Logic notes 3/25/15

<u>Definability</u>

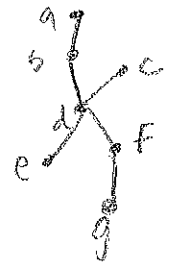
two kinds:

1) if M is a structure, then an element a in M is definable in M if there is a formula $\ell(x)$ such that M satisfies $\ell(a)$ ($M \models \ell(a)$) and a is the only object satisfying ℓ in M.

2) if $A \subseteq M$, then A is a definable class in M if A is the set of all objects that satisfy a certain property for some formula ℓ.

i.e. $A = \{a \in M \mid M \models \ell(a)\}$ for some formula ℓ

Eg. Partial order $\leq$



ℓ:

$x = d$ iff $\forall y (x \leq x \vee x \leq x)$

we can talk about d now because its definable - we can refer in our language to the unique object satisfying ℓ.

now we can define other things by relation to d.

$y = b$ iff $d < y \wedge \exists x \; y \leq x$

We can write that without referring directly to d.

$\forall z$ "$z=d$" $\rightarrow z<y \wedge \exists x\, y<x$)

$\forall z\, (\forall y\, y \leq z \vee z \leq y) \rightarrow z<y \wedge \exists x\, y<x$

So its kosher. (As only kosher b/c we can write it with $\leq$ b/c $\leq$ is definable in our language)

$u=a$ iff $b<u$

$v=c$ iff $d<v \wedge v \neq a \wedge v \neq b$

or $v \neq a \wedge \neg \exists u\, v<u$.

etc. with the rest. So all those points are definable.

$\boxed{\text{Ex 1}}$ Consider Partial orders:

(i) 

(ii) 

Which elements are definable? Provide definitions. Which elements are <u>indiscernable</u>? Give a justification.

If a, b are indiscernable, "interchangable" ~~are~~, that is an isomorphism. Then neither is definable?

$\langle \mathbb{N}, < \rangle$ $(0, 1, 2, \ldots)$

all elements are definable. (a stronger property than being Liebnizian)

you can say that there are exactly 17 things less than $x \to$ how you ~~can~~ define 17.

$$\exists u_0 \ldots u_{16} (u_0 < u_1 < u_2 \ldots u_{16} < x \wedge \forall y \; y < x \to y = u_1 \vee y = u_2 \ldots u_{16})$$

we call this fact that $a < b \to a \neq b$.

$\langle \mathbb{Z}, < \rangle$ $(\ldots -2, -1, 0, 1, 2 \ldots)$

no definable elements. ~~and~~

all ~~a~~ elts are indiscernable.

all sentences if a will be true of b.

~~then~~ $a < b$. Then $\varphi(a) \leftrightarrow \varphi(b)$

Define $\pi(x) = x + k$, where $k = b - a$.

$x < y \leftrightarrow x + k < y + k$

π preserves structure, an automorphism

$\mathcal{M}, \mathcal{N}$

if f is a map from $\mathcal{M}$ to $\mathcal{N}$ $(f: \mathcal{M} \to \mathcal{N})$

- f is a homomorphism iff f preserves structure ($\to$ and $\leftrightarrow$ versions) betw M + N
- f is an isomorphism if f is a homomorphism that is also one-to-one and "onto," i.e. f provides an one-to-one correspondence

between $\mathfrak{M}$ + $\mathfrak{N}$. (i.e. ~~[illegible]~~ every at target model is f (something of same model)

- f is an automorphism iff f is an isomorphism of a structure to itself if $M = N$.

This is why $\pi(x) = x+k$ in the above structure is an <u>automorphism</u>.

so $\langle \mathbb{Z}, < \rangle \models \varphi(a)$ iff $\langle \mathbb{Z}, < \rangle \models \varphi(b)$

If every object in $\mathfrak{M}$ is definable, then $\mathfrak{M}$ is Liebnizian

Q: is the converse true? Is the ability to discern objects the same as the ability to pick them out uniquely?

no.

e.g. look at the model of infinitely many distinct constants PLUS one more object. Language has $c_1, c_2, c_3 \ldots$ Model interprets those constants as different, & then there's one other object that is not = to any of the objects named by the constants, a.

(1) M is Liebnizian easily – if $x \neq x$ in $\mathfrak{M}$, the entr x or y is some constant c_n, & this property distinguishes x from y.

"identity map" is an automorphism?

(2) each c_n is definable, $x = c_n$ defines c_n. But a is not definable, b/c what you want to say is that a is not $=$ to any of the constants, but such a formula would be ∞ long, and ~~the~~ formulas must be finite.

Suppose $\varphi(x)$ is any formula in the language. Since φ is finite, it uses only finitely many constant symbols $c_0 \dots c_k$ for some k.

Now, consider the reduct of M to this smaller language w/ only k constant symbols, ~~&~~ there are ∞ many other points. (these used to be c's).

?

This smaller language (M^{reduct}) has an isomorphism ~~[illegible]~~

swap a with some c_n for $n > k$

an automorphism?

$M^{\text{reduct}} \models \varphi(a)$ iff $M^{\text{reduct}} \models \varphi(c_n)$

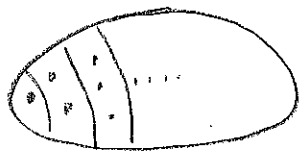
basically – if φ is satisfied by a, then any object not explicitly mentioned by that formula will also satisfy φ.

is the idea that M^{reduct} ~~[illegible]~~ means more than they may

a little confused about the formal use of M^{reduct} here & how we prove things about M with it.

$\boxed{Ex2}$ Consider an eq. relation $\sim$ on a set A w/ exactly one eq. class of size 7, one eq. class of size 2, one of size 3, etc. (faces?)

$(A, \sim)$



What are the definable elements?
What are the definable subsets? (classes?)

$\boxed{Ex3}$ Show that every object in $\langle \mathbb{N}, + \rangle$ is definable

$\forall x (x + y = x)$ defines $\emptyset$.
get 1, then iterate.

Q: Is $<$ definable in $\langle \mathbb{N}, +, \emptyset \rangle$

How do you define a relation?

$a \leq b$ iff $\langle \mathbb{N}, +, 0 \rangle \models \underline{\exists c \; a + c = b}$.
$a < b$ iff $\langle \mathbb{N}, +, 0 \rangle \models \exists c \; c \neq 0 \wedge a + c = b$

Q: is less than definable in $\langle \mathbb{Z}, +, 0 \rangle$?

$a < b$ iff $\langle \mathbb{Z}, +, \emptyset \rangle \models$ _____

no. It's impossible.
let $f(x) = -x$ ← automorphism
$-(x + y) = (-x) + (-y)$
so f is an automorphism of addition. Thus, if

$f(0)=0$ bc $-0=0$

~~The integers~~ if $\langle \mathbb{Z}, +, 0\rangle \models \varphi(a,b)$
then $\langle \mathbb{Z}, +, 0\rangle \models \varphi(-a,-b)$
but $a<b$ is not equivalent
to $-a<-b$. So, therefore,
no $\varphi(a,b)$ can be equivalent
to $a<b$.

i.e. you cannot define $<$ because $<$ is not respected by the automorphism on the model!

$\boxed{\text{Ex 4}}$ Show that $\cdot$ is not definable in $\langle \mathbb{Z}, +\rangle$. i.e., There is no formula $\varphi(a,b,c)$ such that $a \cdot b = c$ iff $\langle \mathbb{Z}, +\rangle \models \varphi(a,b,c)$.

$\langle \mathbb{N}, \cdot \rangle$

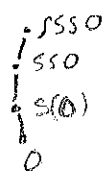
define $a \mid b$ (a divides b) iff $\langle \mathbb{N}, \cdot\rangle \models \exists x(x \cdot a = b)$
define $+$ from $-$ in $\langle \mathbb{Z}, -\rangle$

$x+y=z$ iff $\langle \mathbb{Z}, -\rangle \models y = z-x$
$x-y=z$ iff $\langle \mathbb{Z}, +\rangle \models x = y+z$

PEANO Structure
$\langle \mathbb{N}, S, 0\rangle$ $S=$ "Successor" function
$S(0)=1$, $S(1)=2$, $S(x)=x+1$

Some Axioms:

1. $\forall x \; S(x) \neq 0$
2. $\forall x \forall y \; (S(x) = S(x) \rightarrow x = y)$ — i.e. S is a one-to-one function.
3. $\forall y \; (y \neq 0 \rightarrow \exists x \; y = S x)$



What are the definable elements in

$\langle \mathbb{N}, S, \emptyset \rangle$?

$\emptyset$: $x = \emptyset$.

$x = 1$ means: 1: $S(0) = x$

$x = 2$ means: $x = SS0$

$x = n$ means $x = \underbrace{SS\ldots}_{n \text{ times}} \emptyset$.

in object theory, an object in domain

in metatheory using this #, of this element in $\mathbb{N}$, to ...

This argument depends on the numbers we use in our metatheory matching in some way the #s that are elements of $\mathbb{N}$.

Theorem: $\langle \mathbb{N}, S, \emptyset \rangle$ satisfies "elimination of quantifiers" i.e. for any formula $\varphi(x_1 \ldots x_n)$ in this language, there is a quantifier-free formula $\varphi^*(x_1 \ldots x_n)$ such that $\langle \mathbb{N}, S, \emptyset \rangle \models \varphi(a_1 \ldots a_n) \leftrightarrow \varphi^*(a_1 \ldots a_n)$ i.e. every formula is equivalent to a quantifier free formula.

prove this by induction ~~on~~ on formulas

(1) True for atomic formulas, b/c they are identical to their quantifier-free selves.

(2) ~~Is~~ This property is preserved by boolean combinations.

$$\begin{matrix}\varphi \leftrightarrow \varphi^* \\ \psi \leftrightarrow \psi^*\end{matrix} \longrightarrow (\varphi \wedge \psi) \leftrightarrow (\varphi^* \wedge \psi^*)$$

(3) This is preserved by quantification tableaux...

$\exists x\, \varphi(x, x_1, \ldots x_n)$ eq. to
$\exists x\, \varphi^*(x_1, \ldots x_n)$

put φ^* into ~~A~~ D.N.F

$\exists x (A \vee B)$ b/c $\exists$ distributes over $\vee$.
$\longleftrightarrow$
$\exists x\, A \vee \exists x\, B$

$\exists x$ $\underline{\_ \wedge \_ \wedge \_ \wedge}$ ← atomic formulas.
conjunctive clause

We can eliminate that $\exists x$!

$\exists x (SSSv = SSSy \wedge SSSu = SSSw \wedge SSSu \neq SSu \wedge SSSu \neq x)$

reduces to $x \neq 0 \wedge x \neq S0$

$SSSu = SSSSw \wedge SSSy \neq SSSu \wedge SSSSu \neq y$.

?

ok. So you can eliminate x from this, but how is this generalizable ???

$\boxed{\text{Ex 6}}$ Show "even" and "odd" are definable in $\langle \mathbb{N}, + \rangle$. Conclude that it is not definable in $\langle \mathbb{N}, S \rangle$.

$\boxed{\text{Ex 7}}$ Show "prime" is definable in $\langle \mathbb{N}, \cdot \rangle$