Logic 4/29/15

Incompleteness

Hilbert's program (reaction to set theoretic paradoxes) Goal: provide a secure foundation for arithmetic by providing a finite list of axioms for arithmetic that were consistent and complete. (using only the axioms, ideally?)

Peano arithmetic was a proposed axiomatization. $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$ axioms: familiar statements plus "the induction scheme" (for: $\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x+1)) \Rightarrow \forall x\, \varphi(x)$ (very powerful)

Hilbert wanted to prove that P.A. was complete and consistent (though it's not finite)?

If there were a finite, complete, consistent set of axioms for arithmetic, then they'd eventually settle all questions systematically.

If not, there are questions not settled by the axioms - we'd have to justify x or ~~other~~ some other way if we can justify it at all.

KG Theorem (Proof by Kleeny (sp?))

If T is any true, completely axiomitizable theory of arithmatic, then T does not prove all true statements (i.e. is incomplete).

Suppose T is true and has a complete list of axioms. Generate all proofs from those axioms.

Then we can solve the halting problem, i.e. given a turing machine P and an input k, we can form the statement "P halts on k."

Q. but is P expressable in the language T?
Only if so is this a reduction.

But we know that the halting problem is not decidable. ⊥

Fine version of KG (KGG):

. If T is any true, completely axiomitizable theory, then there is some TM w/ some input k such that P does not halt on k, but T does not prove this.

Proof: if T proves all instances of non halting, then we can solve the halting problem.

Given P, K, half time run P(K), half time look for a proof that P(K) doesn't halt. So we solve the halting problem, ↯.

So there MUST be some formula that does not halt but the TM does not prove it.

Theorem G: for any computably axiomatizable true theory T in the language of arithmetic, there is a polynomial $P(x_1 \dots x_n)$ with integer coefficients such that $P(x_1 \dots x_n) = 0$ has no solution in the integers, but T does not prove this.

(consequence of MRDP theorem)

Godel's incompleteness theorem:

key idea arithmetization of syntax → all finite combinatorial mathematical objects can be represented as numbers.

i.e. we can extract all information of any object from its numerical representation.

Gödel coding:

Formal language of arithmetic (+, ·, ≤, ÷, ≤, ∀, ∃, ∧, ∨, ... etc.)

assign each symbol a Gödel code, a number (e.g. + = 2, ∀ = 3 ... etc

now, finite sequences of symbols have their own Gödel code

* e.g. "$\forall x \; x+0=x$"

$n_0, n_1, n_2 \ldots n_k$ might be the individual codes of the symbols in *, you can give a single number that represents that sequence, e.g. $2^{n_0} \cdot 3^{n_1} \cdot 5^{n_2} \cdots p_k^{n_k}$, which definitely represents *.

So: all of the syntactic operations on formulas are representable in the language of arithmetic via Gödel coding.

Thus whether a given program halts on an input is expressable in arithmetic, b/c it can be expressed as a number (I don't really understand why which Gödel should show this?)

So syntactic properties are expressable too.

"z is the code of an axiom of T"
"a consequence of T"

$Pr_T(\ulcorner \varphi \urcorner) \longrightarrow$ "φ is provable in T" **

** is a consequence of a syntactic property.

also expressable in arithmetic, b/c you're ~~the~~ "just saying that $\exists y$ " y is the code of a proof of $\ulcorner \varphi \urcorner$ in T." and that's expressable b/c our notion of proof is syntactic.

Con T ("T is consistent") is expressable b/c it just says: "It's not the case that T proves a $\perp$". i.e. $\neg Pr_T(\ulcorner 0 = 0 \wedge 0 \neq 0 \urcorner)$

So assertions about numbers can be assertions about arithmetic.

<u>Fixed pt. Lemma (Carnap):</u>

For any formula $\varphi(x)$ in the language of arithmetic, there is a sentence ψ such that $PA \vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner)$

Gödel coding fixed

ψ says that it's code has property φ.

$\psi \leftrightarrow \varphi(n)$ where n is the code of ψ.

Proof: fix $\varphi(x)$

define $sub(\ulcorner \varphi(x) \urcorner, n) = \ulcorner \varphi(\underbrace{1+\cdots+1}_{n}) \urcorner$

Let $\theta(x) = \varphi(sub(x, x))$

Let $n = \ulcorner \theta(x) \urcorner$

Let $\psi = \theta(n)$

$PA \vdash$

$$\begin{aligned}
\psi &\longleftrightarrow \theta(n) \\
&\longleftrightarrow \ell(\mathrm{Sub}(n,n)) \\
&\longleftrightarrow \ell(\mathrm{Sub}(\ulcorner\theta(x)\urcorner, n)) \\
&\longleftrightarrow \ell(\ulcorner\theta(n)\urcorner) \\
\psi &\longleftrightarrow \ell(\ulcorner\psi\urcorner) \quad \checkmark
\end{aligned}$$

So this allows us to have self reference. - indeed, it's everywhere.

ψ says: "ℓ holds of the Gödel code of ψ"

ψ asserts that ℓ holds of its Gödel code.

~~Theorem~~ <u>Gödel's 1st Incompleteness Theorem</u>

If T is a true, representable/recursively axiomatizable, ~~consistent~~ theory of arithmetic, then T is incomplete.

T must contain PA

<u>Proof:</u>

By the fixed point lemma, there is a sentence ψ such that $PA \vdash \psi \longleftrightarrow \neg Pr(\ulcorner\psi\urcorner)$.

$\psi \longleftrightarrow$ "ψ is not provable"

If $T \vdash \psi$, then $T \vdash Pr_T(\ulcorner\psi\urcorner)$, so

$T \vdash Pr_T(\ulcorner\psi\urcorner) \wedge \neg Pr_T(\ulcorner\psi\urcorner)$.

So T is inconsistent.

So T does not prove ψ. So ψ is true!

Godel's 2nd incompleteness Theorem:

If T is true and representable and extending PA, then $T \nvdash \text{Consistent}(T)$.

Proof:

Fix ψ as before. $PA \vdash \psi \leftrightarrow \neg Pr_T(\ulcorner T \urcorner)$

where ψ = "I am not provable in T"

$PA \vdash \psi \rightarrow \text{Consistent}(T)$, since if there's anything at all not provable by a theory, then the T is consistent.

Claim: $PA \vdash \text{Consistent}(T) \rightarrow \psi$, i.e.

$PA \vdash \psi \leftrightarrow \text{Consistent}(T)$.

need to go back, check this reasoning.

Proof: we argued before that if T is consistent, then $T \nvdash \psi$.

Formalize this argument inside T, which includes PA. We get a proof that $\text{Consistent}(T) \rightarrow \neg Pr_T(\ulcorner \psi \urcorner)$, but that's eq. to ψ,

so $T \vdash \text{Con}\,T \rightarrow \psi$.

but it doesn't!

so $T \nvdash \text{Con}\,T$

Hilbert–Bernays derivability conditions (aka Löb?)

1) • If T proves φ, then $T \vdash Pr_T(\ulcorner \varphi \urcorner)$

2) • $T \vdash Pr_T(\ulcorner \varphi \urcorner) \rightarrow Pr_T(\ulcorner Pr_T(\ulcorner \varphi \urcorner) \urcorner)$

3) • $T \vdash Pr_T(\ulcorner \varphi \urcorner) \wedge Pr_T(\ulcorner \varphi \rightarrow \sigma \urcorner) \rightarrow Pr_T(\ulcorner \sigma \urcorner)$

Löb's Theorem:

If $T \vdash Pr_T(\ulcorner \varphi \urcorner) \to \varphi$

Then $T \vdash \varphi$.

3) is very easy. 1) is easy as you have (3)?

(2):

Formalize the previous implication inside T

From (1), (2), and (2):

$\psi \iff \neg Pr_T(\ulcorner \psi \urcorner)$

(1) $T \vdash Pr_T(\ulcorner \psi \to \neg Pr_T(\ulcorner \psi \urcorner) \urcorner)$

(3) $T \vdash Pr_T(\ulcorner \psi \urcorner) \to Pr_T(\ulcorner \neg Pr_T(\ulcorner \psi \urcorner) \urcorner)$

by (2): $T \vdash Pr_T(\ulcorner \psi \urcorner) \to Pr_T(\ulcorner Pr_T(\ulcorner \psi \urcorner) \urcorner)$

So $T \vdash Pr_T(\ulcorner \psi \urcorner) \to Pr_T(\ulcorner Pr_T(\ulcorner \psi \urcorner) \wedge Pr_T(\ulcorner \psi \urcorner) \urcorner)$, so you've proved that if ψ is provable, I can prove $\bot$, which proves that $\neg \psi \to \neg Con(T)$. So $T \vdash Con_T \to \psi$.

$T \vdash \neg \psi \to \neg Con\, T$

$T \vdash Con_T \to \psi$.

Gödel assumed that T was ω-consistent. He only needed to assume that it was ω-consistent.

<u>Theorem (Gödel/Rosser Theorem)</u>

Every consistent representable theory T extending PA is incomplete

Proof: Use Rosser sentence (σ)

$PA \vdash \sigma \longleftrightarrow$ "for any proof of $\ulcorner \sigma \urcorner$ in T there is a shorter proof of $\ulcorner \neg\sigma \urcorner$"

If $T \vdash \sigma$ (the proof has some length k), then T must prove that there is a shorter proof of $\ulcorner \sigma \urcorner$, of length $< k$.

But then $T \vdash \neg\sigma$. $\perp$, T is inconsistent.

If $T \vdash \neg\sigma$, then there's a proof of length k

$T \vdash$ There is a proof of σ w/ no shorter proof of $\neg\sigma$. So the whatever proof of σ has to be shorter than our proof of $\neg\sigma$.

So $T \vdash \neg\sigma$. $\perp$.

So T does not prove σ or $\neg\sigma$, incomplete.

So we cannot find complete axiomatizations of Mathematics. So we should expect a pervasive independence phenomenon.

$\forall T\ \exists$ statement σ such that σ is not settled by T.

Eg. ZFC axiom
continuum hypothesis, not settled, etc.

Since you can never prove the consistency of T ~~&~~ from the axioms of T, then no consistent theory can be complete.

Goodstein sequences

$a_2 = 42 = 2^{2^3+1} + 2^{2+1} + 2$ (complete base 2)

$a_3 = 3^{3^3+1} + 3^{3+1} + 3 - 1 = 3^{3^3+1} + 3^{3+1} + 2$

$a_4 = 4^{4^4+1} + 4^{4+1} + 1$

$a_5 = \ldots$

$a_6 = 6^{6^6+1} + 6^{6+1} - 1 = \ldots$

$\vdots$

<u>Goodstein's Theorem</u>

$\forall a_2$

$\exists n \quad a_n = 0$?!?

<u>Theorem (Kirby & Paris)</u>

Goodstein's Theorem is not provable in PA ... It's true but not provable in PA. It ~~[illegible]~~ is provable in set theory but not arithmetic.

<u>Hydra Theorem:</u>

(1) every strategy succeeds in killing the hydra. ~~[illegible]~~ wow!

(2) the assertion of (1) is not provable in PA (just in set theory).

PA is not sufficient?