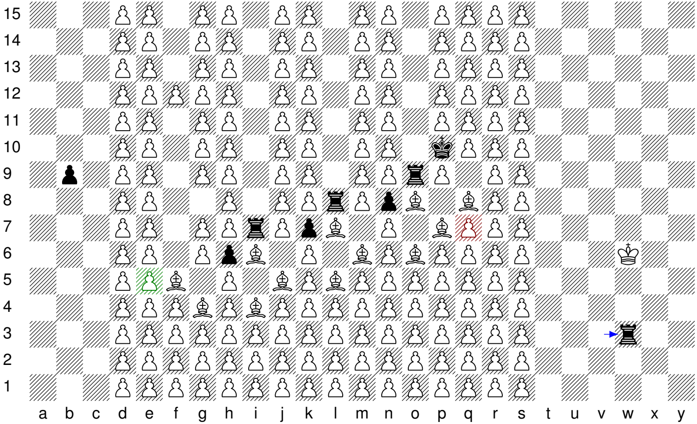

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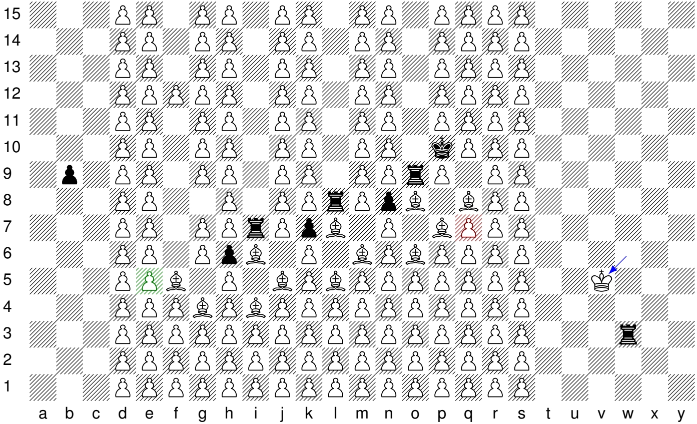

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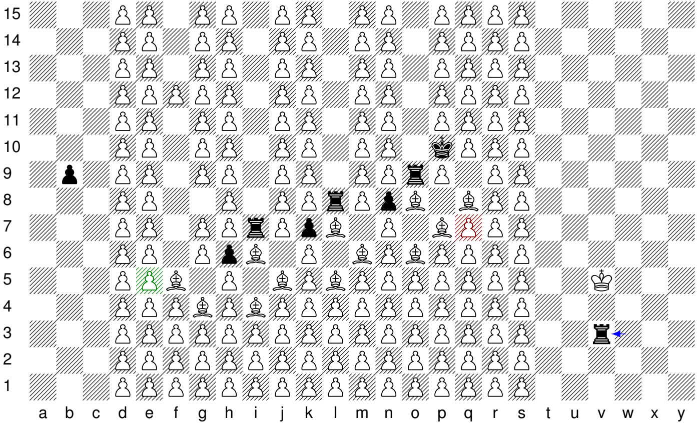

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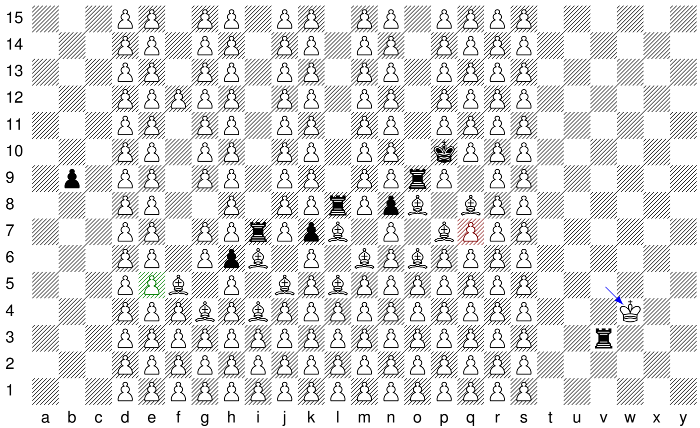

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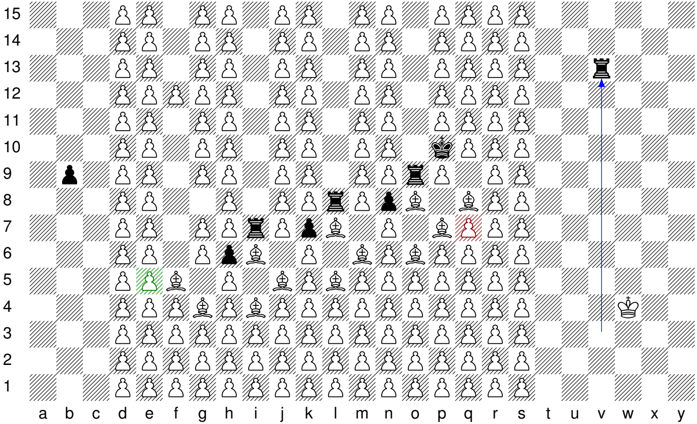
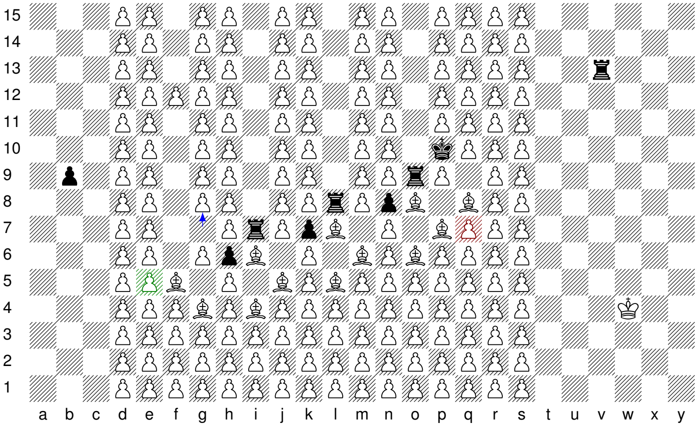


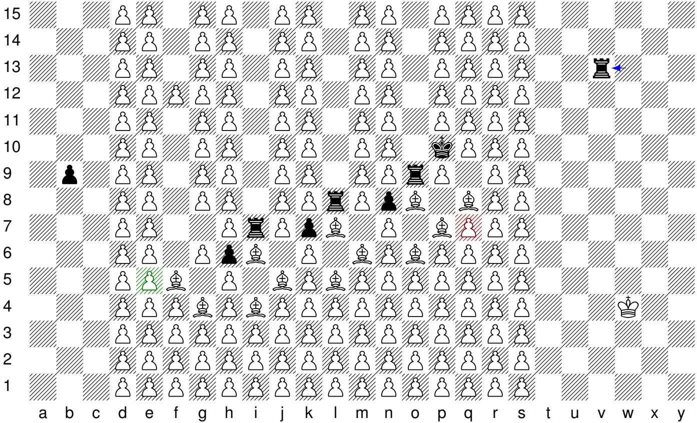
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$

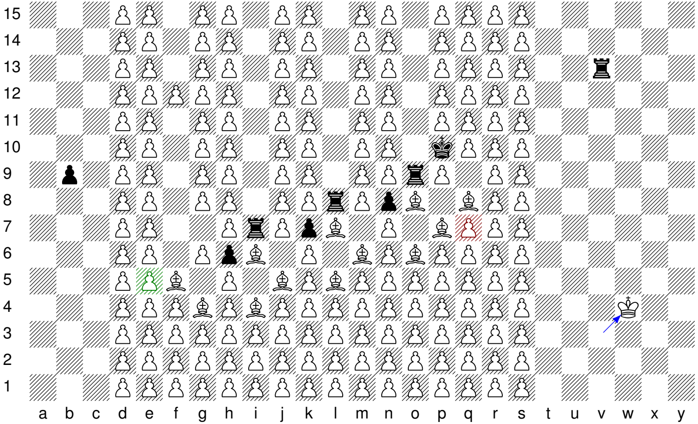


White advances pawn.

Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$



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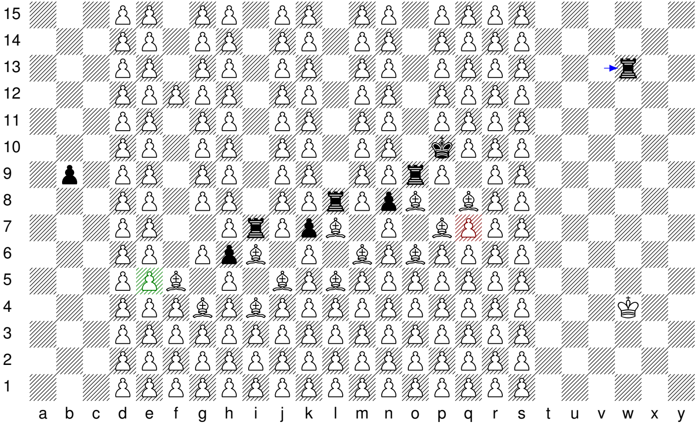

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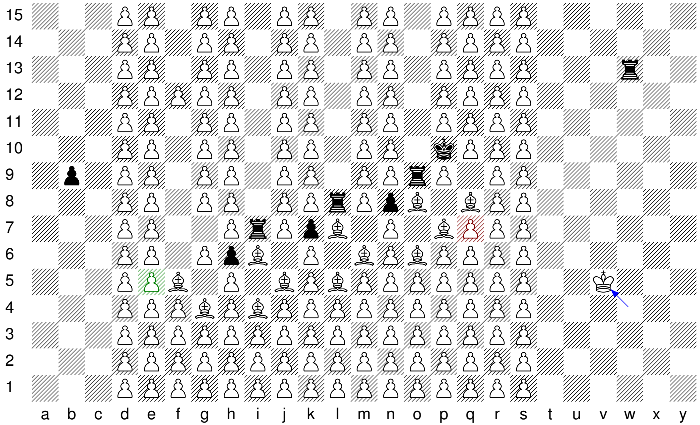

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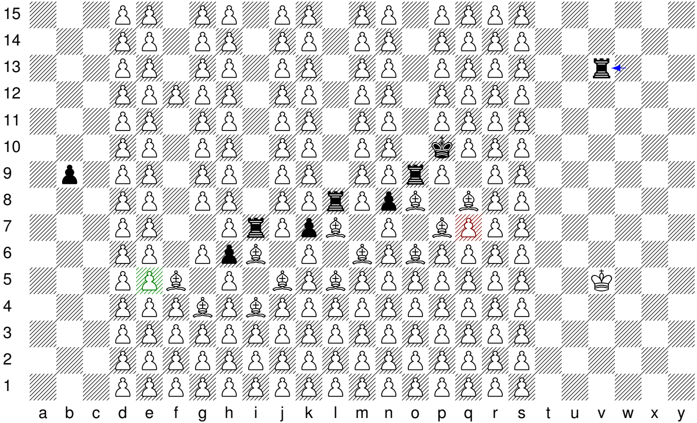

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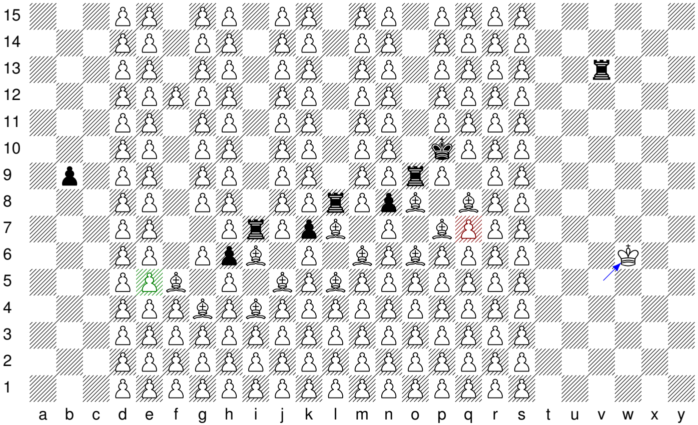

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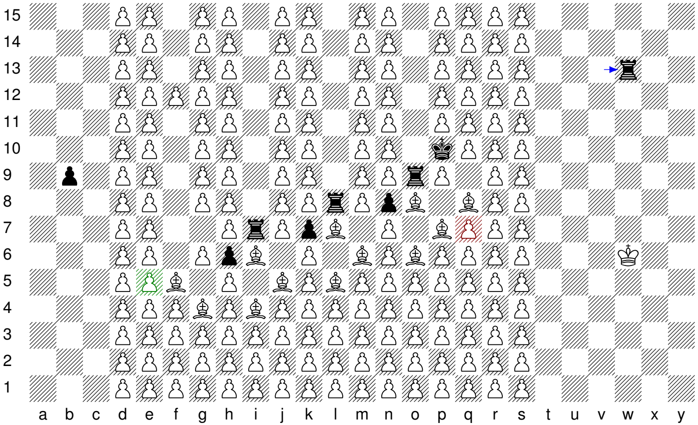

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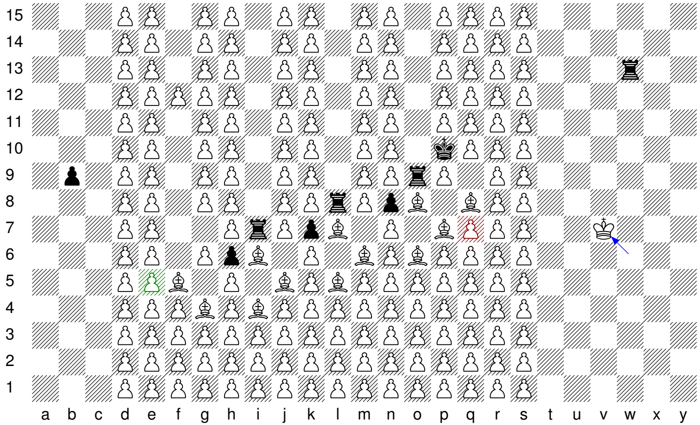

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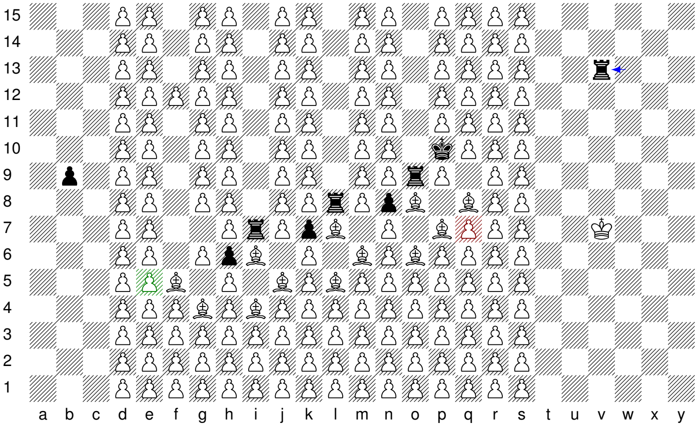

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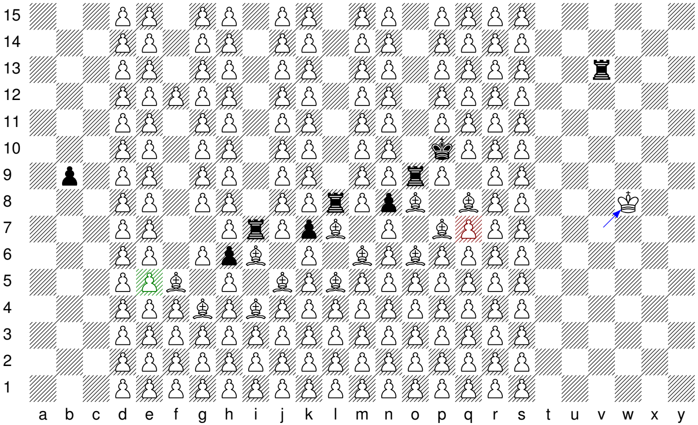

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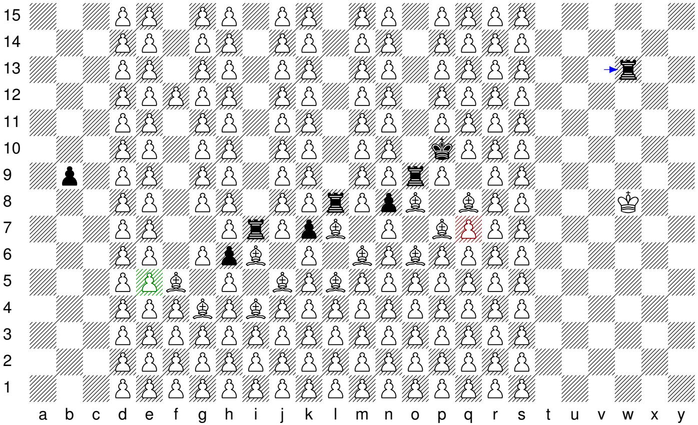

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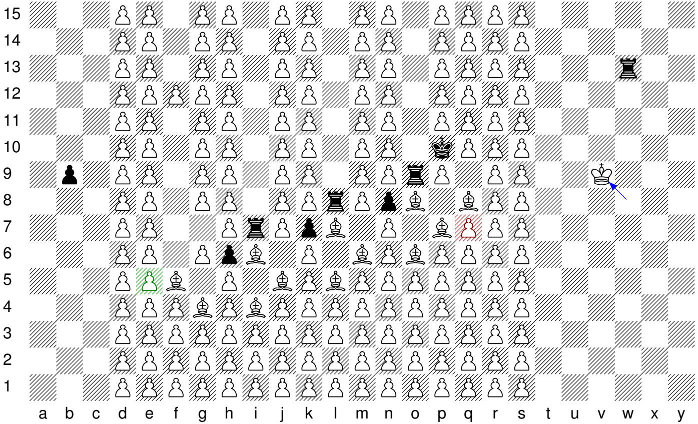

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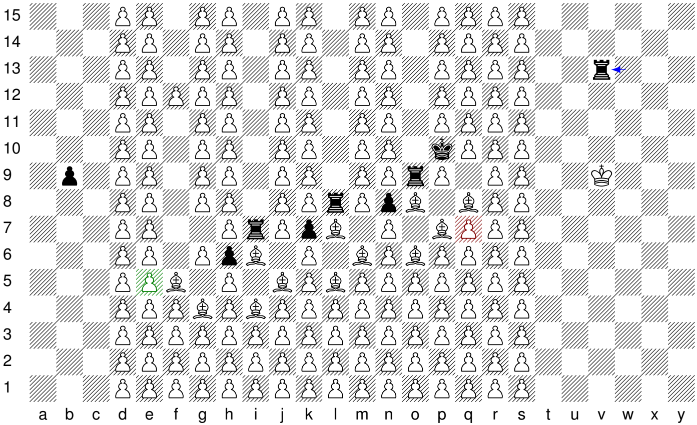

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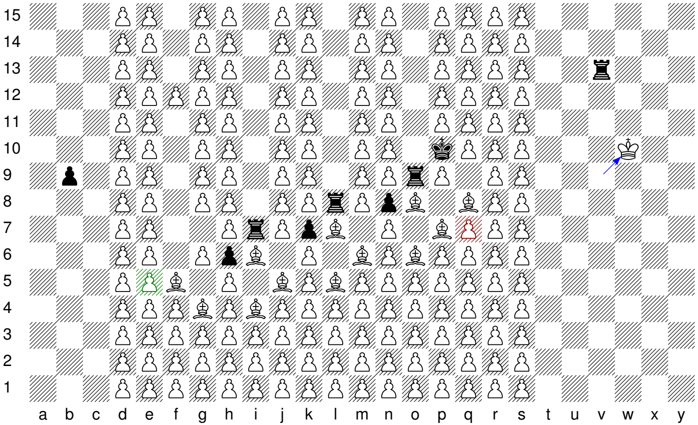

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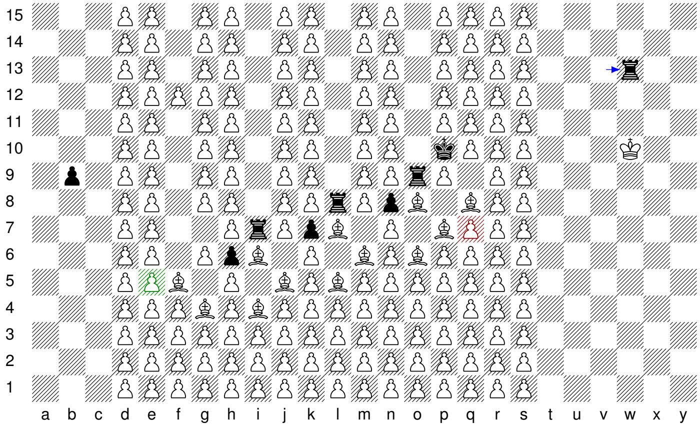

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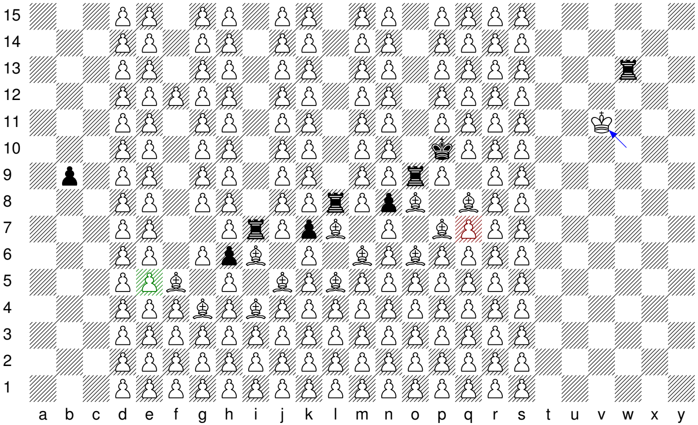

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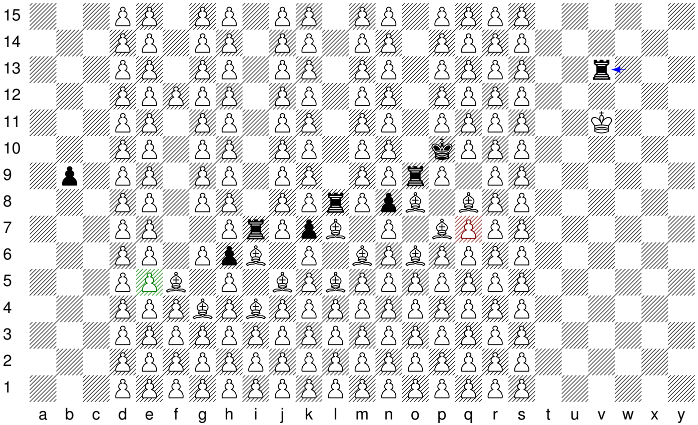

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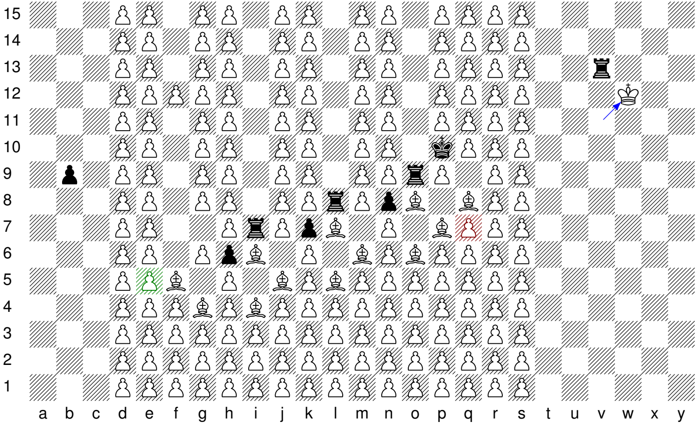

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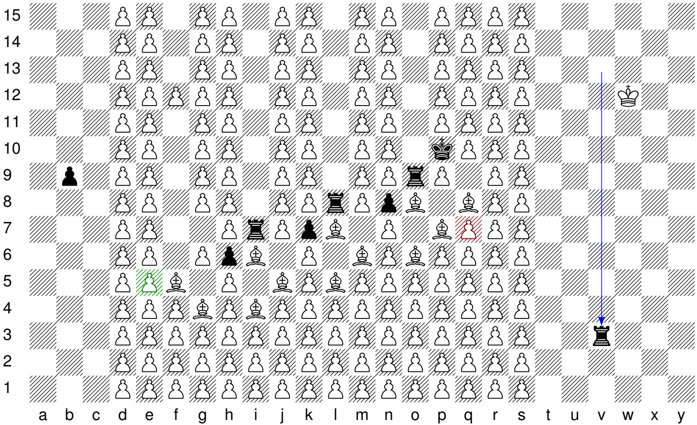
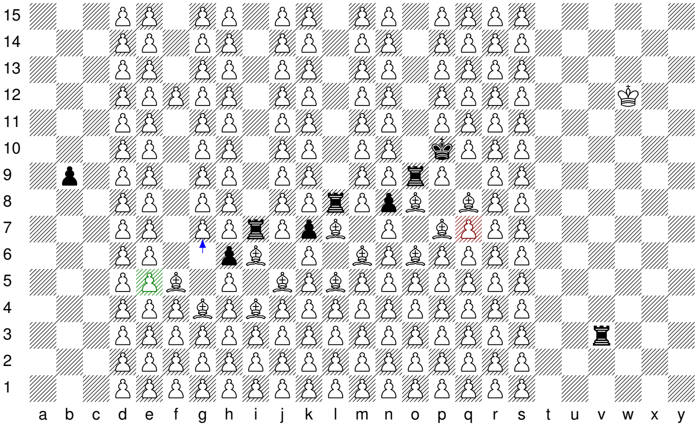


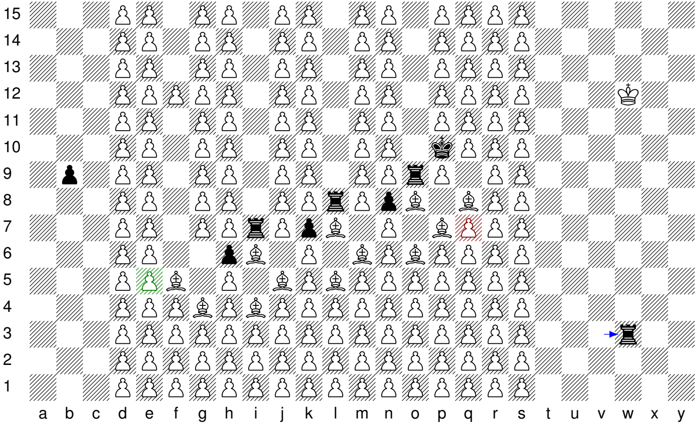
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$

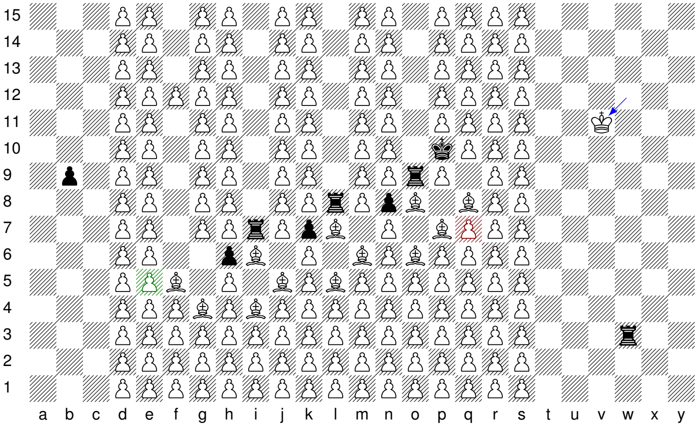


White advances pawn.

Iterated lock and key, value $\omega^2 \cdot 4$



Iterated lock and key, value $\omega^2 \cdot 4$



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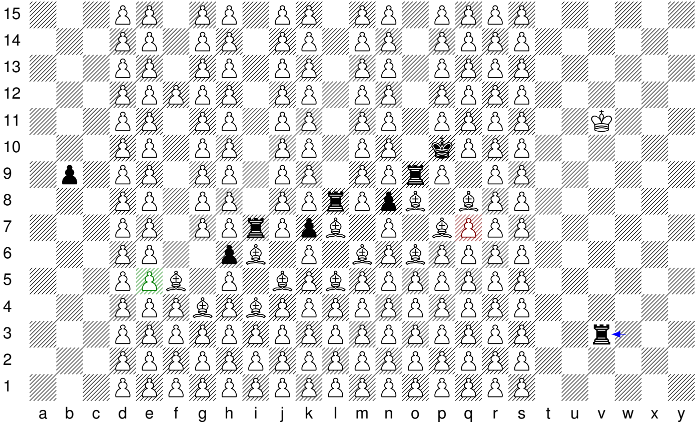

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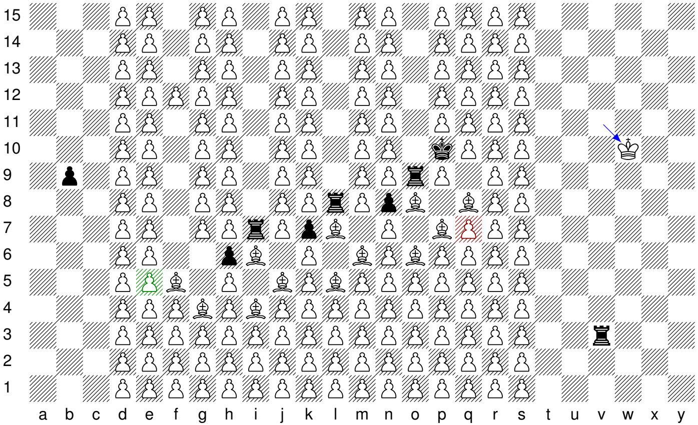

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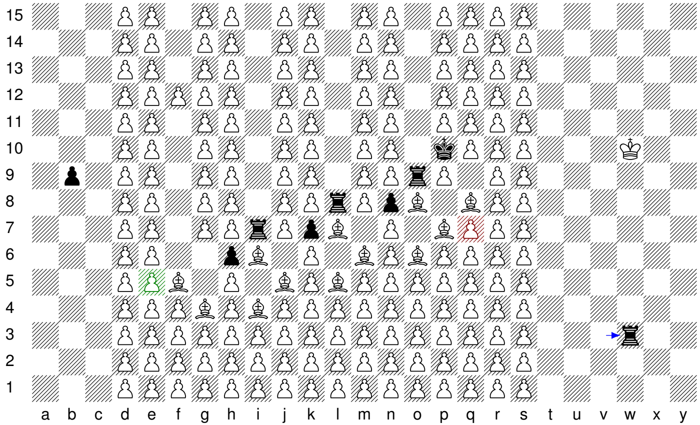

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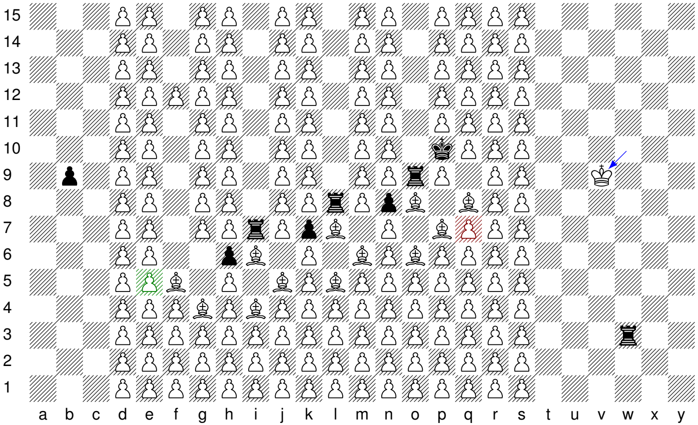

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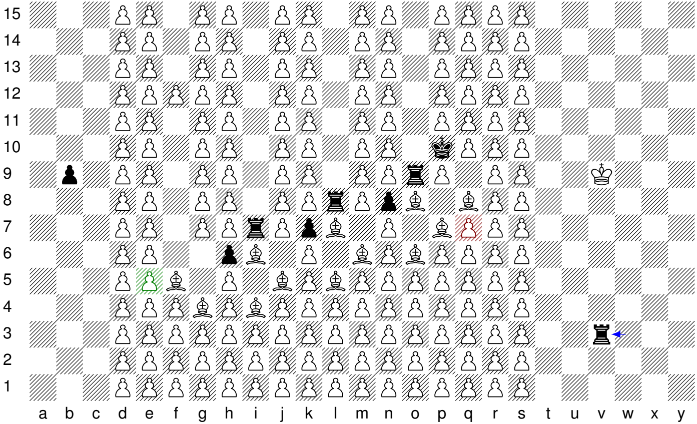

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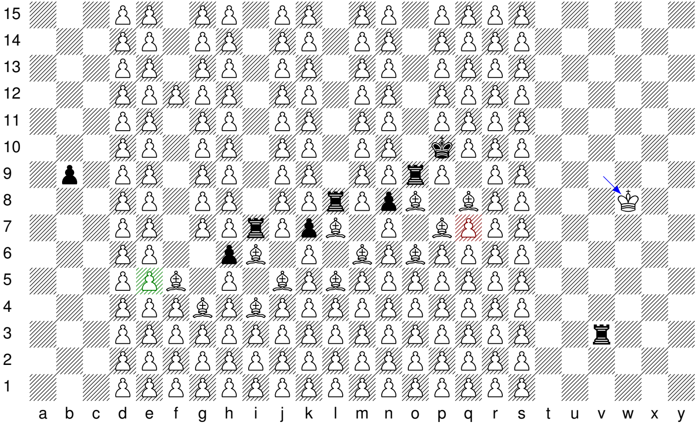

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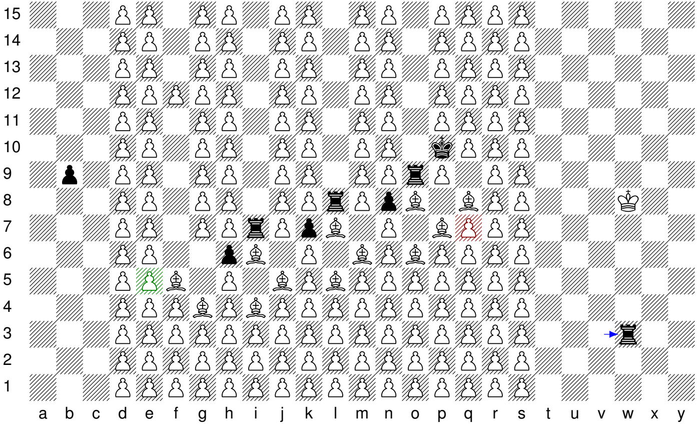

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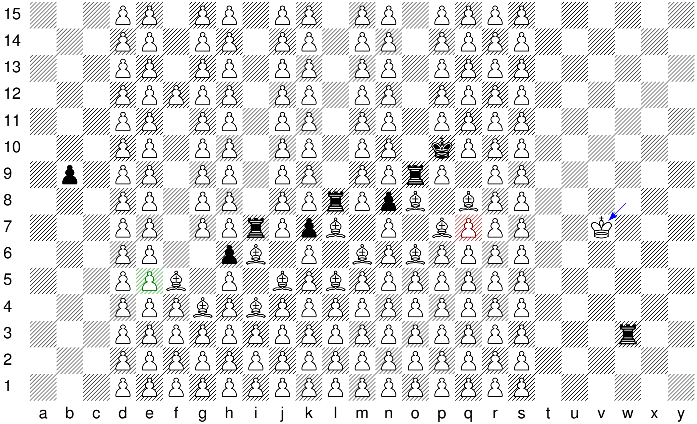

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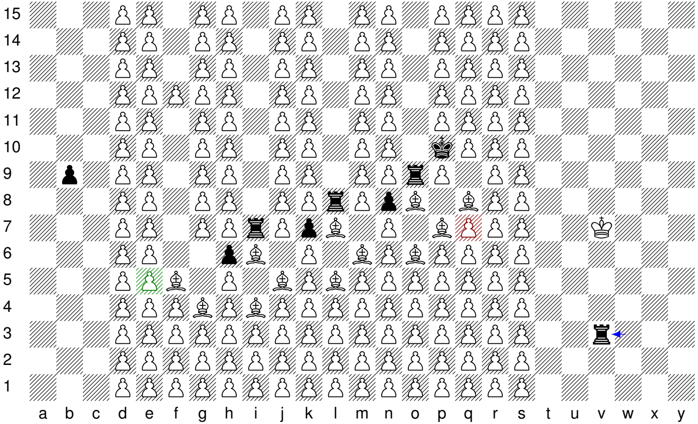

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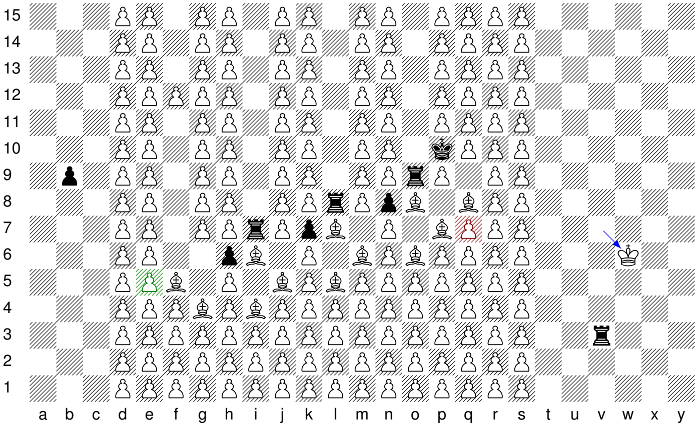

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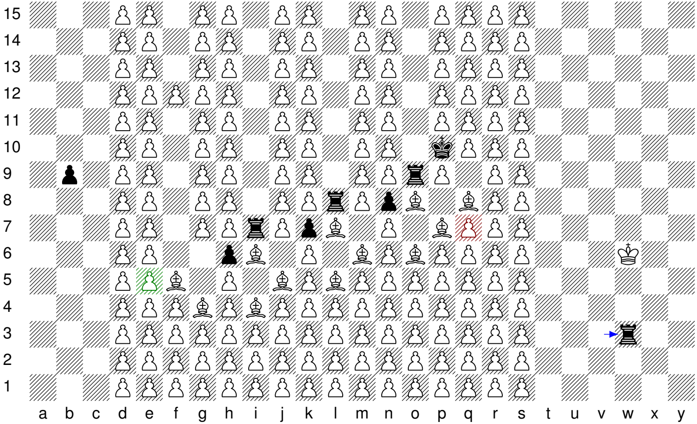

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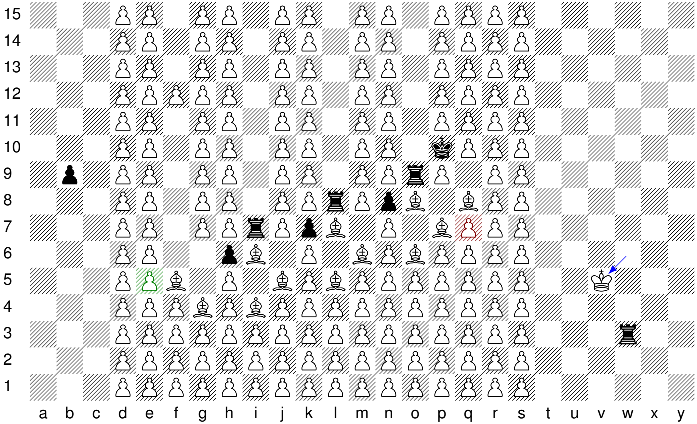

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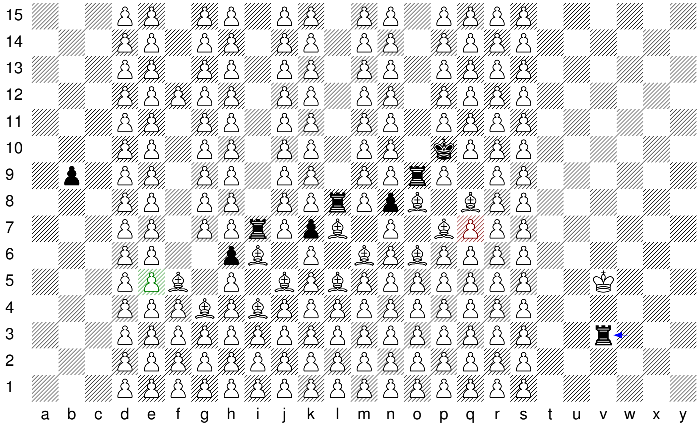

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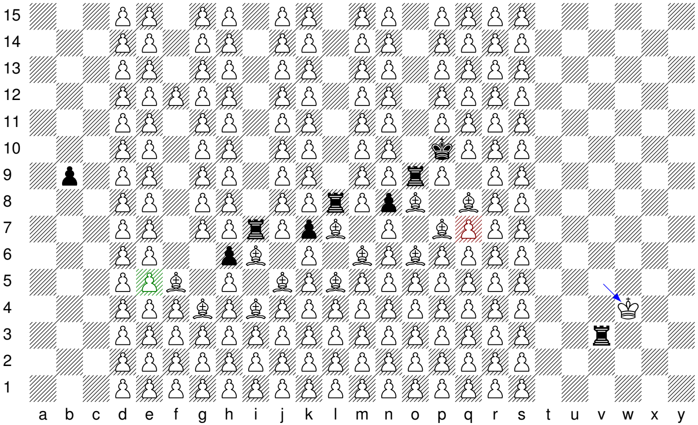
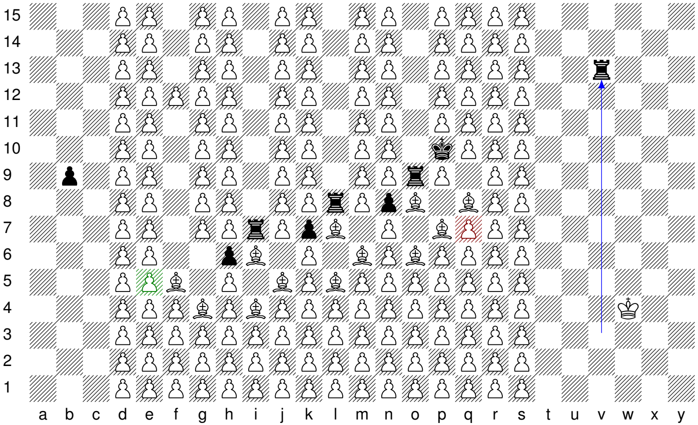


Iterated lock and key, value $\omega^2 \cdot 4$



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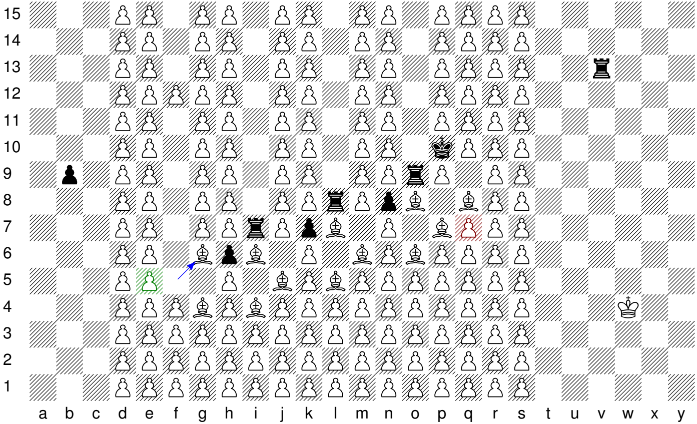

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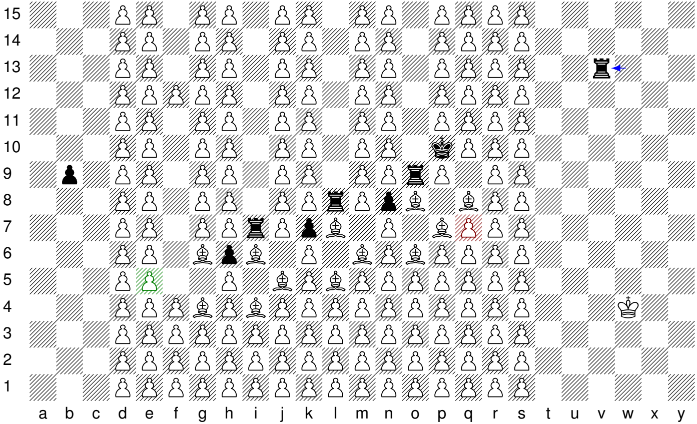

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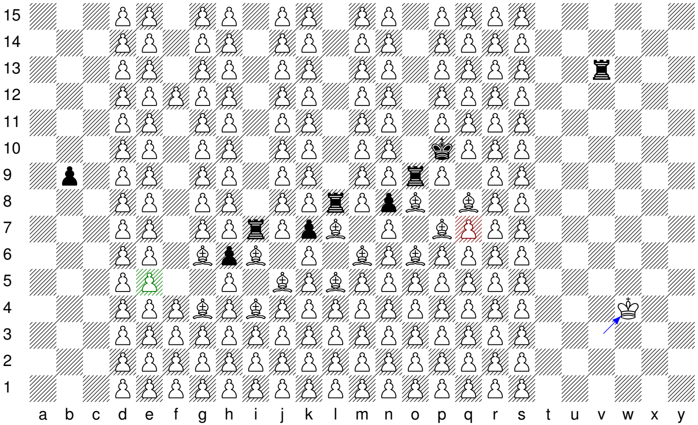

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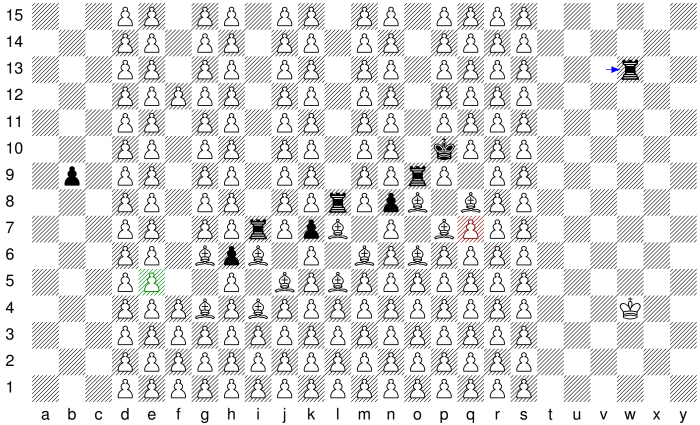

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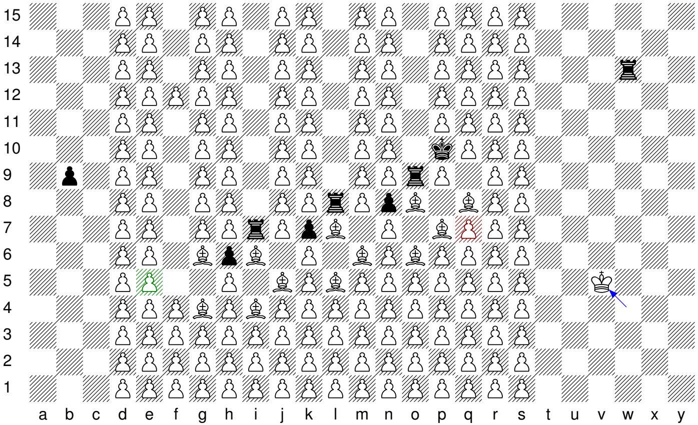

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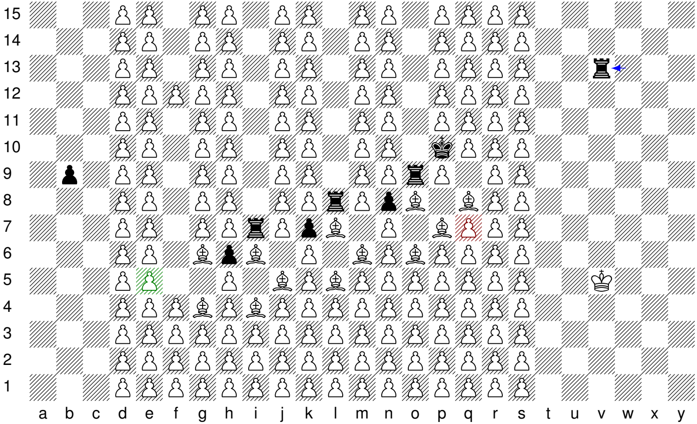

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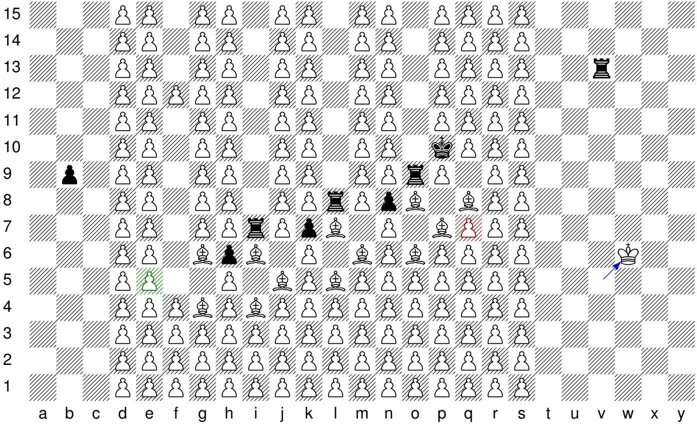

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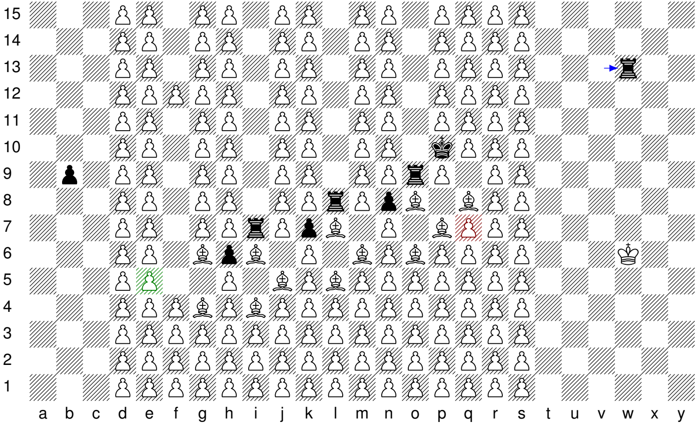

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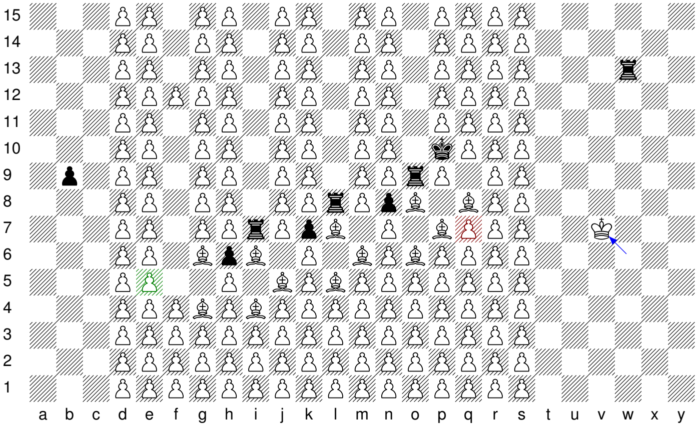

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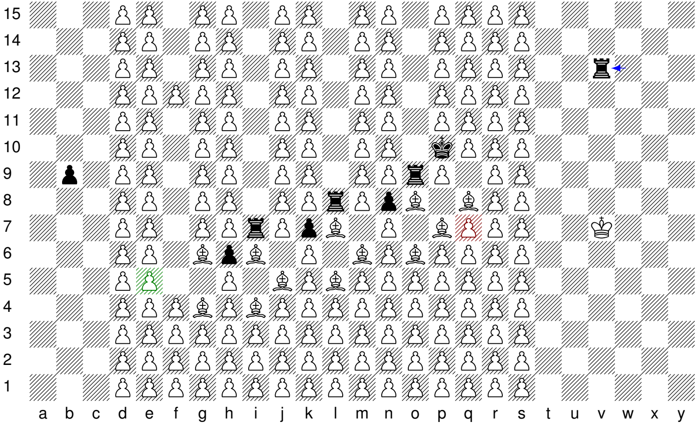

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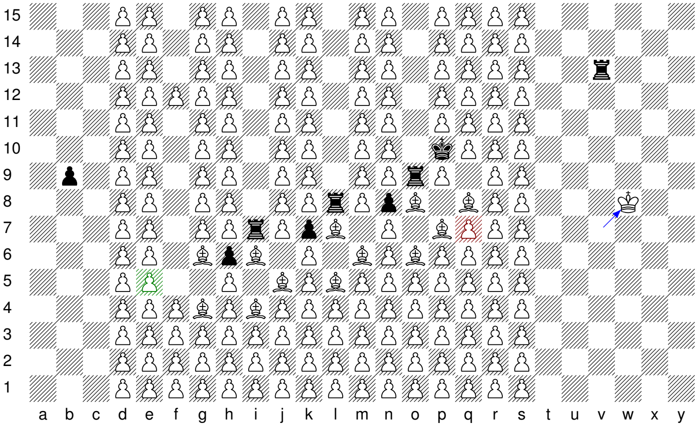

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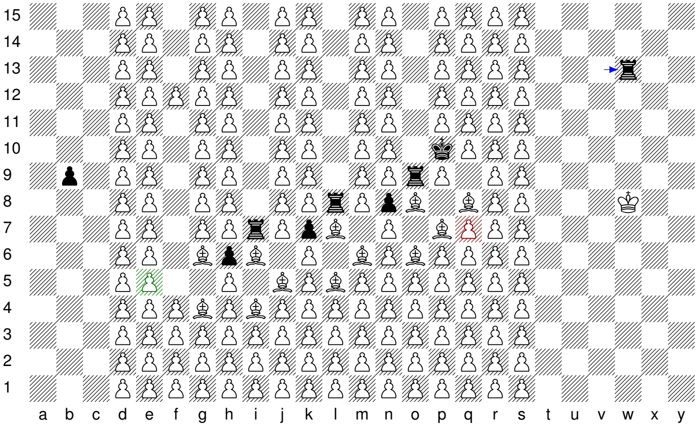

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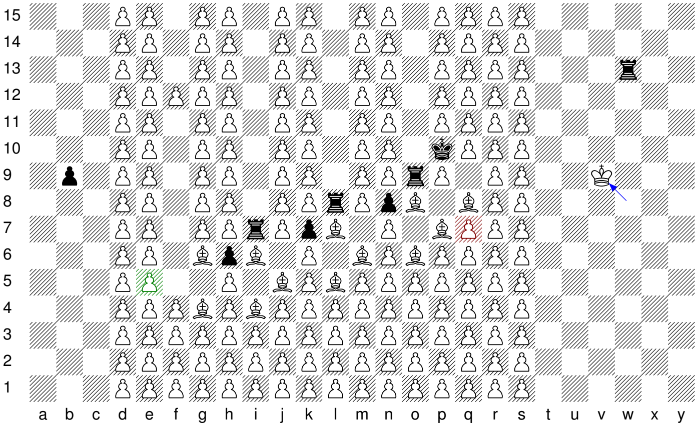

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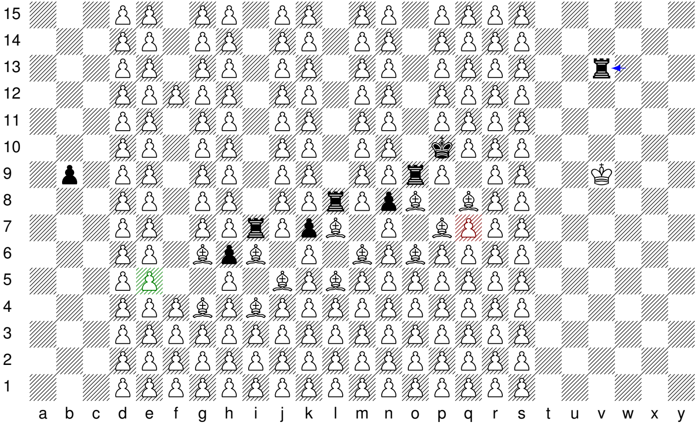

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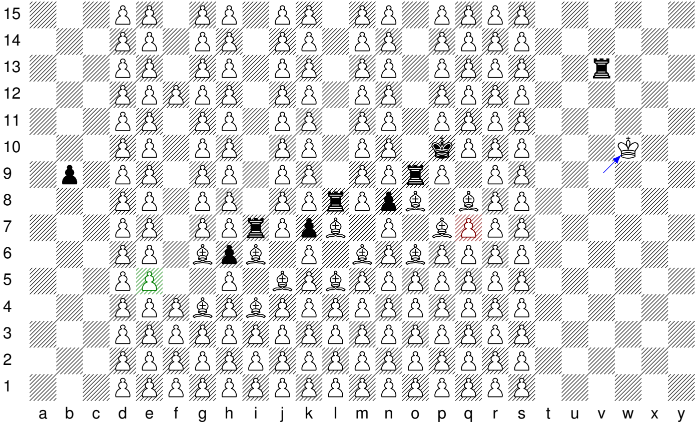

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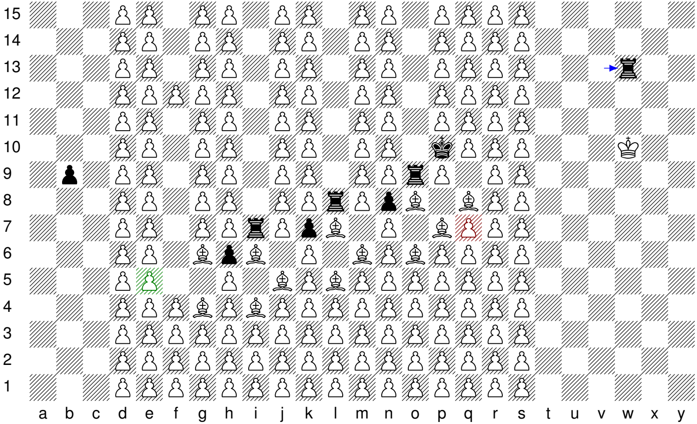

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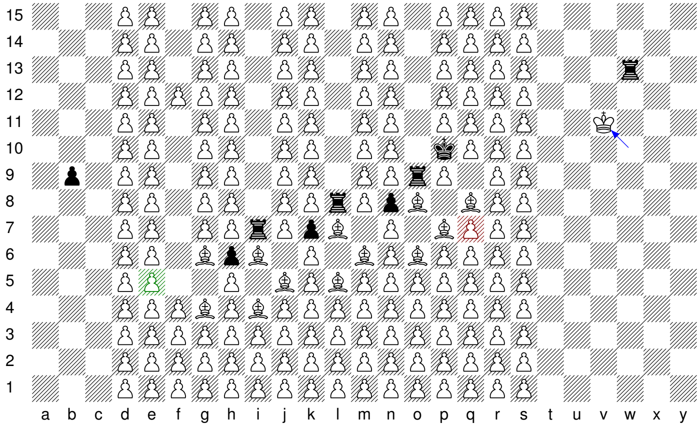

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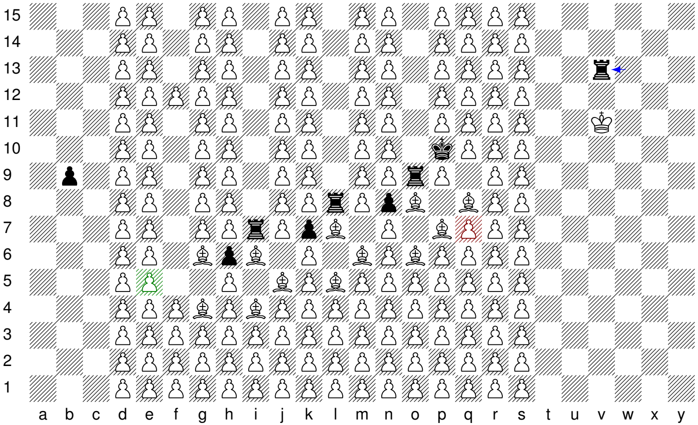

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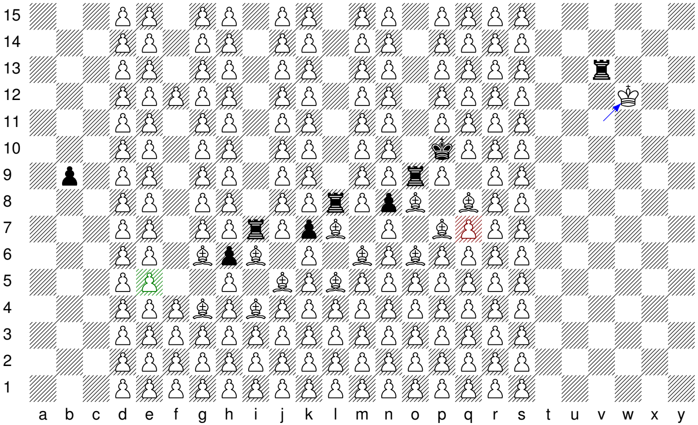

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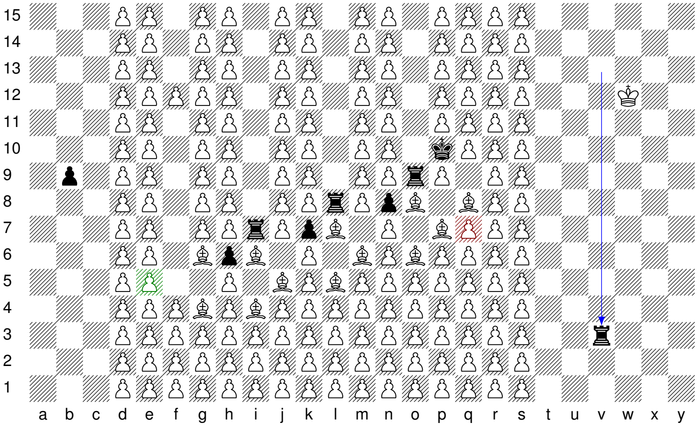

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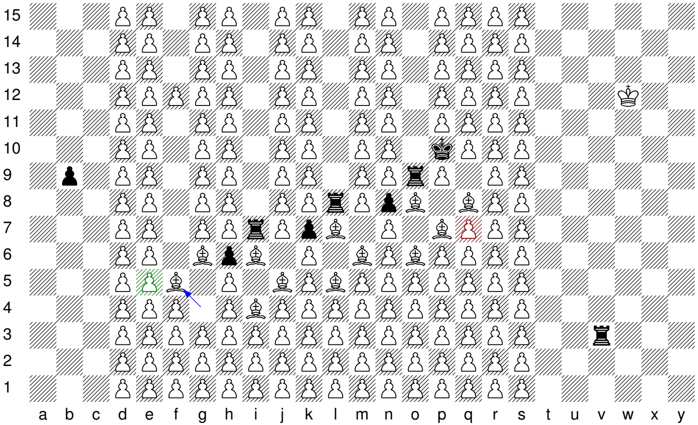

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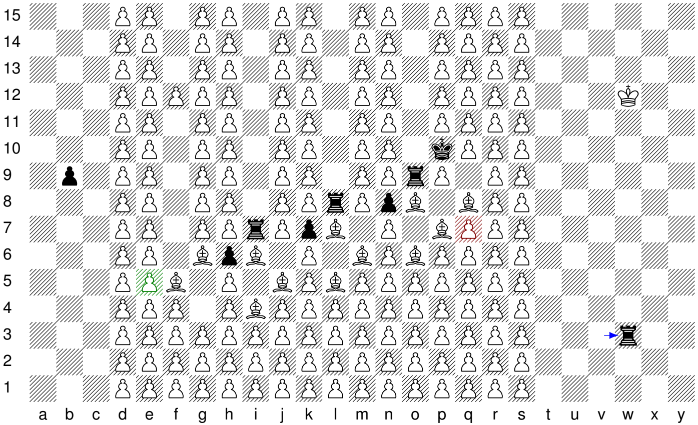

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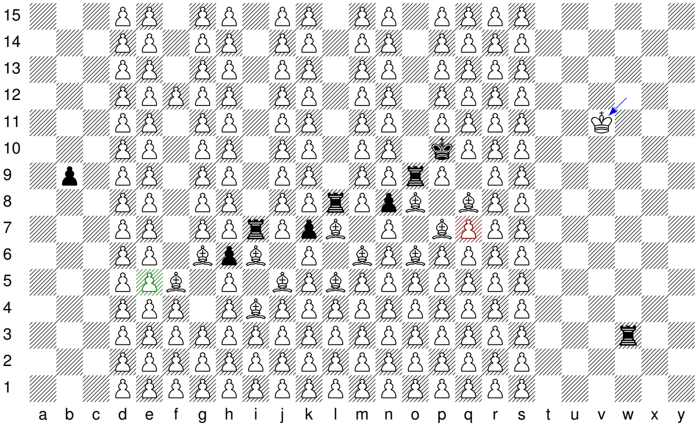

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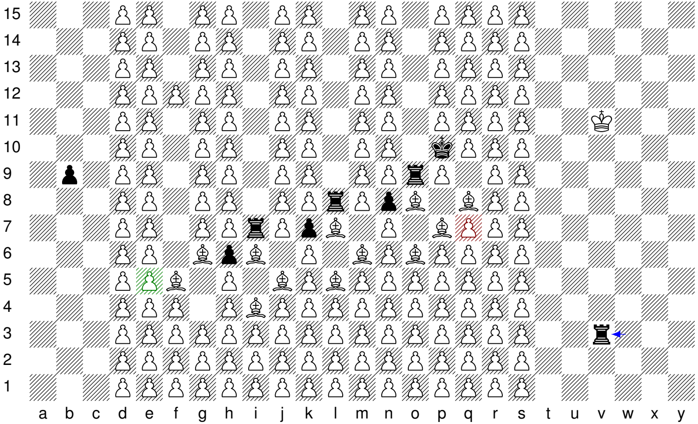

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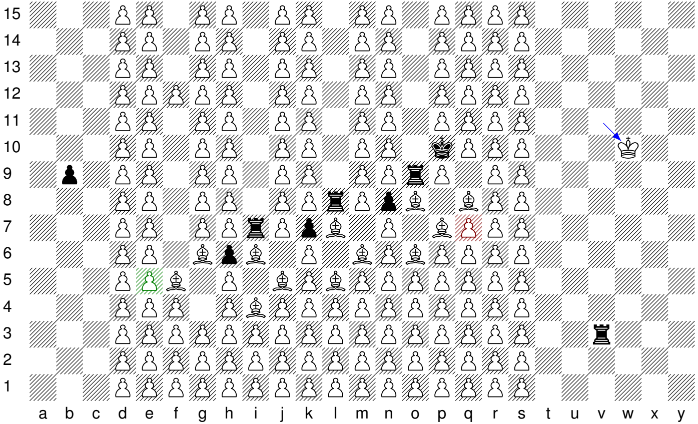

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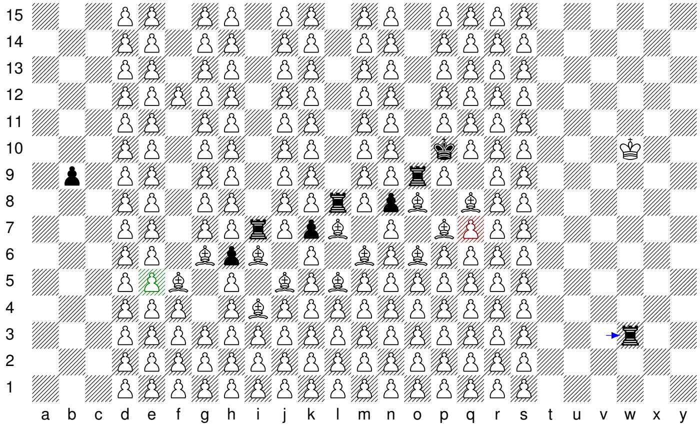

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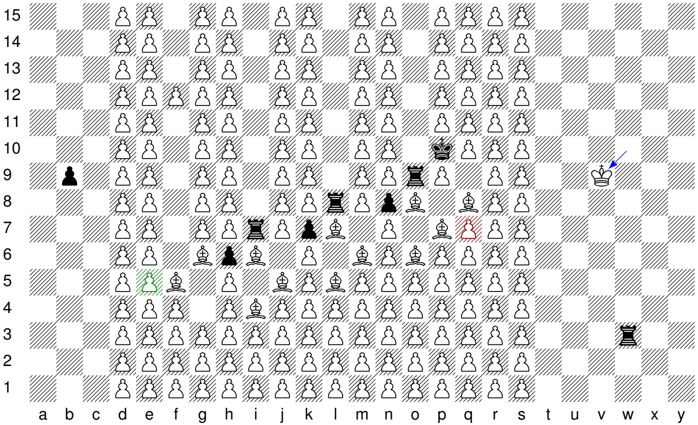

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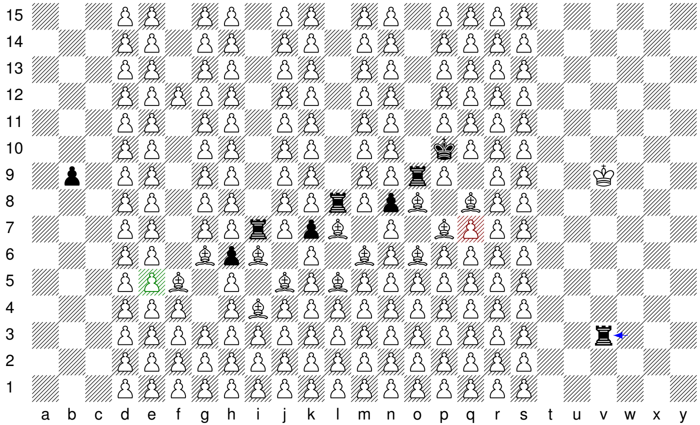

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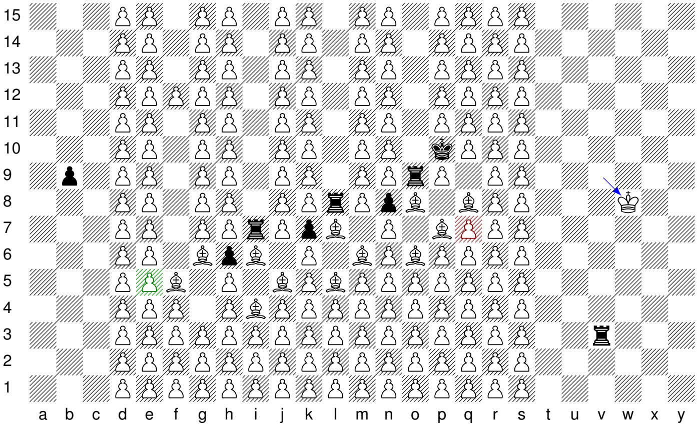

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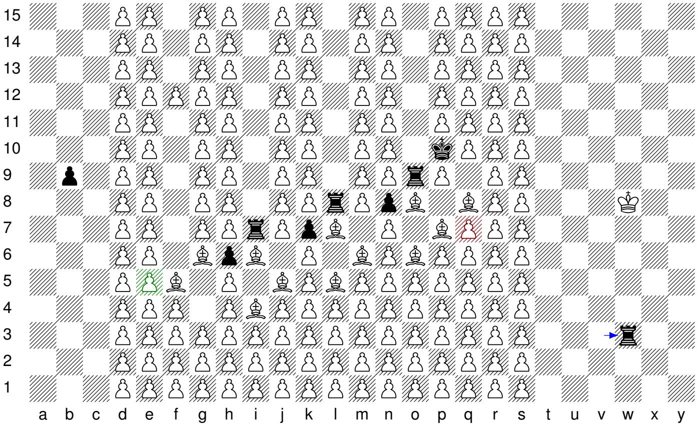

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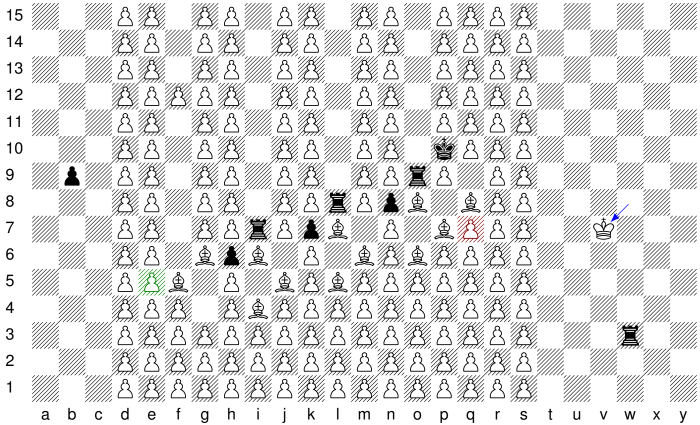

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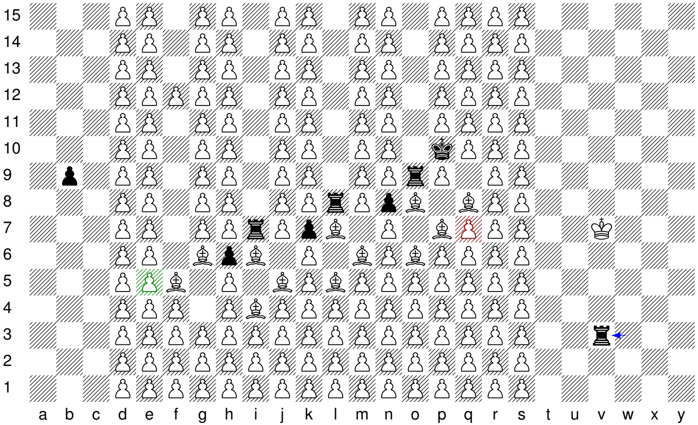

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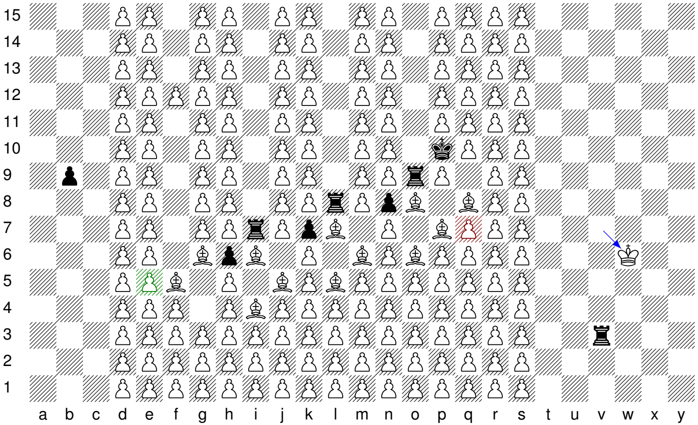

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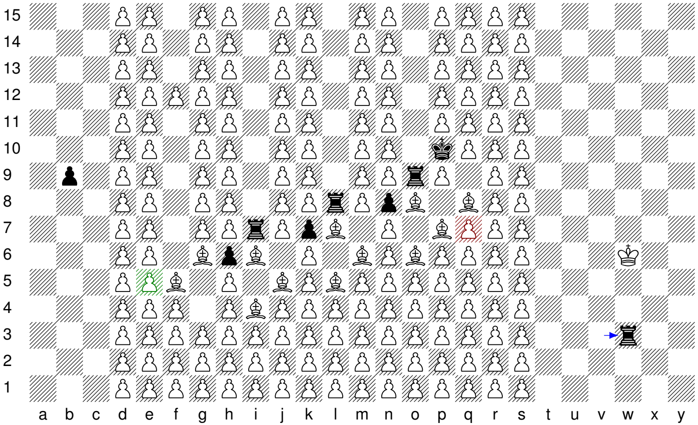

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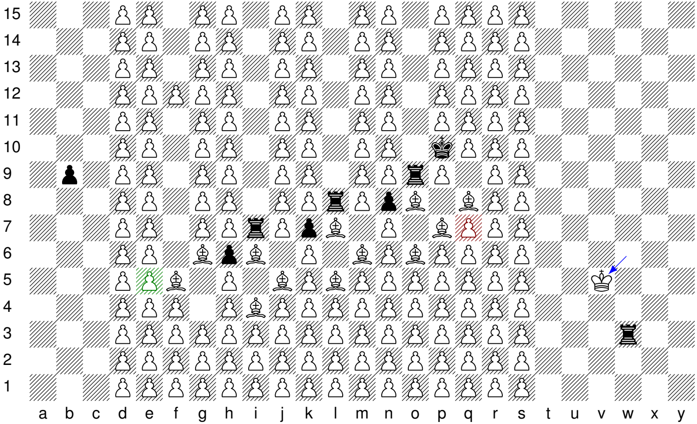

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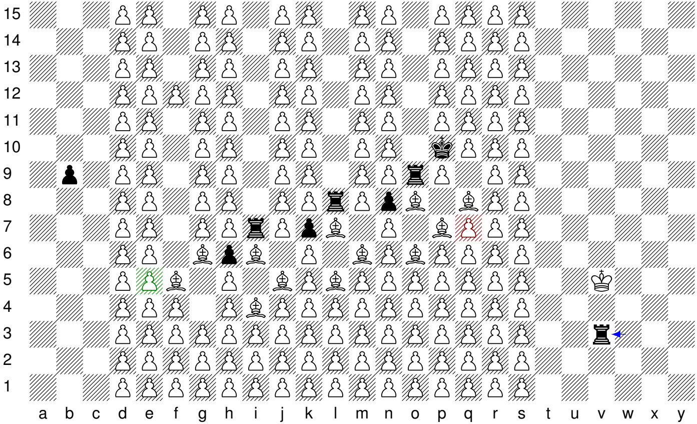

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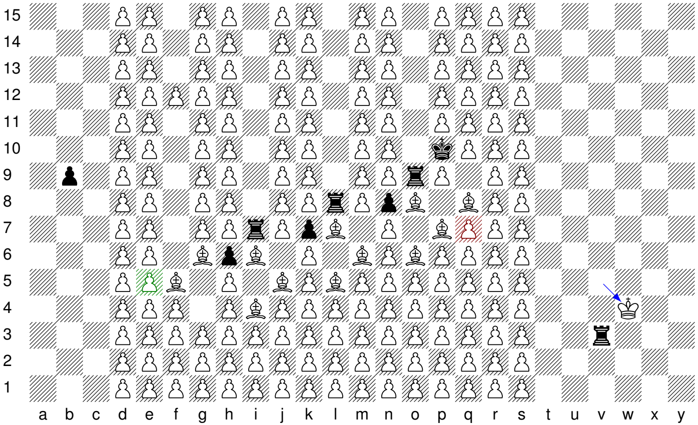

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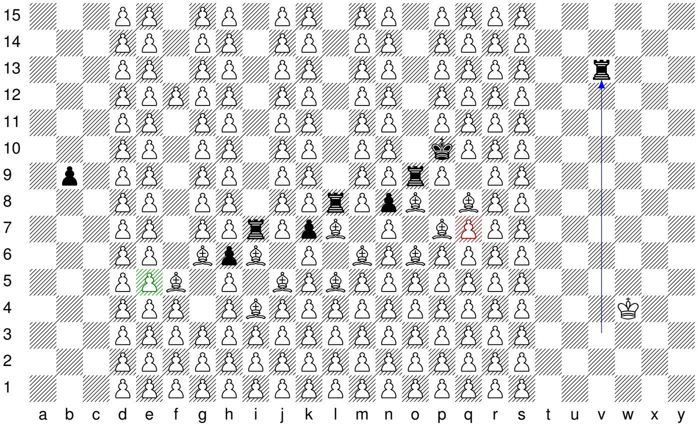

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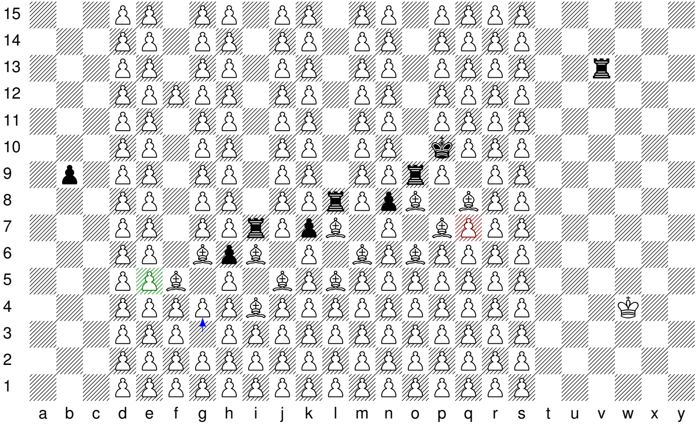

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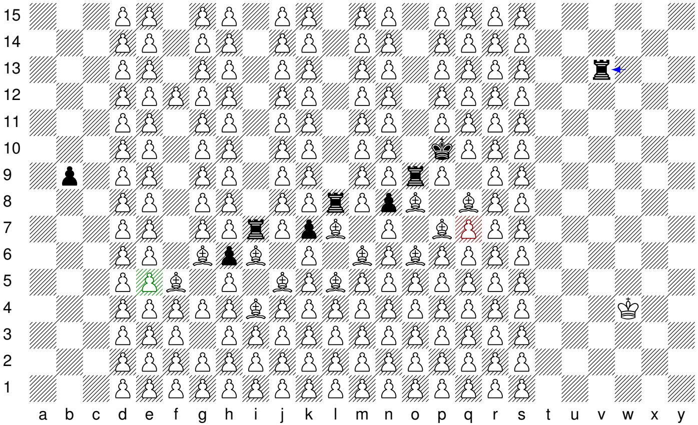

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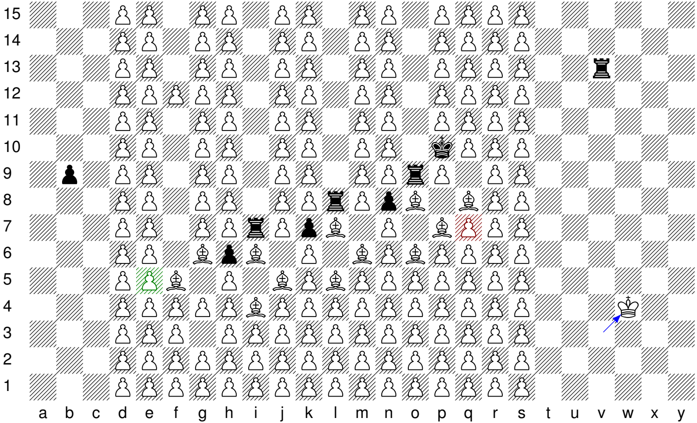

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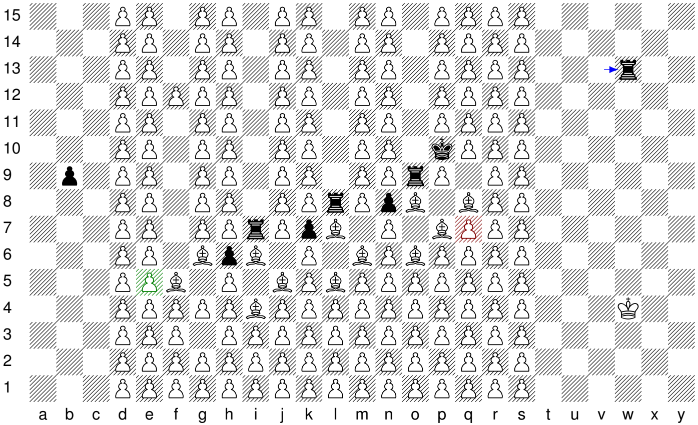

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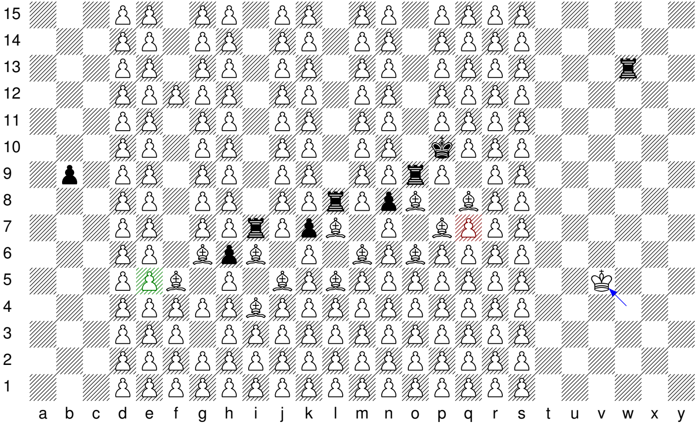

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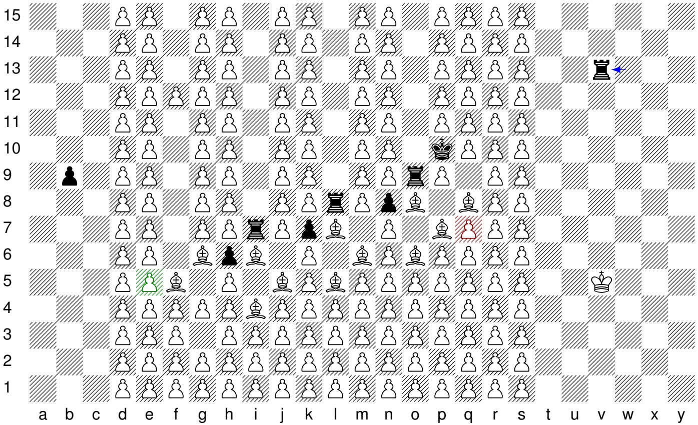

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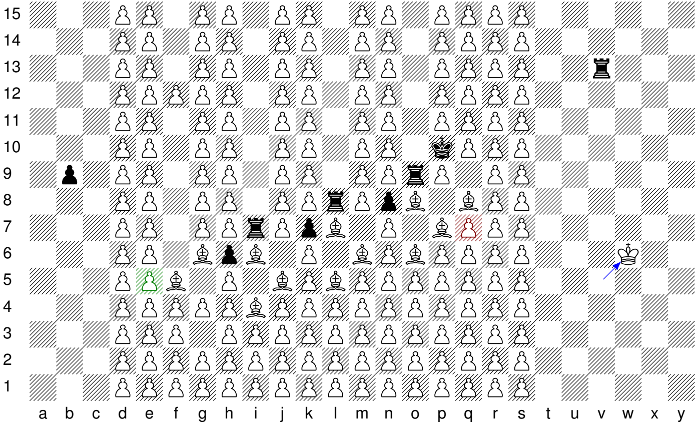

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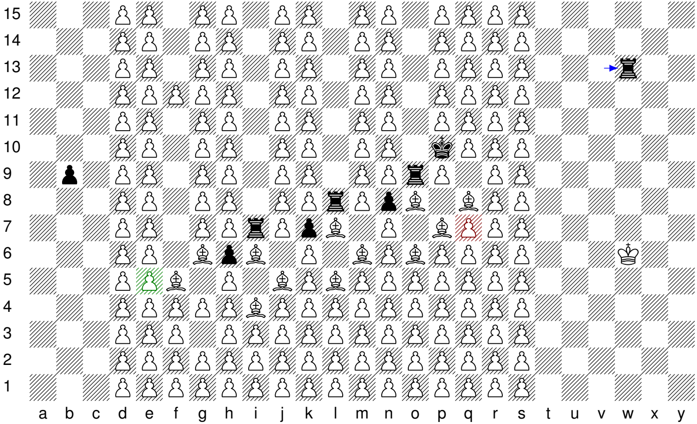

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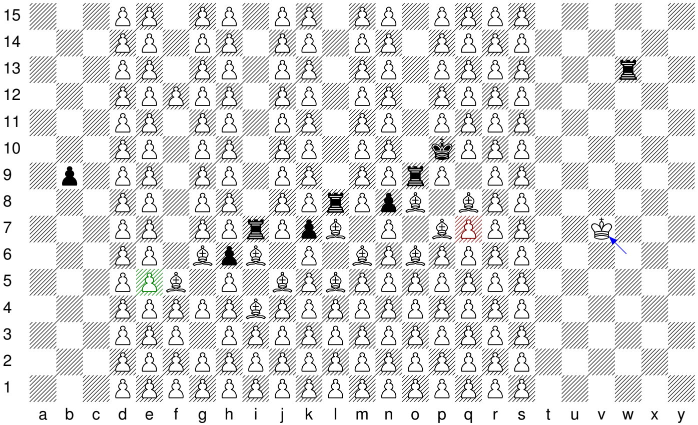

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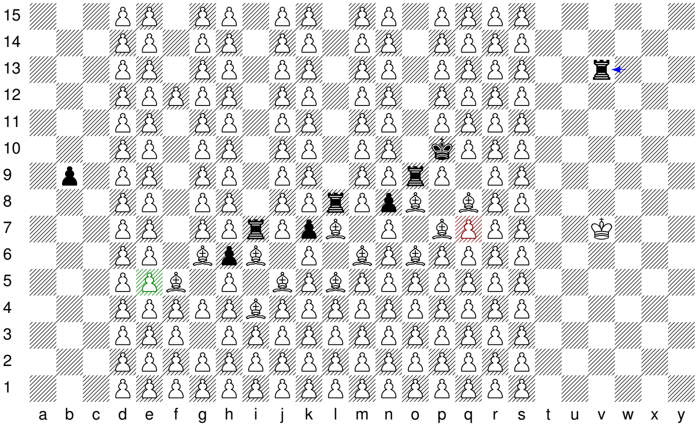

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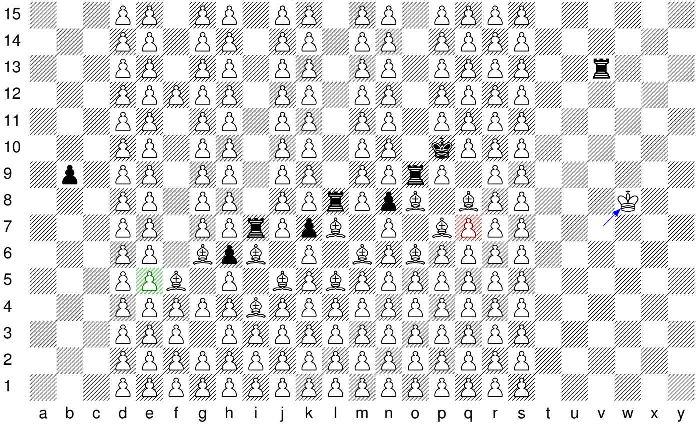

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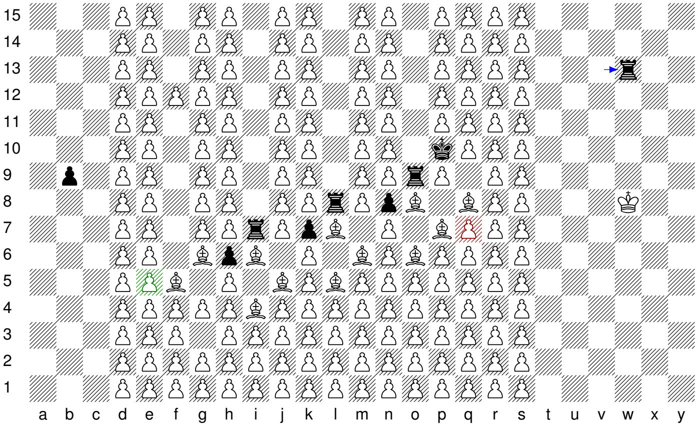

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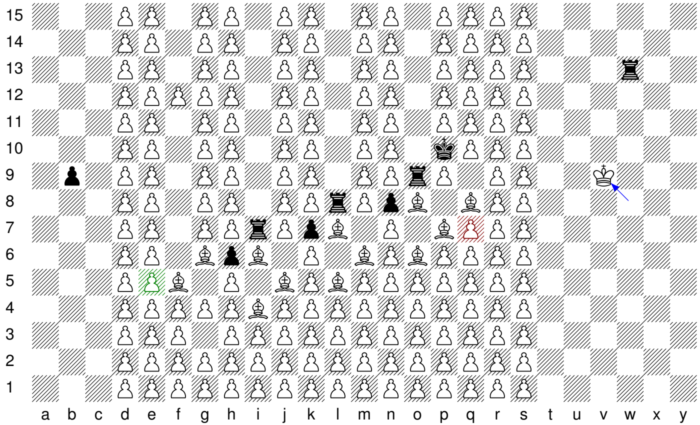

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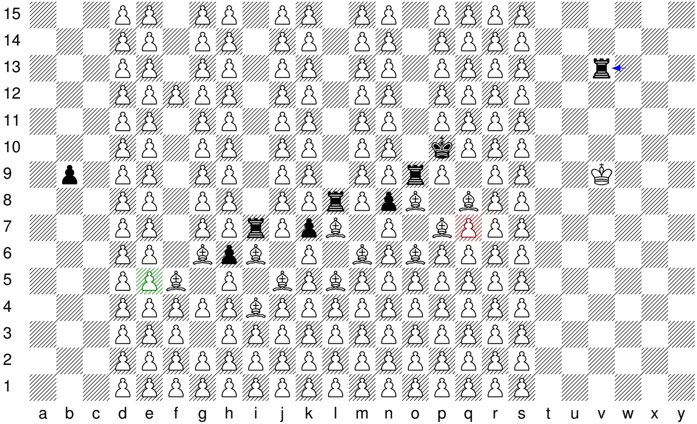

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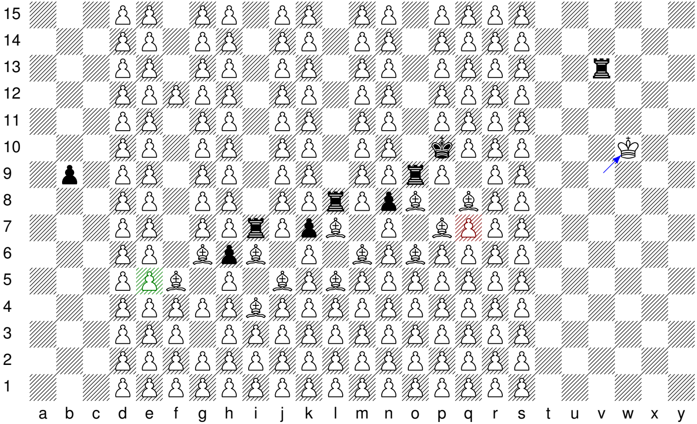

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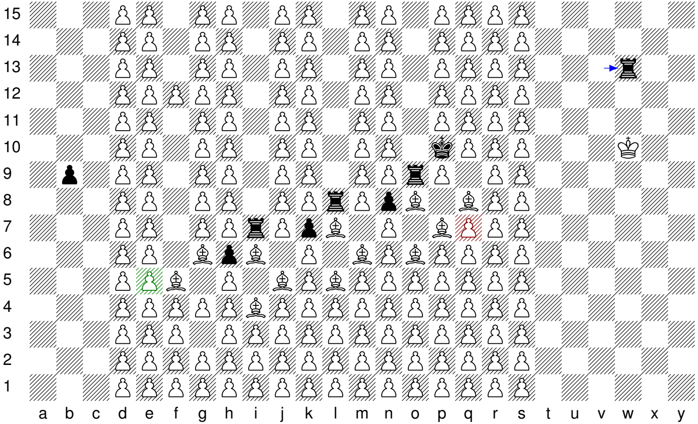

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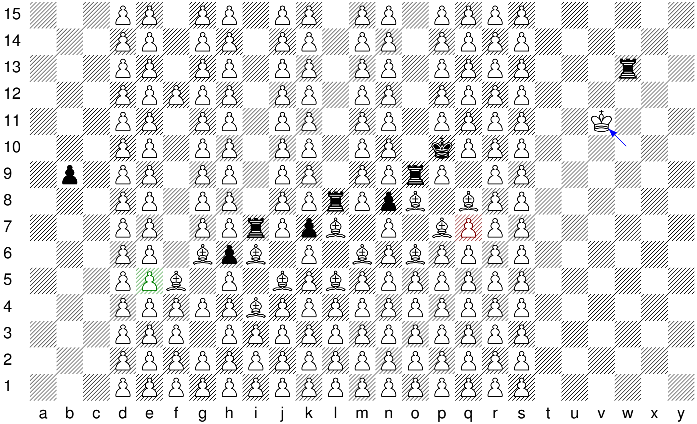

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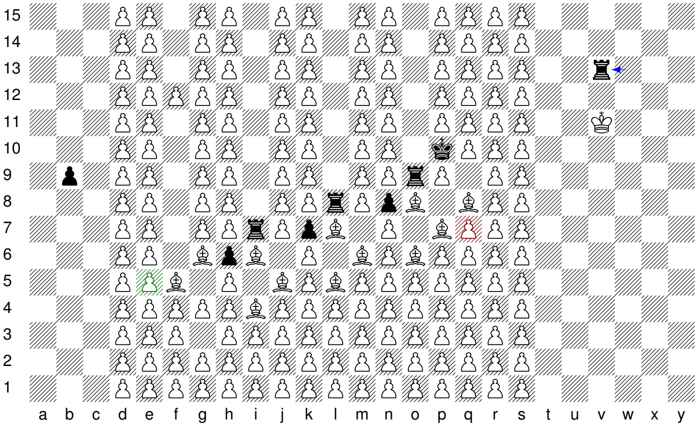

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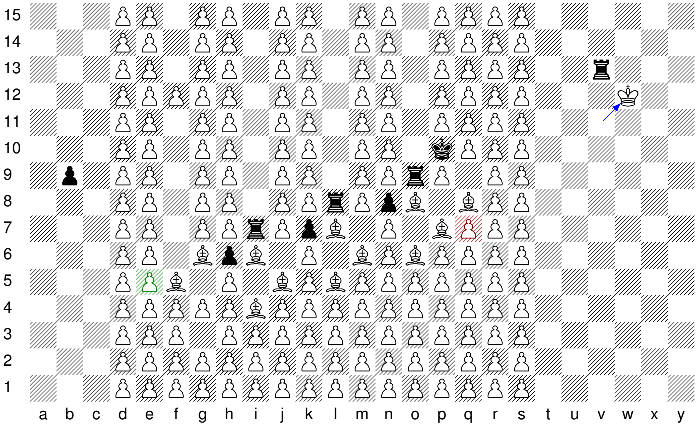

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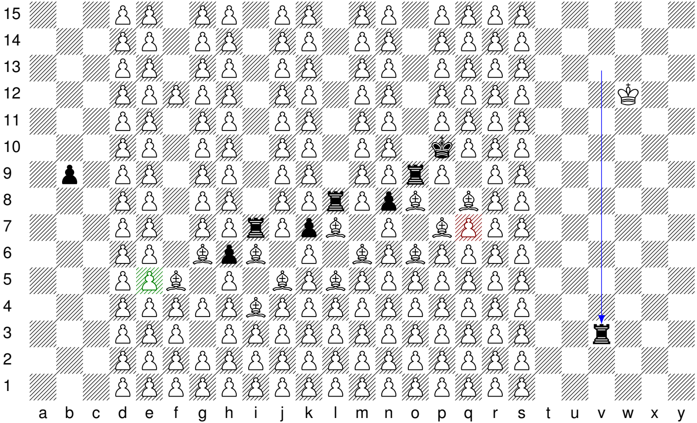

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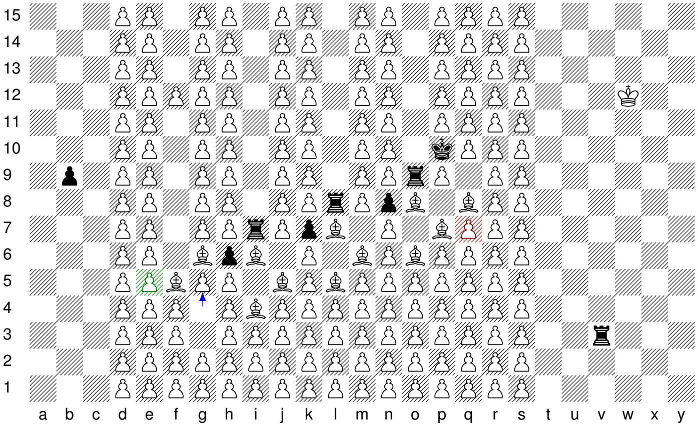

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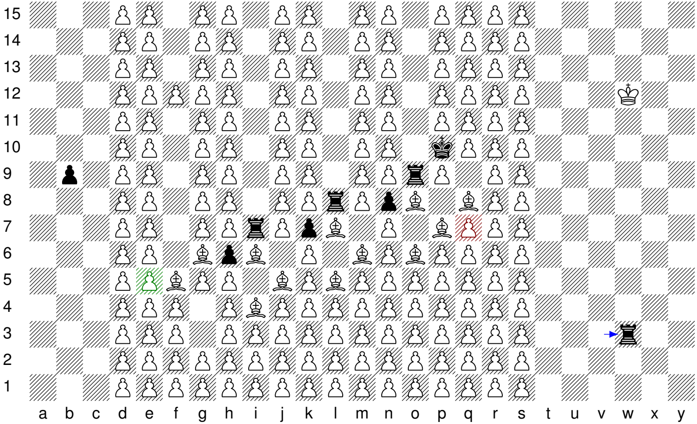

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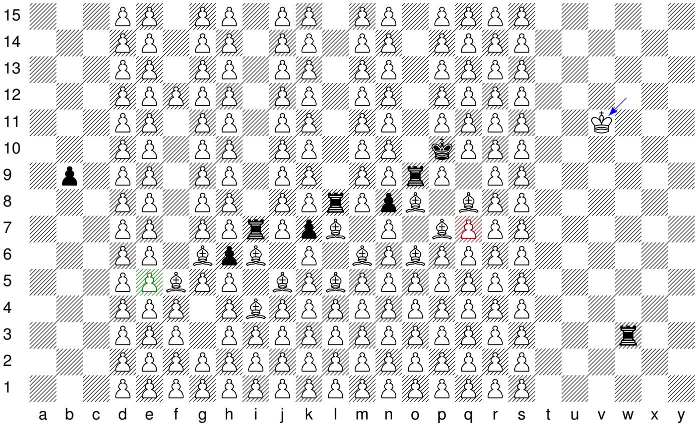

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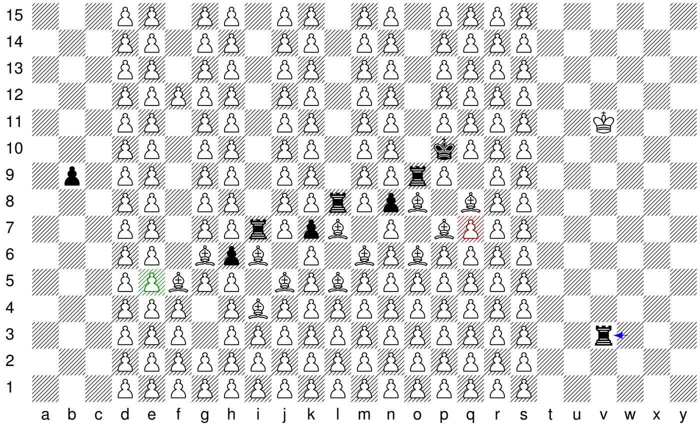

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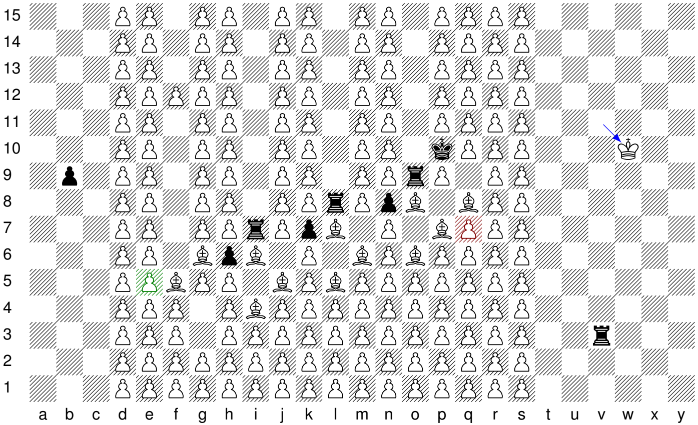

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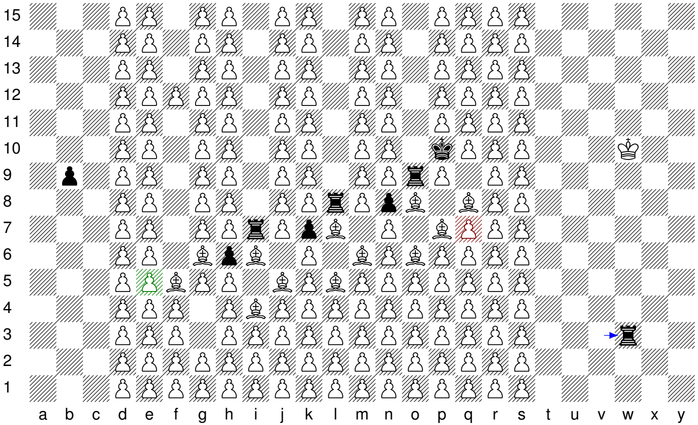

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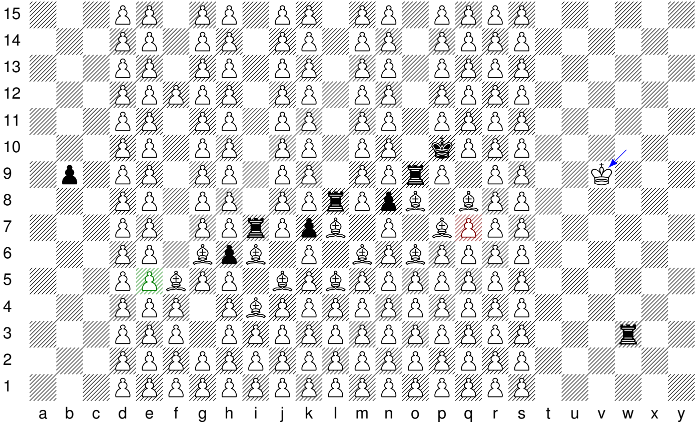
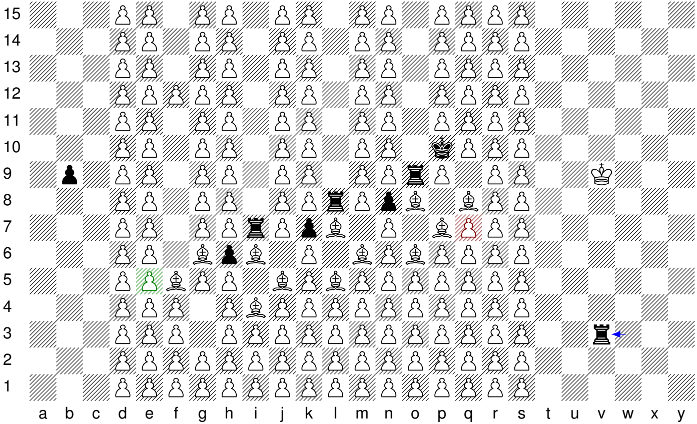


Iterated lock and key, value $\omega^2 \cdot 4$



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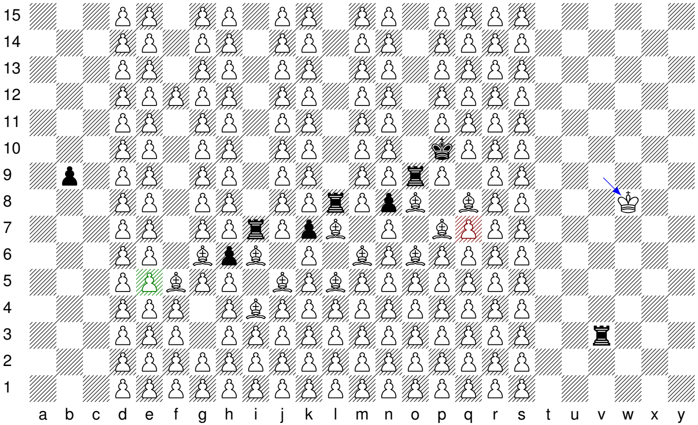

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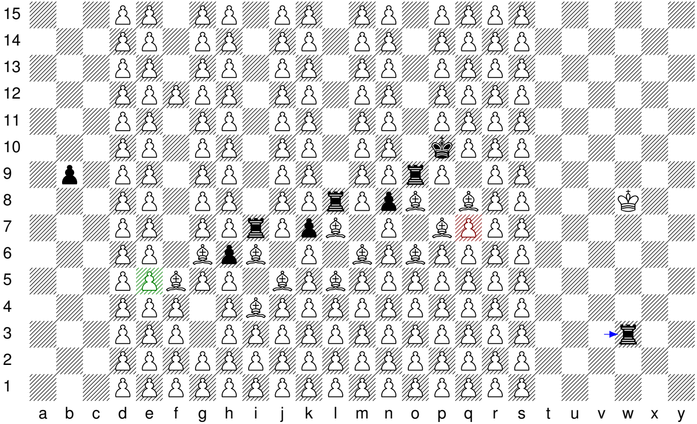

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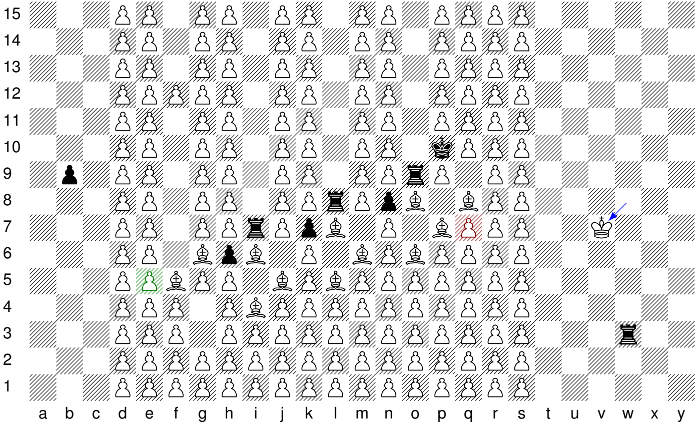

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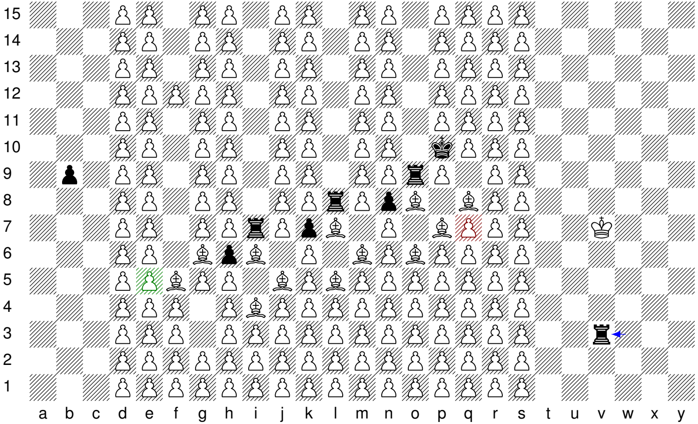

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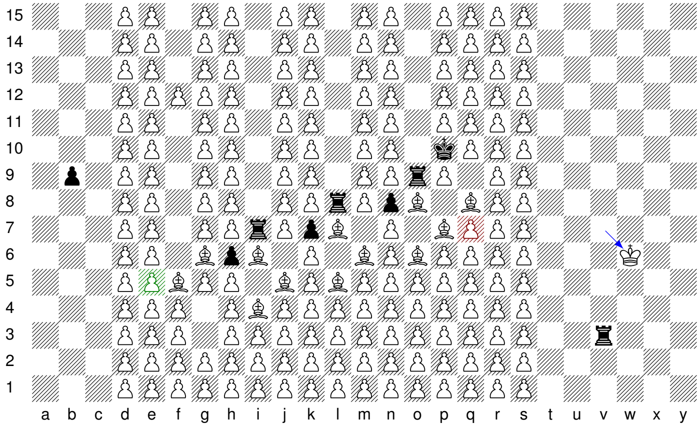

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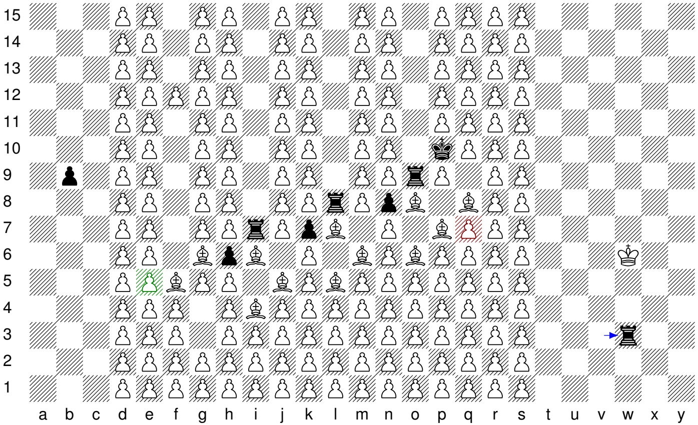

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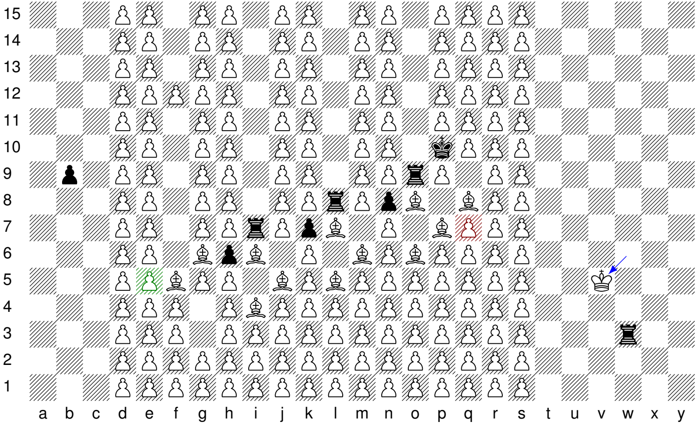

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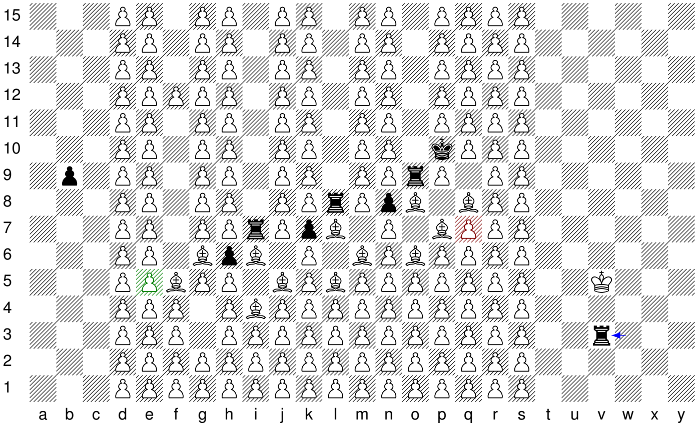

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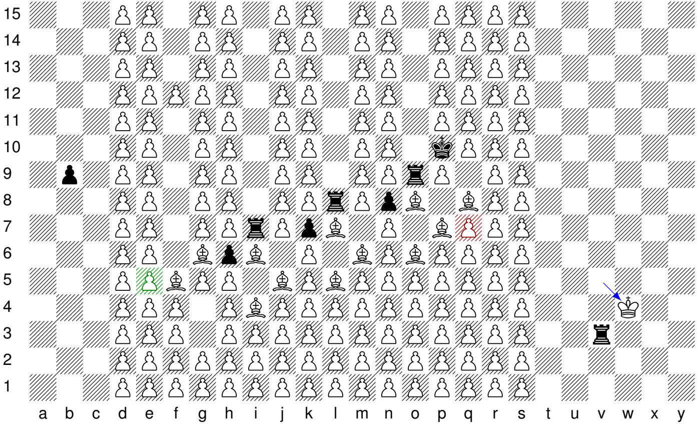

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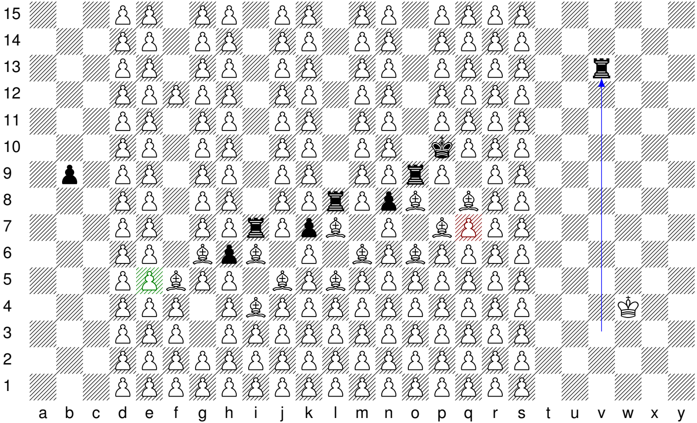
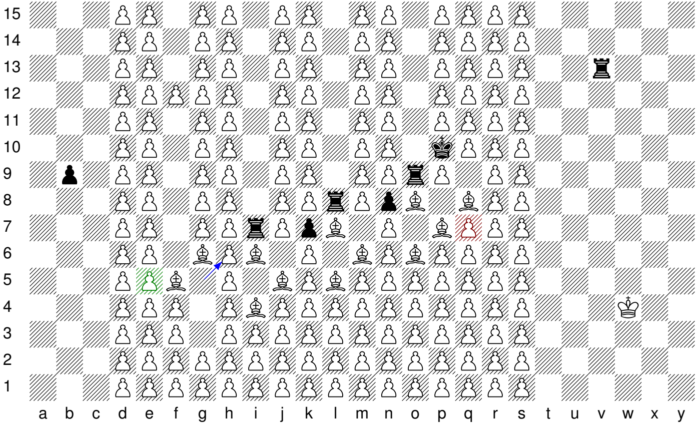


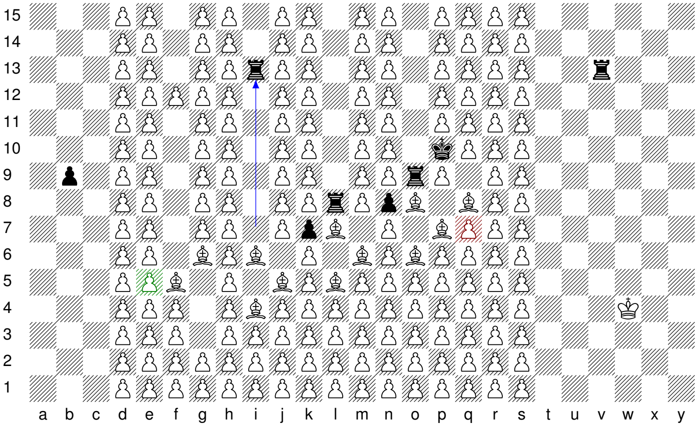
Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$



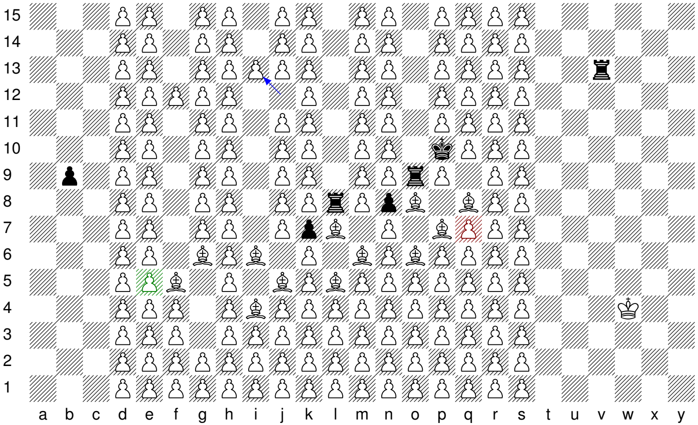
Black key pawn attacks second tower.

Iterated lock and key, value $\omega^2 \cdot 4$



Second tower ascends.

Iterated lock and key, value $\omega^2 \cdot 4$



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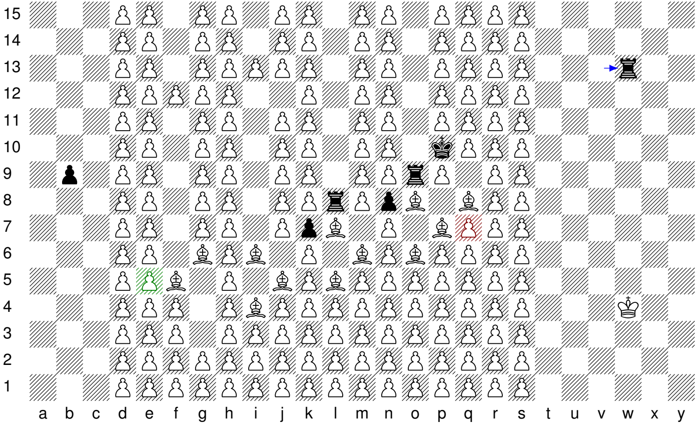

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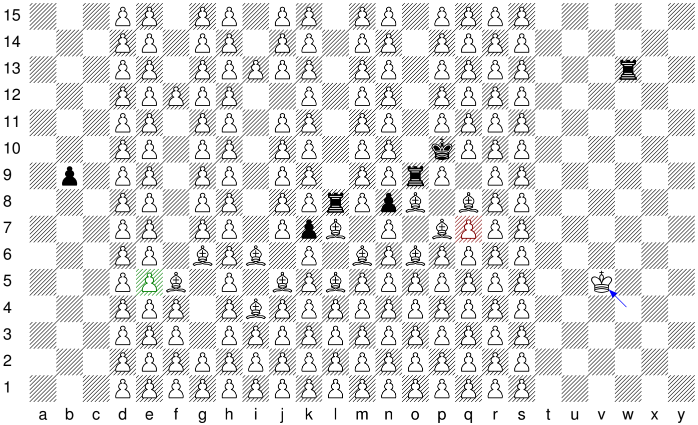

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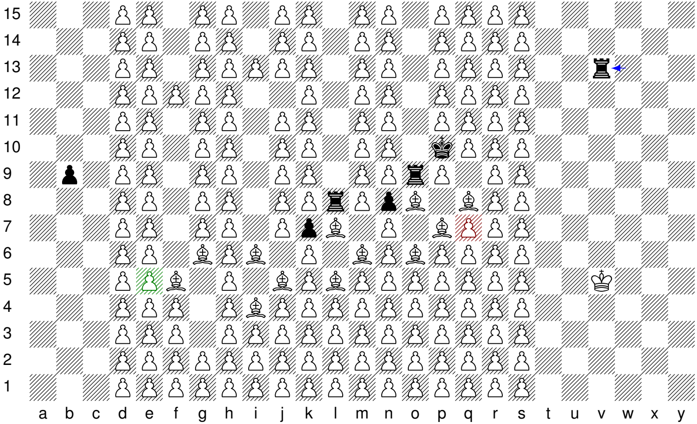

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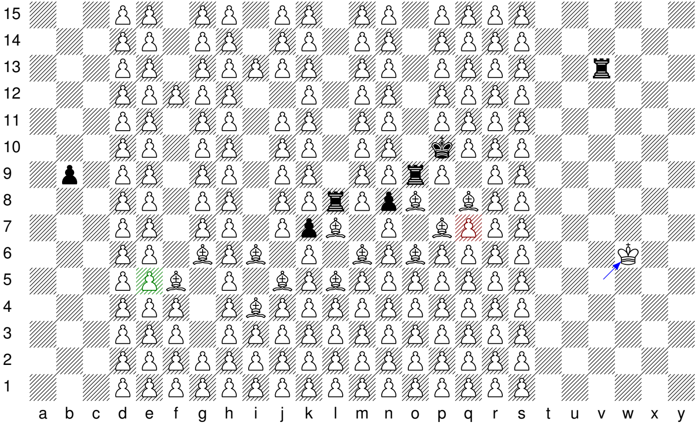

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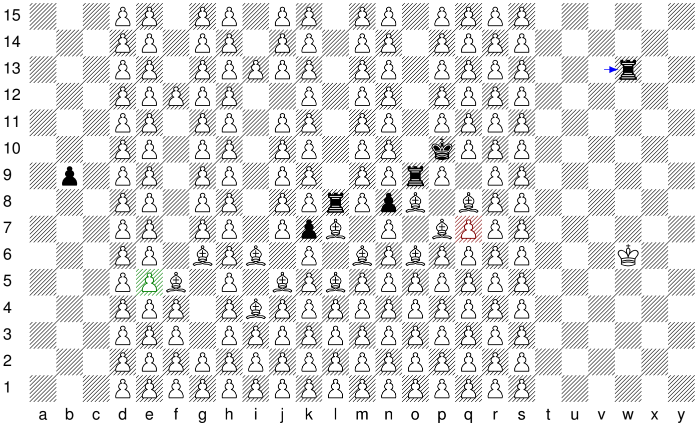

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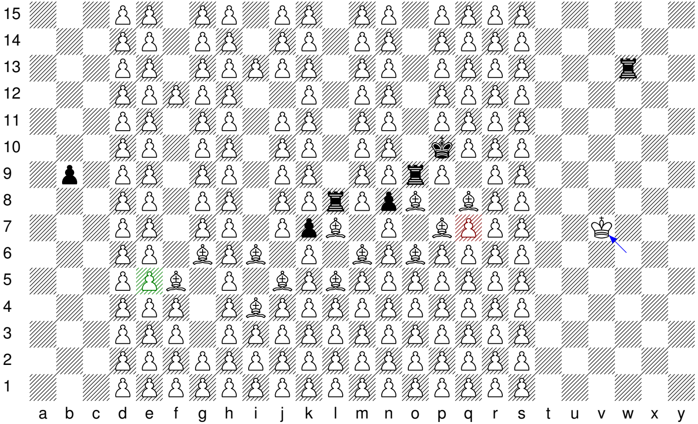

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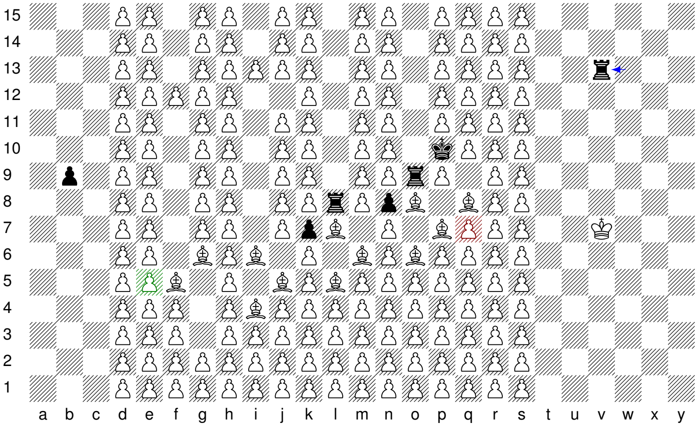

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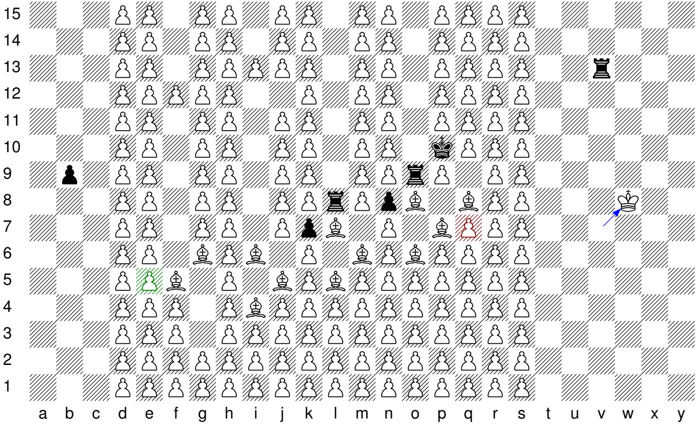

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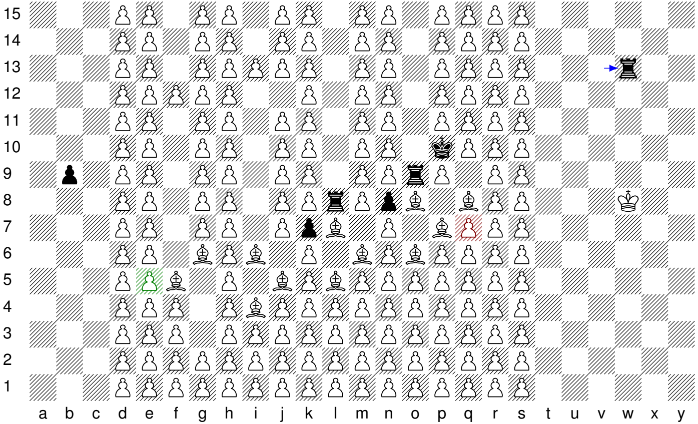

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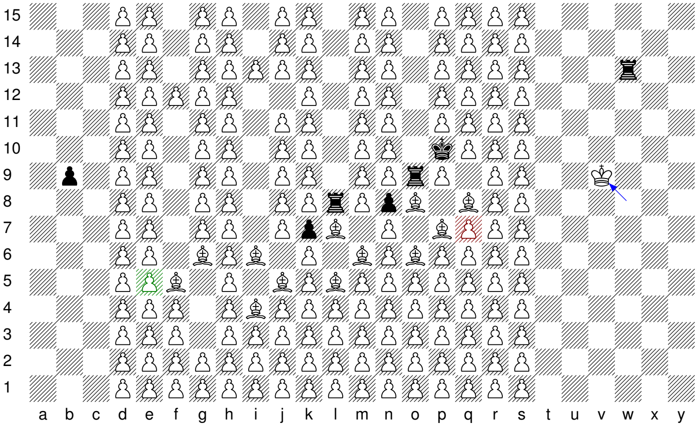

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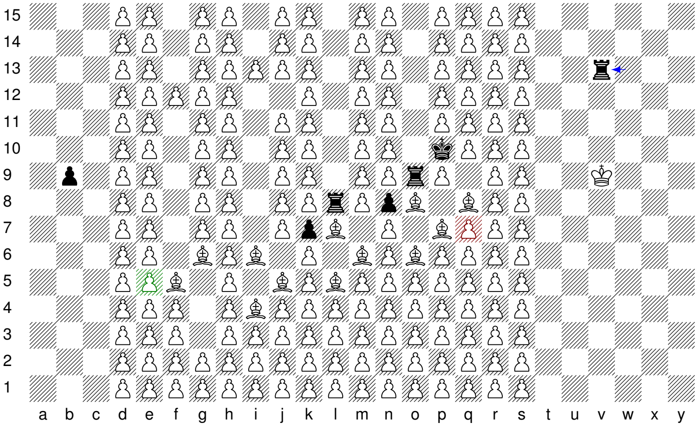

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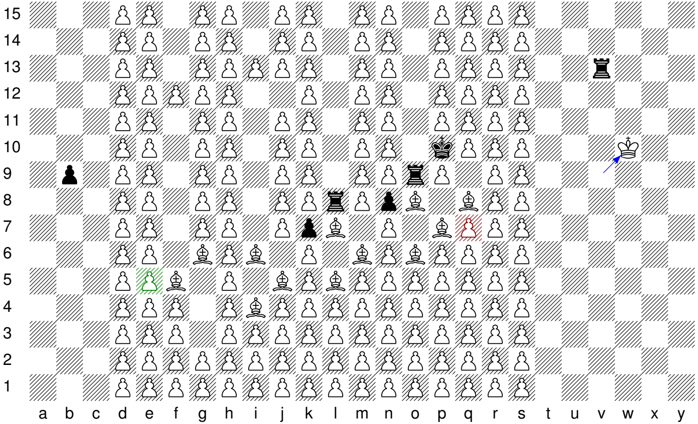

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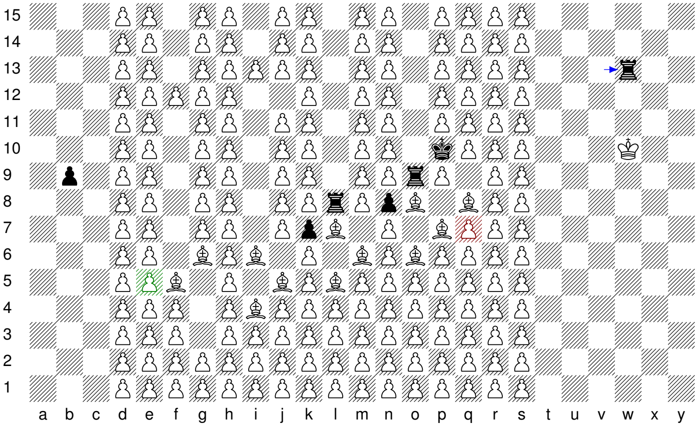

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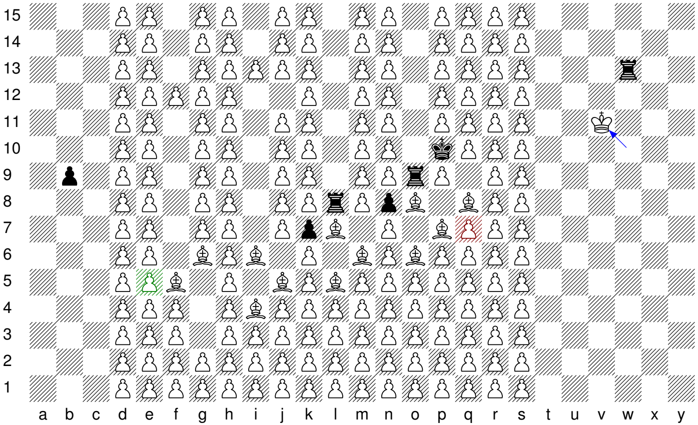

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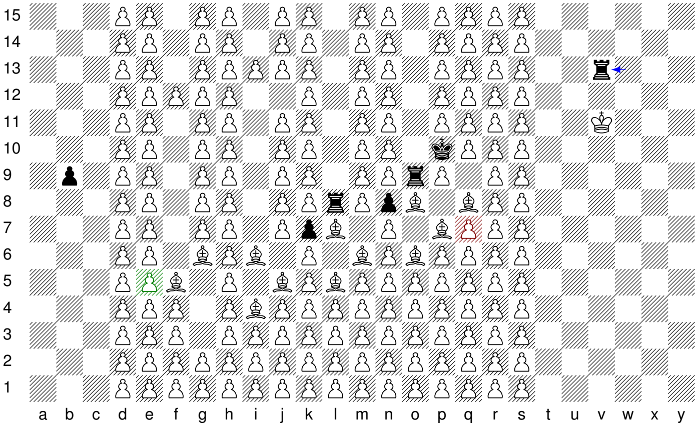

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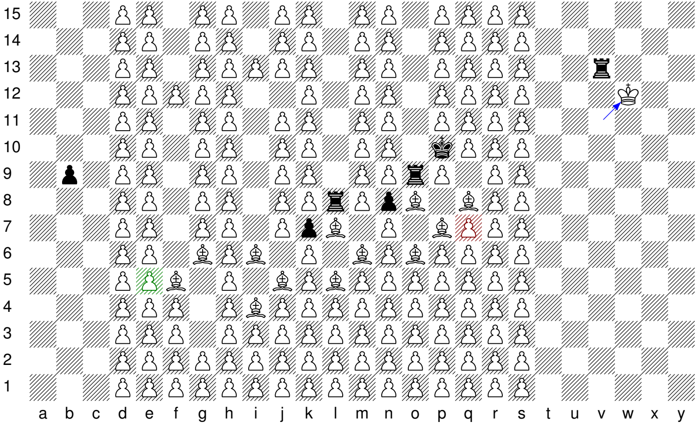

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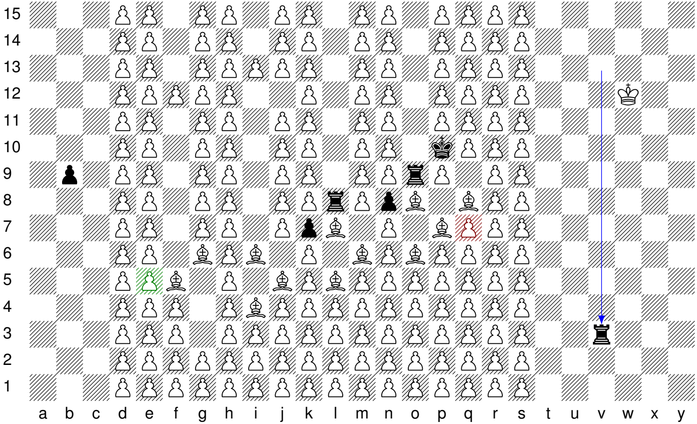
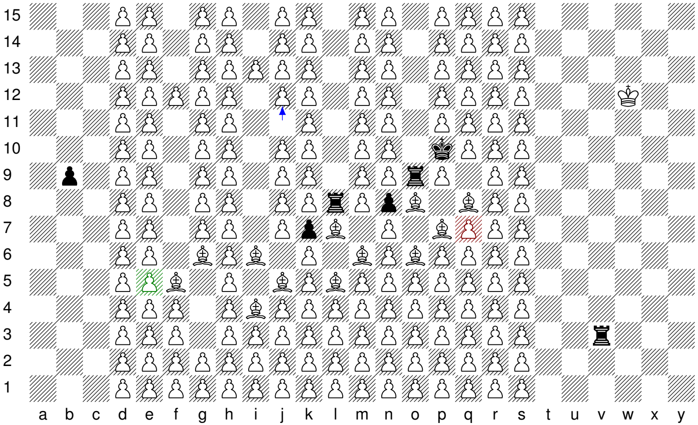


Black should actually move arbitrary distance right.

Iterated lock and key, value $\omega^2 \cdot 4$



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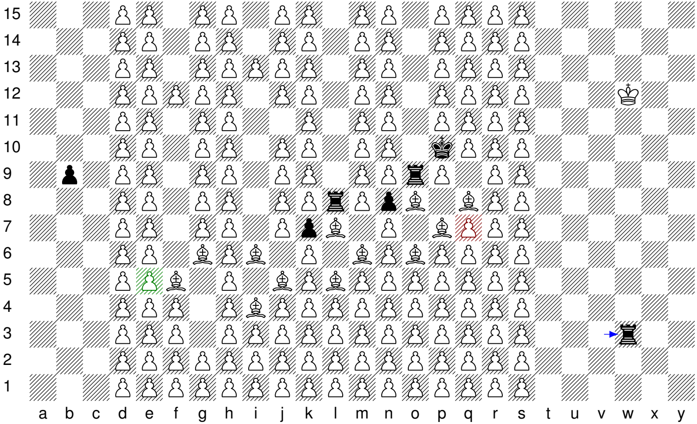

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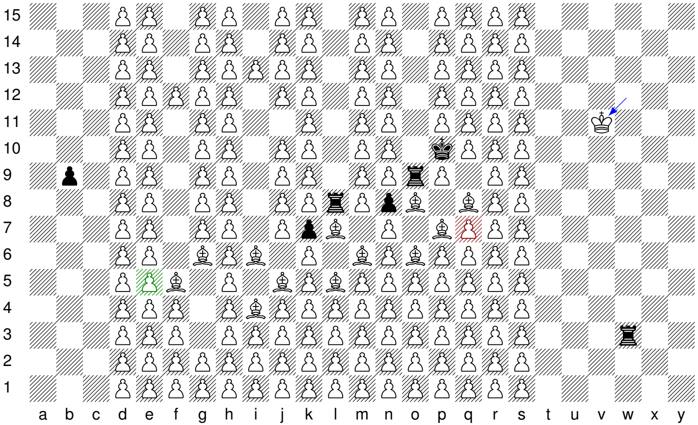

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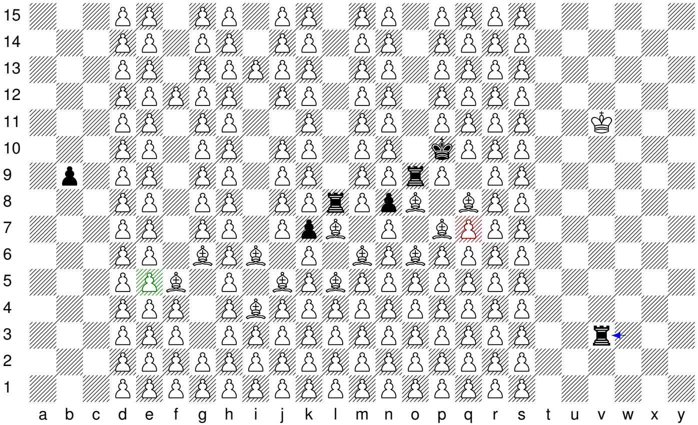

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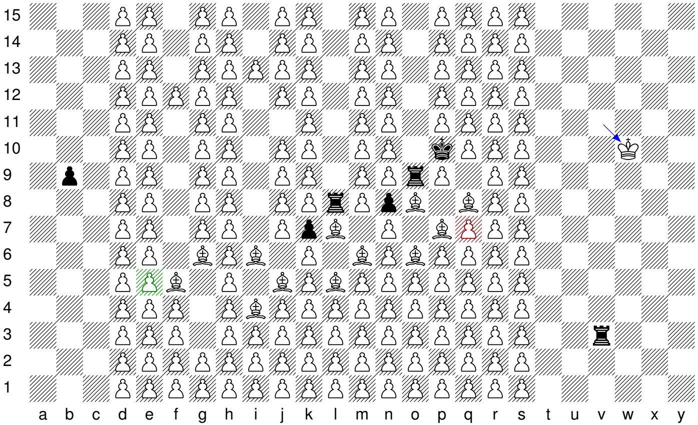

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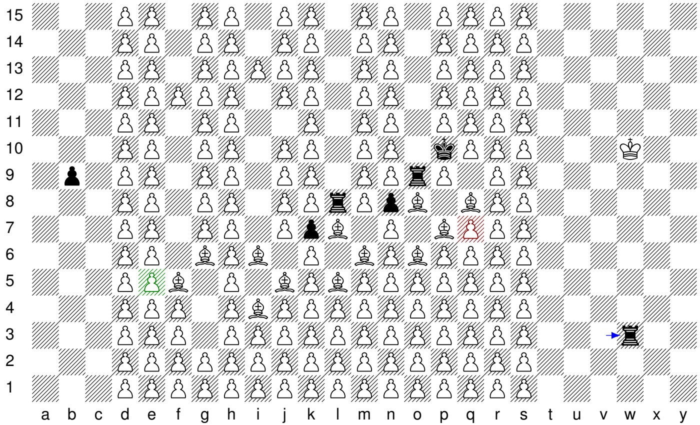

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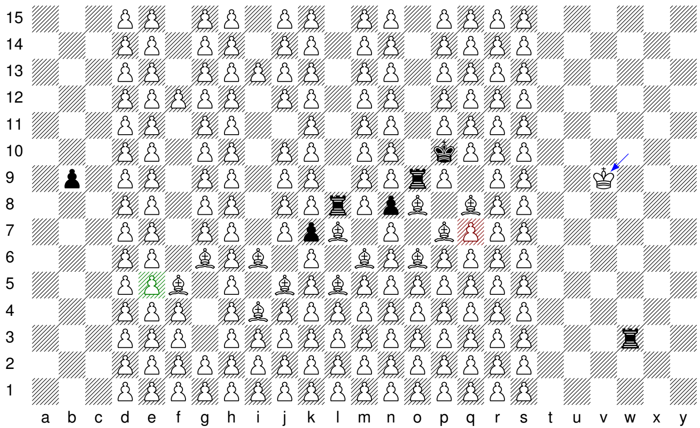

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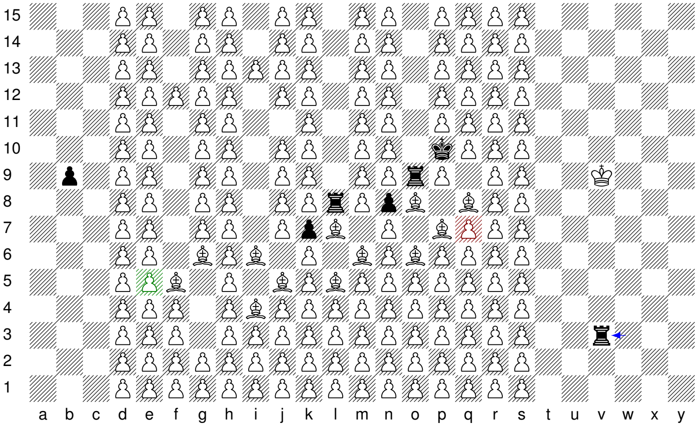

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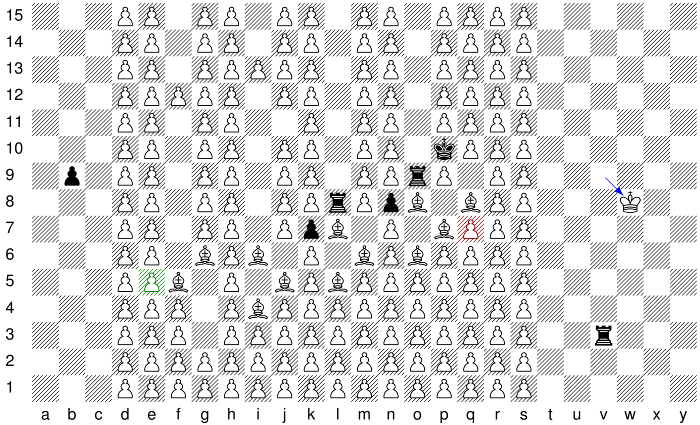

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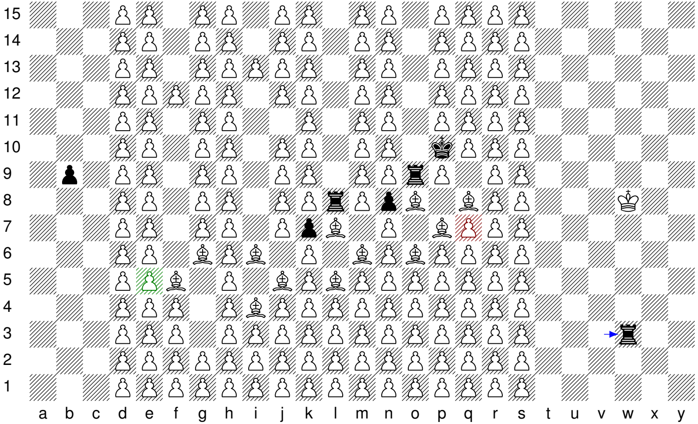

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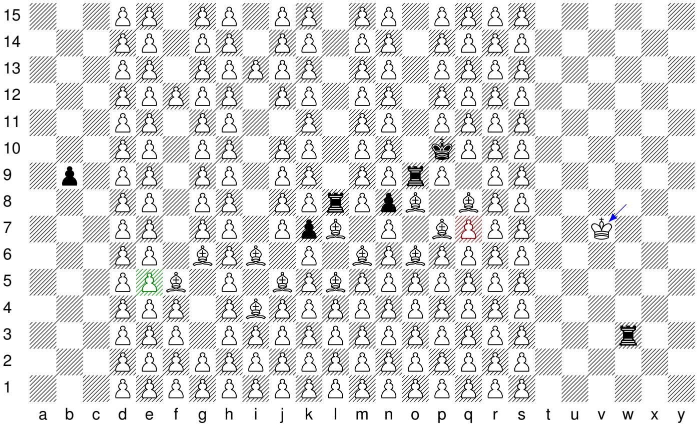

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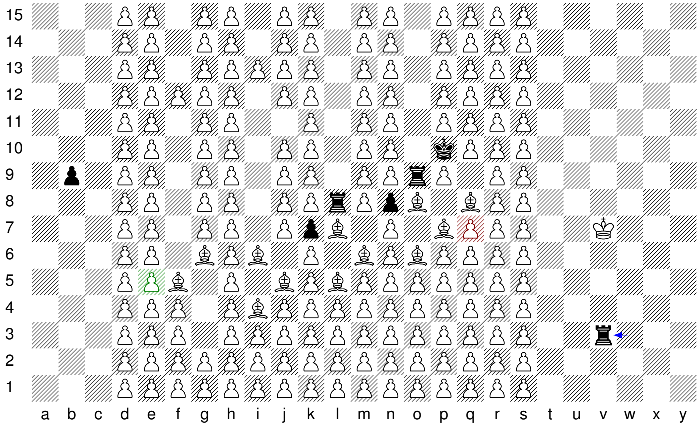

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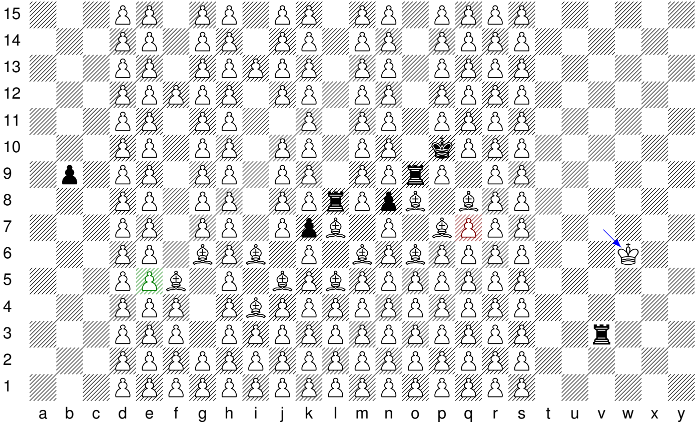

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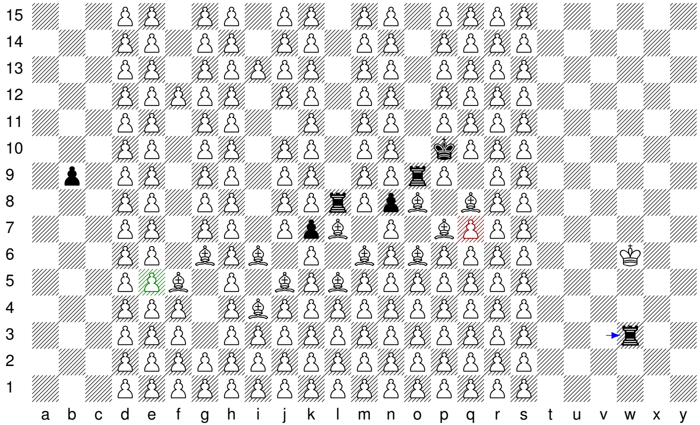

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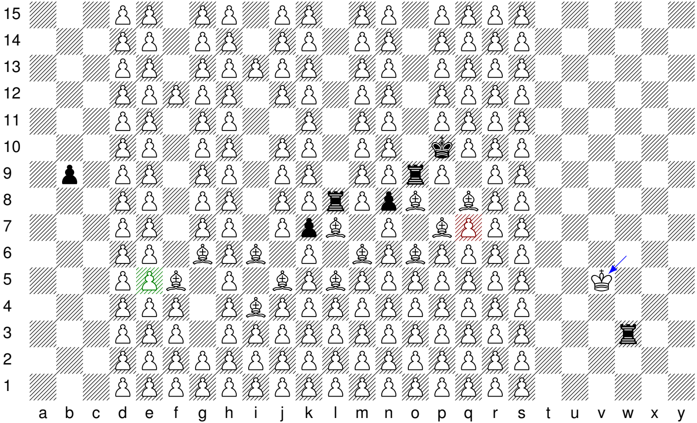

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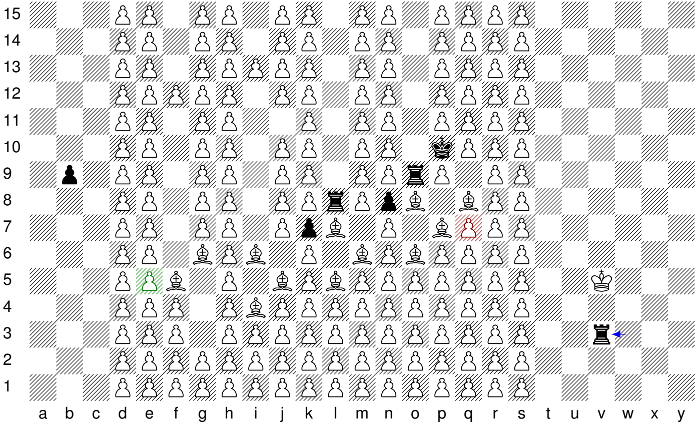

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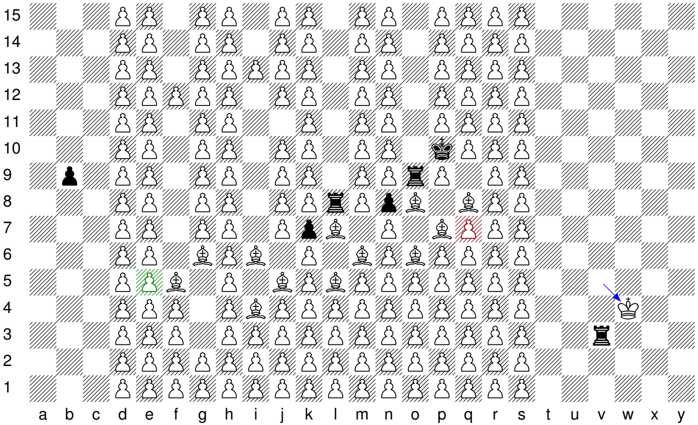

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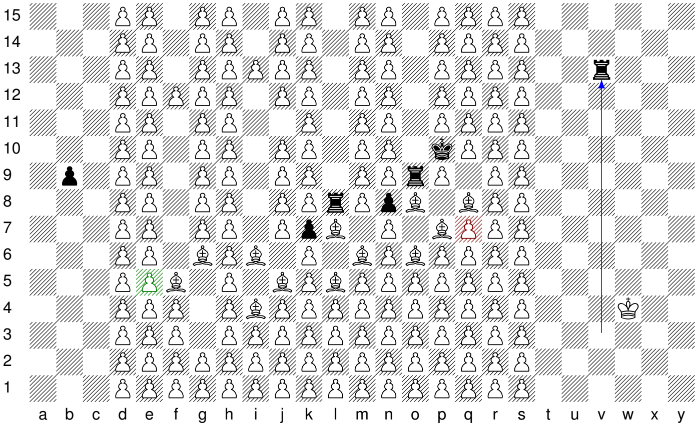
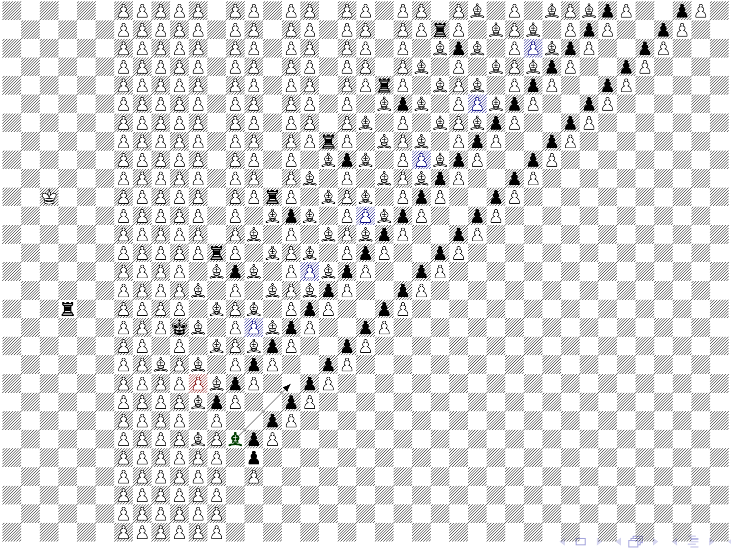


Black should actually move arbitrary distance right.

State of the art: value ω^3



The omega one of chess

$\omega_1^{\mathfrak{Ch}} =$ supremum of values of finite positions in infinite chess

$\omega_1^{\underset{\sim}{\mathfrak{Ch}}} =$ supremum of values of all valued positions

Question

How big is the omega one of chess?

$$\omega_1^{\mathfrak{Ch}} \leq \omega_1^{CK}$$

$$\omega_1^{\underset{\sim}{\mathfrak{Ch}}} \leq \omega_1$$

The omega one of chess

Conjecture (Evans, Hamkins)

The omega one of chess is as large as it could be.

$$\omega_1^{\mathfrak{Ch}} = \omega_1^{CK} \qquad\qquad \omega_1^{\underset{\sim}{\mathfrak{Ch}}} = \omega_1.$$

The best lower bounds are still very small...

$$\omega \cdot n + k < \omega_1^{\mathfrak{Ch}} \leq \omega_1^{CK}$$

$$\omega^3 \cdot 4 < \omega_1^{\underset{\sim}{\mathfrak{Ch}}} \leq \omega_1$$

But we conjecture that the upper bounds are met.

We haven't been able yet to prove this in two-dimensional infinite chess (but meanwhile, in *three* dimensions...)

Computability

Let’s explore a few computability issues that arise in connection with infinite chess.

Deterministic computable strategies

A position is computable if there is a computable function giving the locations and number of the pieces. (Complications...)

Consider that the players will both play according to deterministic computable strategies.

Question

Will that requirement affect our judgement of whether a position is winning?

Requiring computable play matters

Yes. Requiring players to play according to a deterministic computable procedure can affect our judgement of whether a given computable position is drawn or a win.

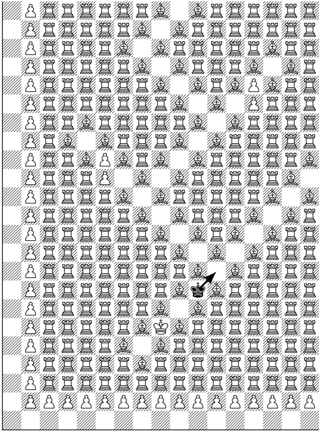
Theorem

There is a position in infinite chess such that:

1. *The position is computable.*
2. *The position is drawn. Neither player has a winning strategy, and either player can force a draw.*
3. *But among computable strategies, the position is a win for white. White has a computable strategy defeating every computable strategy for black.*

A position where computable play matters

Fix a computable tree $T \subseteq 2^{<\omega}$ with no computable branch.



White forces black to climb the tree; black hopes to avoid the dead-end traps.

The mate-in-*n* problem

There is a genre of chess problems, the mate-in-*n* problems

- White to mate in 2
- Black to mate in 3

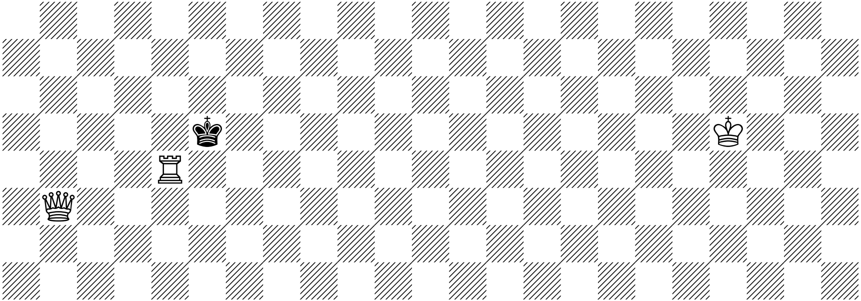
Consider the corresponding class of problems in infinite chess.

The mate-in-*n* problem

Given a finite position in infinite chess, can a designated player force a win in at most *n* moves?

A mate-in-12 problem

White to move on an infinite, edgeless board.



Can white force mate in 12 moves?

The mate-in-*n* problem

Question

Is the mate-in-*n* problem computably decidable?

A naive formulation of white to mate-in-3, with white to move:

> *There is a white move, such that for every black reply,*
> *there is a white move, such that for every black reply,*
> *there is a white move that delivers checkmate.*

$$\exists w_1 \forall b_1 \exists w_2 \forall b_2 \exists w_3 \cdots$$

Very high arithmetical complexity, $2n$ alternating quantifiers.

Infinite game tree

In finite $n \times n$ chess, one can search the entire game tree, which is finite.

In infinite chess, the game tree is not only infinite, but infinitely branching. The unbounded pieces (queens, rooks, bishops) may have infinitely many possible moves.

Thus, one cannot expect computably to search the entire game tree, even to finite depth.

The naive, brute-force-search manner of deciding the mate-in-n problem is simply inadequate.

The infinitary mate-in-n problem is decidable

Theorem (Brumleve, Hamkins, Schlicht)

The mate-in-*n* problem of infinite chess is decidable.

Further, there is a computable strategy for optimal play from any mate-in-*n* position.

Two proof methods:

- The structure of chess $\mathfrak{Ch}$, the space of all chess positions, with chess concept relations, is an *automatic* structure, whose theory is therefore decidable.
- The structure of chess is encodable in Presburger arithmetic $\langle \mathbb{Z}, +, < \rangle$, whose theory is decidable.

The winning-position problem still open

Our theorem does not show that the winning-position problem is decidable.

Question (Richard Stanley)

Is the winning-position problem for infinite chess decidable?

This question remains open, and is MathOverflow-hard.

The point is that a player may have a winning strategy from a position, but it is not mate-in-*n* for any *n*. So it is not clear that the winning positions are even c.e., or even arithmetic or hyperarithmetic.

These positions are precisely the positions with transfinite game value.

3D chess

Let us turn to three-dimensional chess.

3D chess

The history of 3D chess spans more centuries than you might expect:

- 19th century: Kieseritzkys Kubikschack in 1851.
- early 20th century: Maacks raumschach chess clubs, Hamburg from 1919 (first $8 \times 8 \times 8$, then $5 \times 5 \times 5$)
- late 20th century: I played $8 \times 8 \times 3$ chess as a child.

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- late 20th century: I played 8 × 8 × 3 chess as a child.
- 23rd century: Spock and Kirk play on several Star Trek episodes.

Infinite 3D chess

We consider *infinite* three-dimensional chess, 3D chess on infinite boards with no boundary.

One must specify the piece movement; there is room for reasonable disagreement.

The omega one of 3D infinite chess

Theorem (Evans, Hamkins)

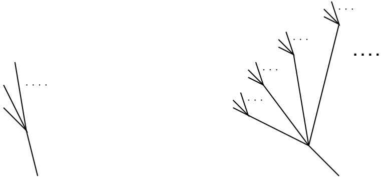
The ω_1 of infinite 3D chess with infinite positions is true ω_1, as large as it could possibly be.

$$\omega_1^{\underset{\sim}{\mathfrak{Ch}}_3} \quad = \quad \omega_1$$

Every countable ordinal therefore arises as the game value of a position in three-dimensional infinite chess.

The ω_1 of 3D infinite chess

The basic idea is to code an arbitrary well-founded tree $T \subseteq \omega^\omega$ as a position in infinite chess.

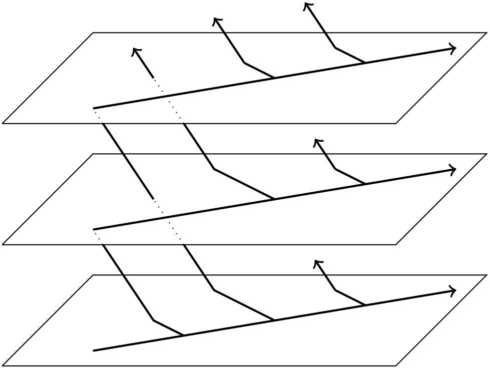


Every such tree admits an ordinal ranking function.

We create a chess position that is like climbing through such a well-founded tree.

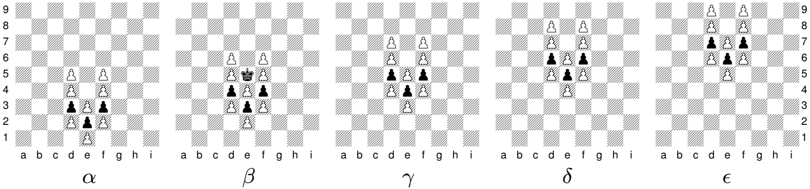
The difficulty is that the tree must be infinite-branching to achieve large ranks. How to force black to make a choice?

Embedding a tree into 3 space



We embed the tree, with the infinite-branching nodes of T simulated individually on separate layers

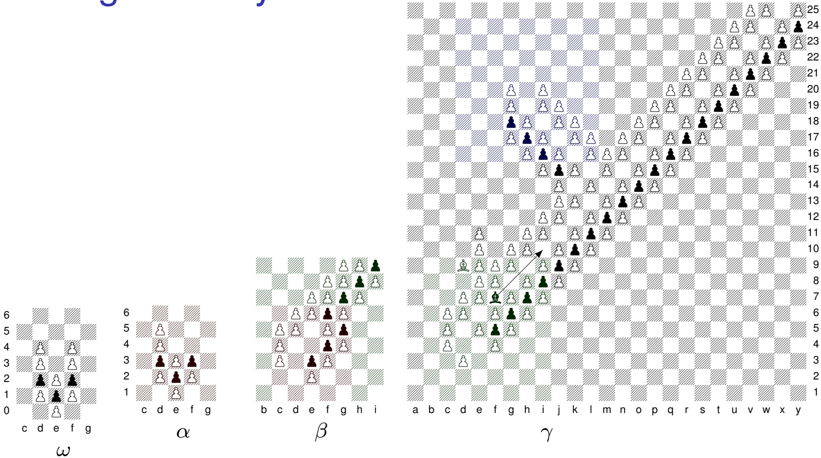
Ascending stairs in 3D chess



The black king is forced to ascend the stairs via

1.αe4+ Kγe6 2.βe5+ Kδe7 3.γe6+ Kϵe8 4.δe7+.

Branching node layer in 3D infinite chess



Black has mate-in-2 elsewhere. Play is tightly forced. Transition from stairwell to branching layer γ; additional pawns in layer above and below will confine black king to the channel.

The omega one of infinite 3D chess

Summary of the argument:

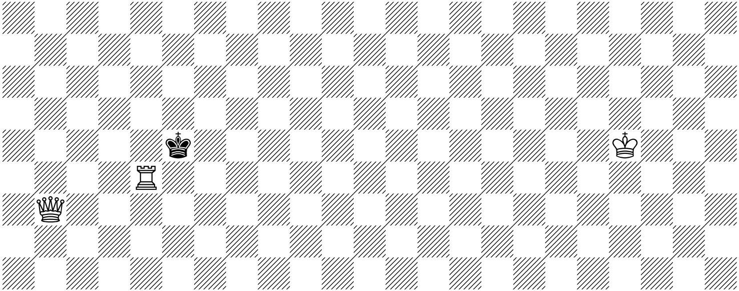
- Every well-founded tree T can be embedded into a position of infinite 3D chess.
- Chess play from those positions corresponds to climbing-through-T.
- Thus, the rank of T is a lower bound for the game value of the position.
- Consequently, the values of positions in infinite 3D chess are unbounded in ω_1.

The ω_1 of infinite 3D chess, therefore, is true ω_1, as large as it could possibly be.

$$\omega_1^{\underset{\sim}{\mathfrak{Ch}}_3} = \omega_1$$

Solution to the mate-in-13 problem

Lastly, let's give the solution of the mate-in-13 problem.



White to move.

White mates in 13, but not 12. The kings must come together.

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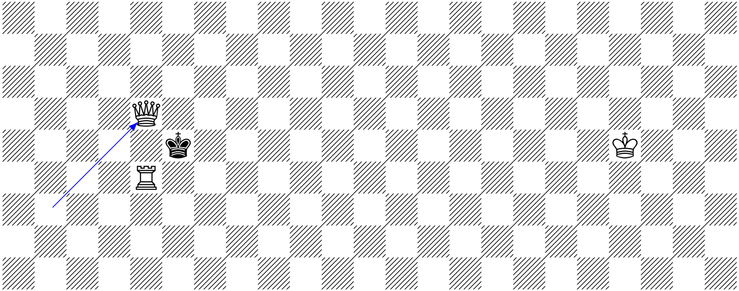

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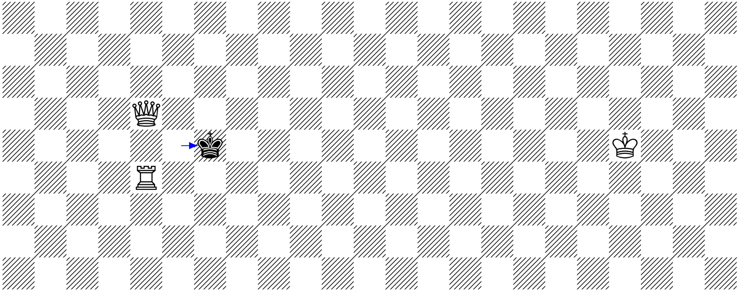

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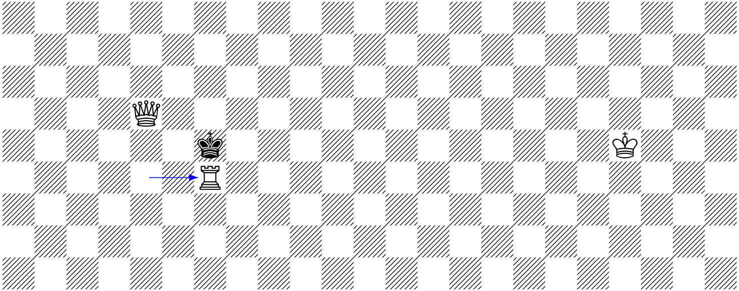

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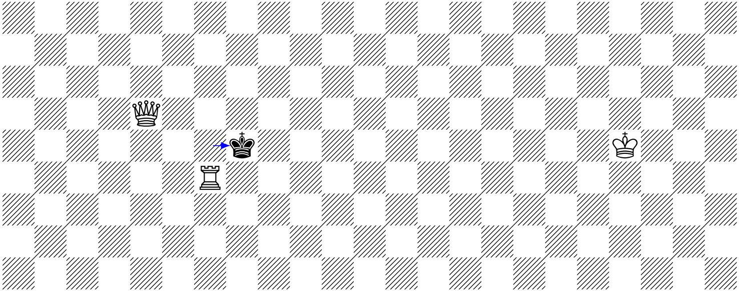

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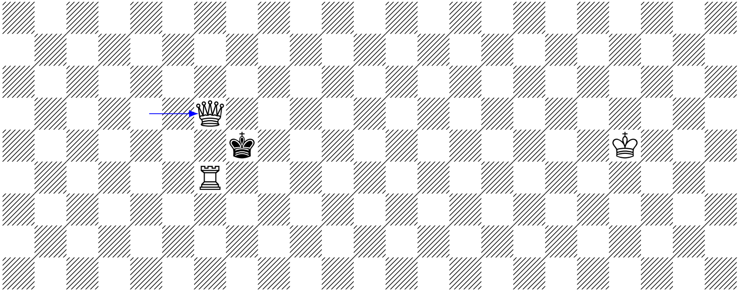

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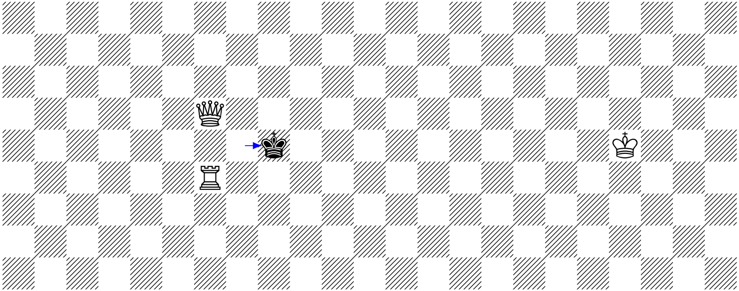

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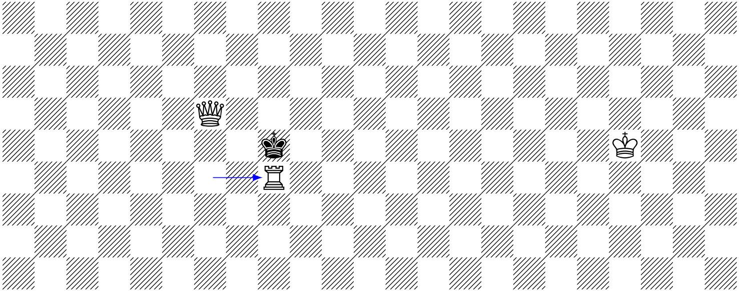

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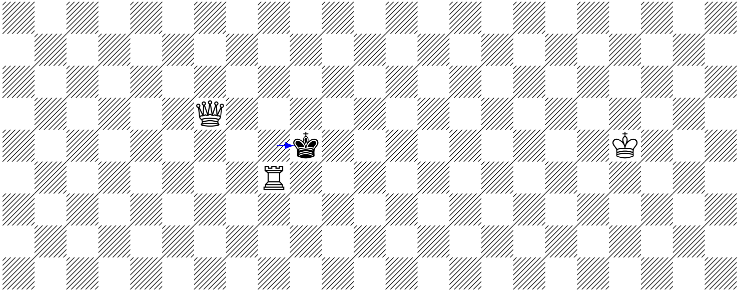

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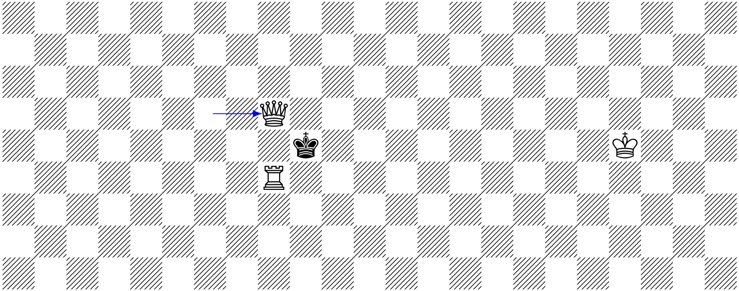

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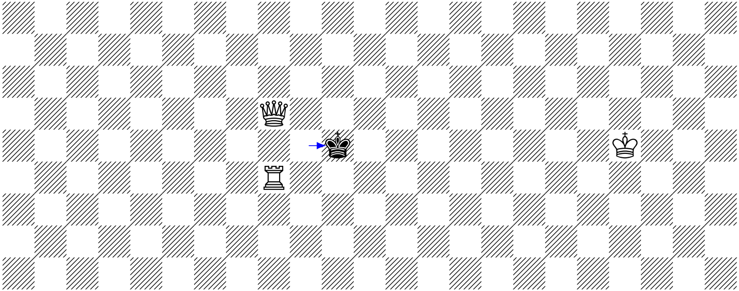

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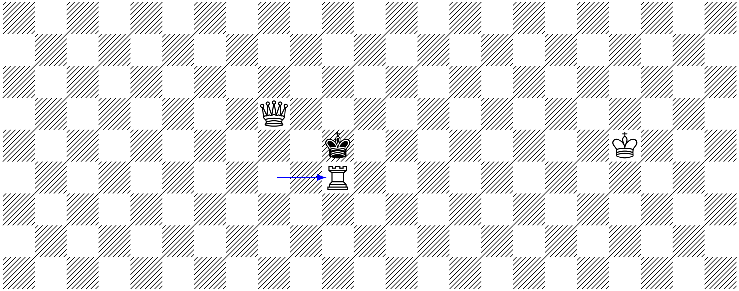

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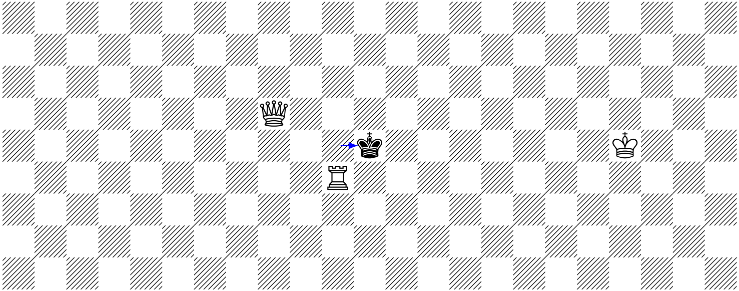

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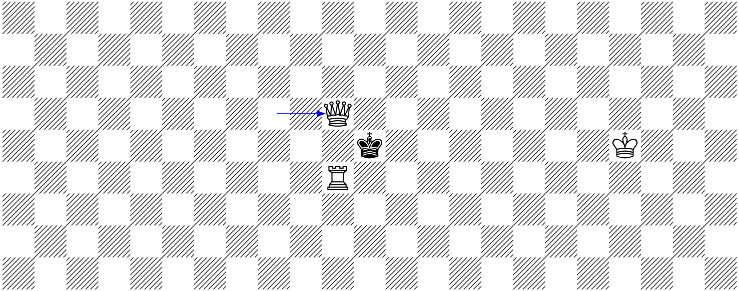

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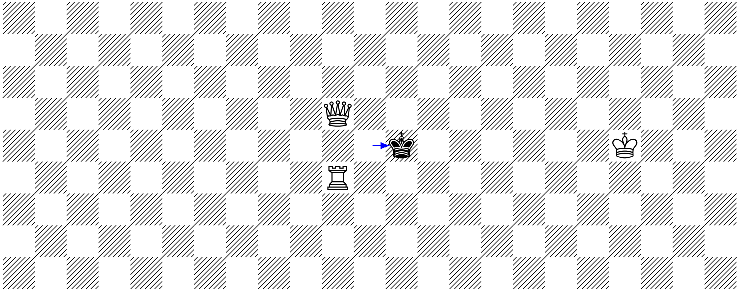

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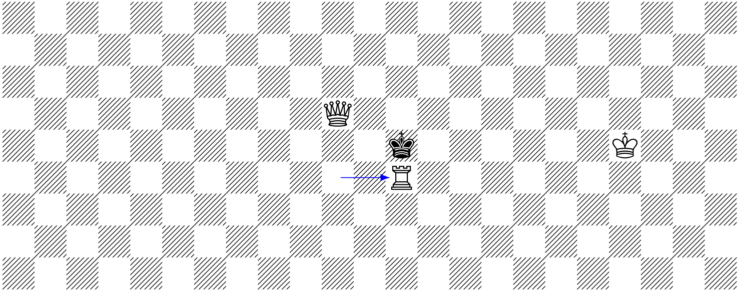

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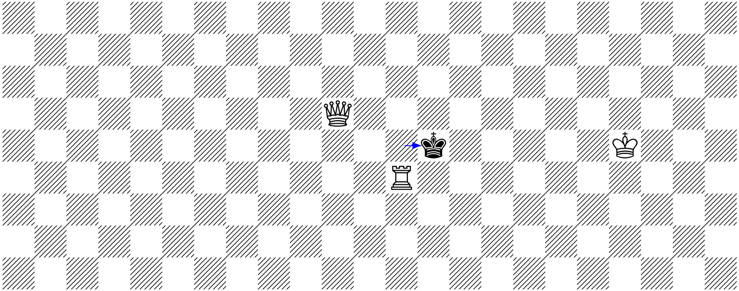

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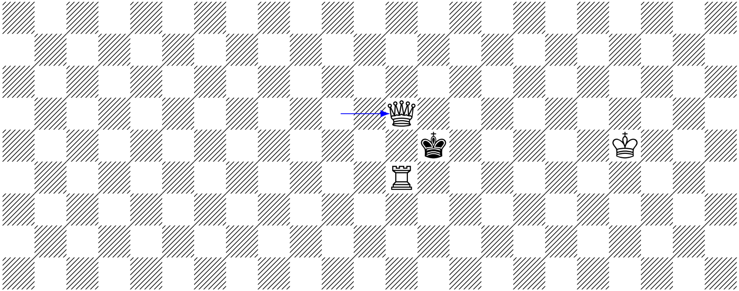

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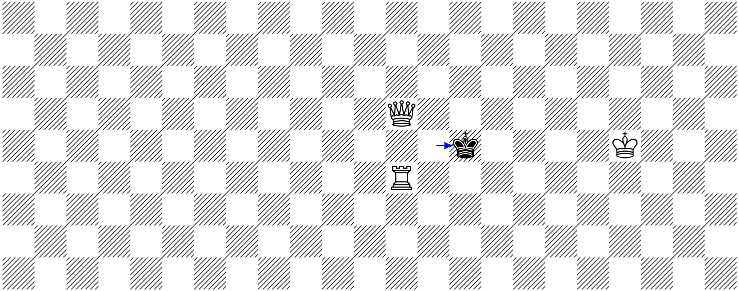

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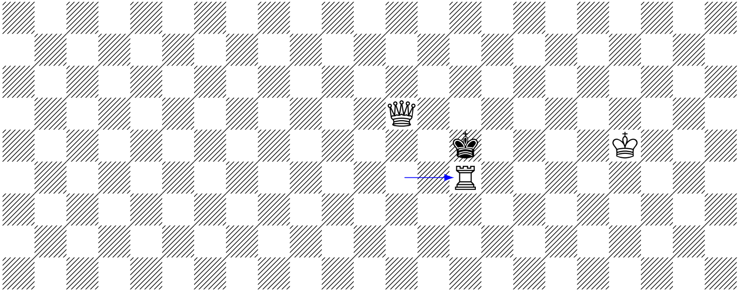

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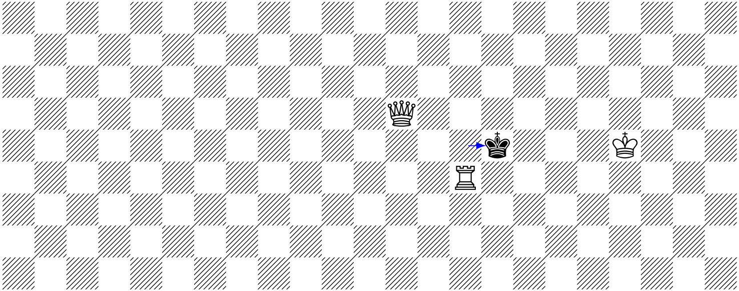

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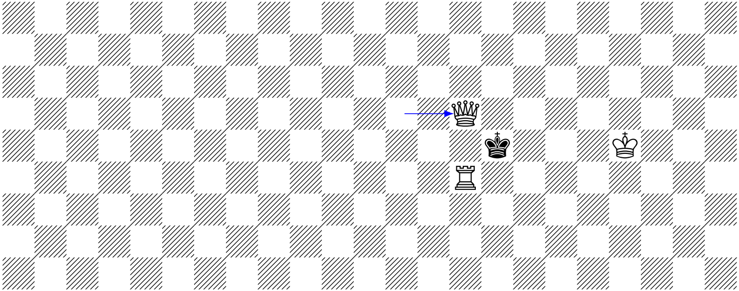

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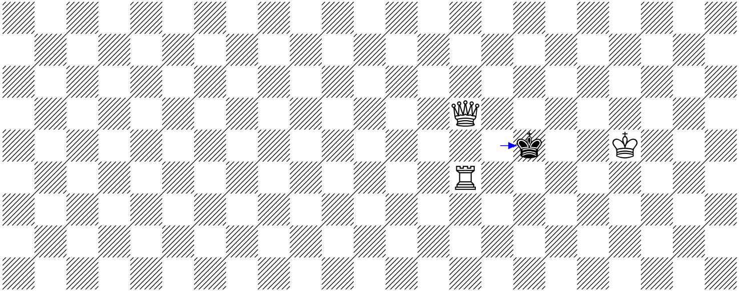

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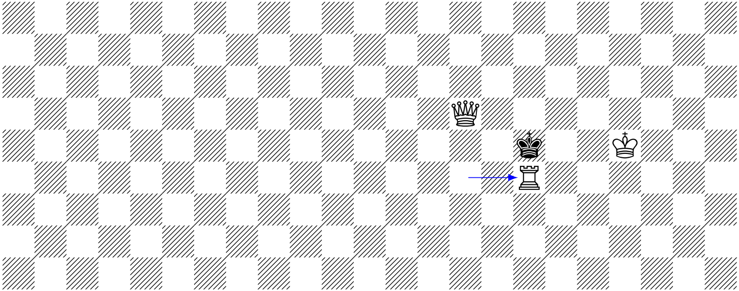

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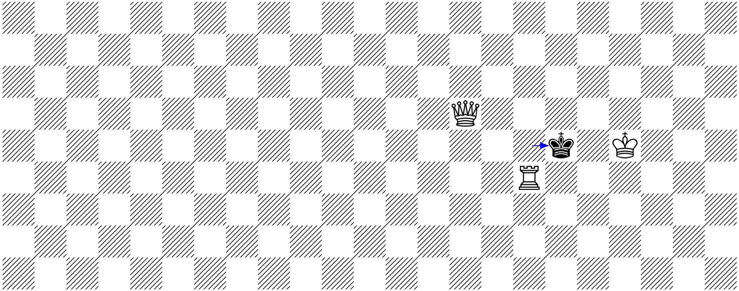

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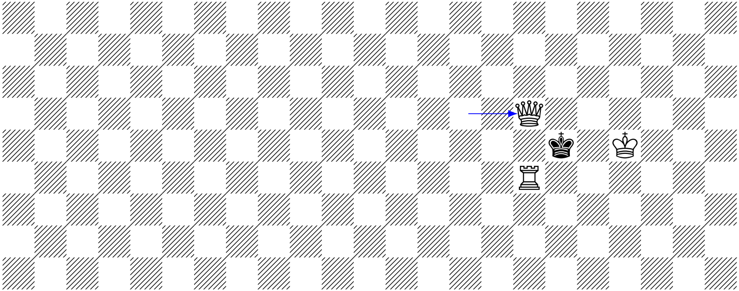


Checkmate

White mates in 13, but not 12. The kings must come together.

Thank you.

- C. D. A. Evans, J. D. Hamkins, “Transfinite game values in infinite chess,” to appear in Integers Journal.
- D. Brumleve, J. D. Hamkins, and P. Schlicht, “The mate-in-n problem of infinite chess is decidable,” in How the world computes, S. Cooper, A. Dawar, and B. Löwe, Ed., Springer Berlin Heidelberg, 2012, vol. 7318, pp. 78-88.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
The City University of New York

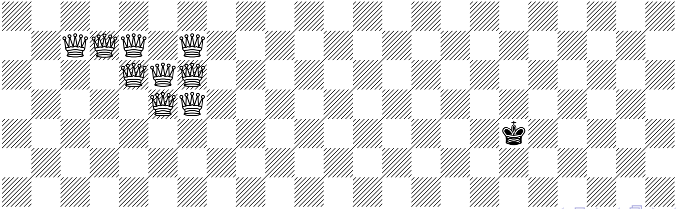
New starting procedure

From the December 2011 IBM *Ponder This* problem.

White places any number of queens on the board. Black then places his king anywhere, and play commences.

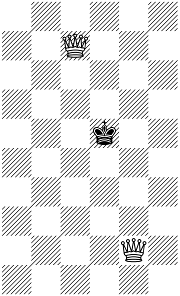
Question

How many queens suffice to ensure checkmate?



Two queens suffice

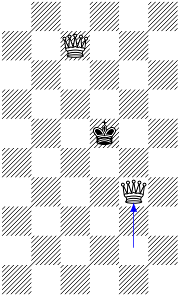
First, trap the king in a column. Then move a knight's move away, then again a knight's move away.



Black to move.

Two queens suffice

First, trap the king in a column. Then move a knight's move away, then again a knight's move away.



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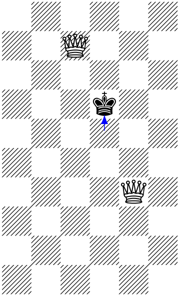

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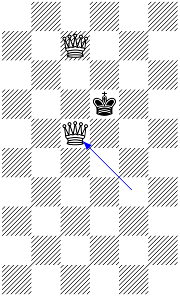
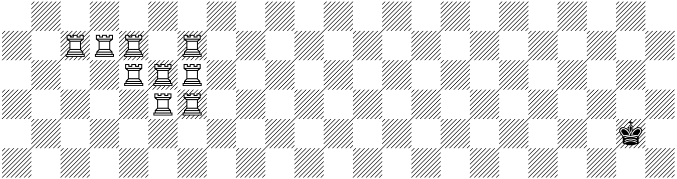


Checkmate

How many rooks do you need?

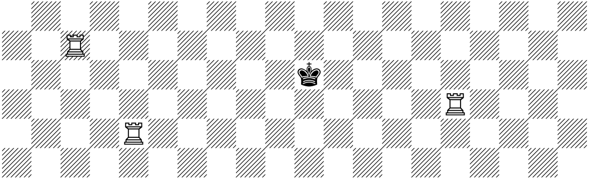
Consider the similar problem, where white places rooks, and then black places a king.



How many rooks suffice?

Three rooks are necessary

Clearly, with three rooks, we can do it.



Black to move.

Two rooks are insufficient, since there is no checkmate position with two rooks.

Three rooks are necessary

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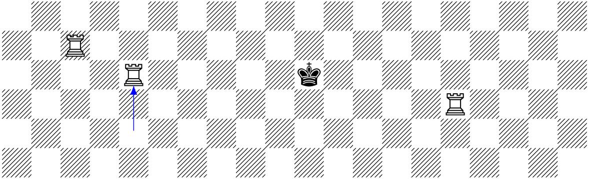
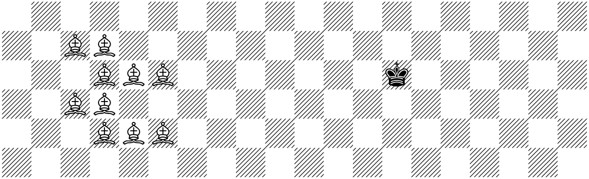


Checkmate

Two rooks are insufficient, since there is no checkmate position with two rooks.

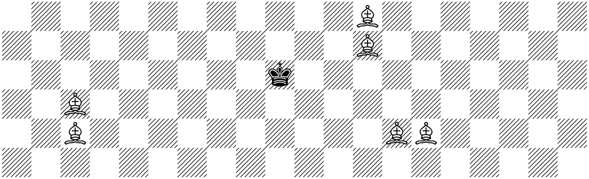
How many bishops



How many bishops do you need?

Six bishops required

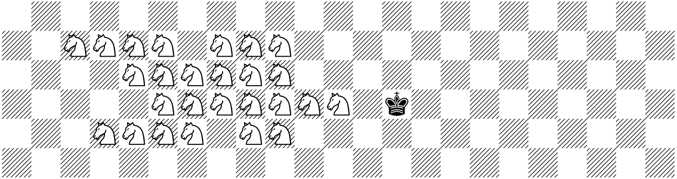
Six suffice: use two pairs of two to close two walls. Extra bishop (of correct color) delivers checkmate



Black to move.

Five do not suffice: the black king will stay on the color having at most two bishops of that color. There is always a free move (or stalemate), since double check cannot arise.

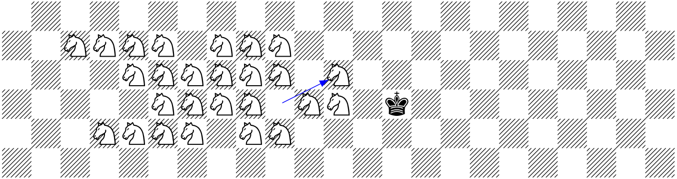
How many knights?



Black to move.

Is this enough to ensure checkmate?

How many knights?



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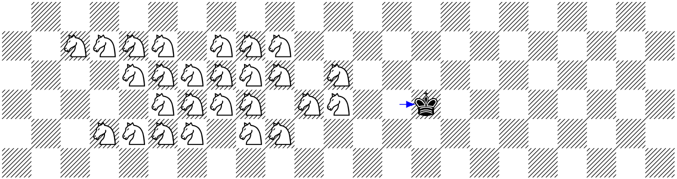

Is this enough to ensure checkmate?

No, black can outrun the knights, since two knights are insufficient to checkmate.

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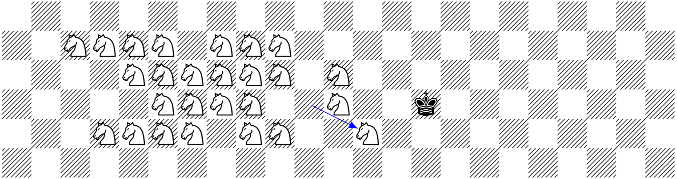

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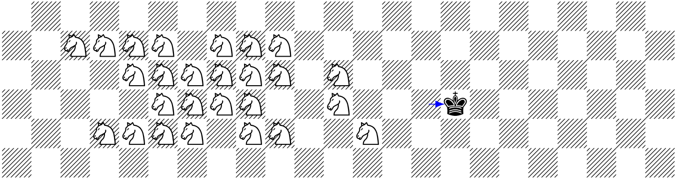

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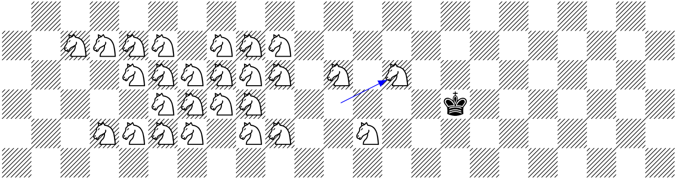

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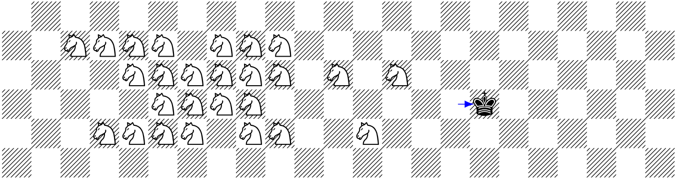

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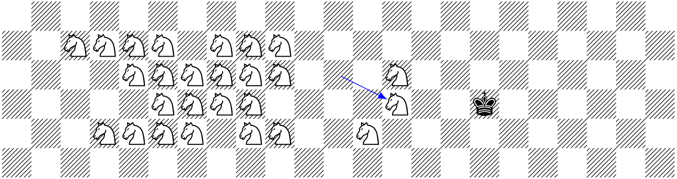

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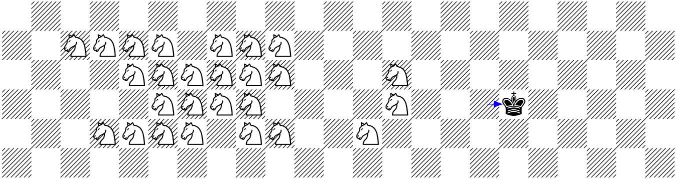

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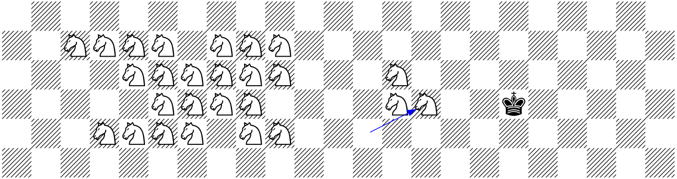

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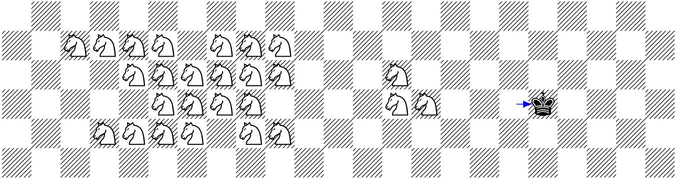

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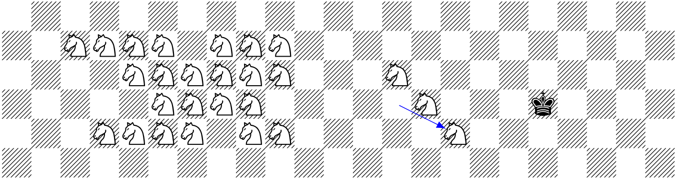

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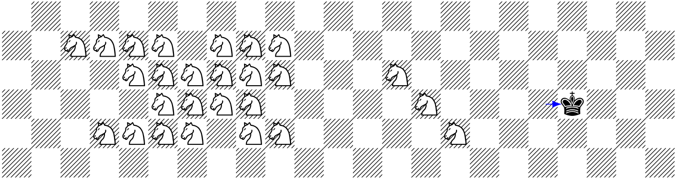

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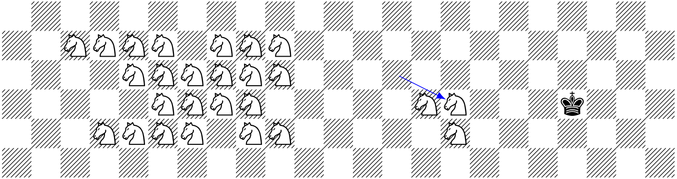
How many knights?



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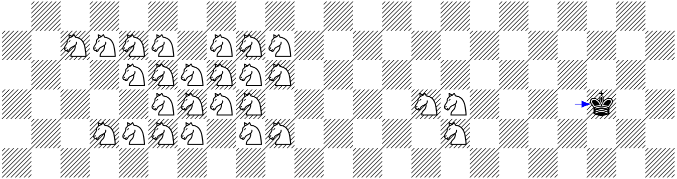

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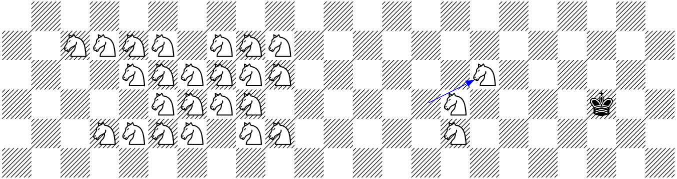

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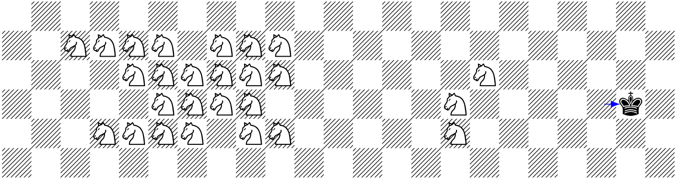

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