Pluralism-inspired mathematics, including a recent breakthrough in set-theoretic geology

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Set-theoretic Pluralism 2016
Symposium I, University of Aberdeen

Set-theoretic geology

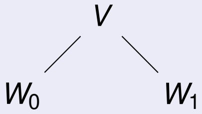
In set theory, forcing is naturally viewed as a method of building *outer* as opposed to *inner* models of set theory.

Set-theoretic geology inverts this perspective by studying how the set-theoretic universe V was obtained by forcing, by studying the fundamental structure of the grounds of V.

A fundamental question

Question (Reitz 2006)

Suppose that the set-theoretic universe V was obtained by forcing over two different grounds $V = W_0[G_0] = W_1[G_1]$. Must there be a common deeper ground?



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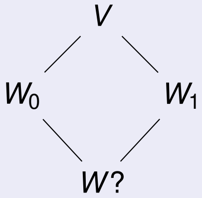

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Downward directedness

Downward-directedness expresses a fundamental property of the structure of grounds.

The question could have been asked 50 years ago.

My view is that it is difficult for us to claim a deep undertanding of forcing, if we lack the knowledge whether or not the forcing-theoretic ground models exhibit this basic, elementary feature.

Some previous knowledge

We knew many instances where it was true.

- True in every model where we were able to determine answer.
- True in the *bottomless* models of Reitz—no bedrock.
- (Fuchs,JDH,Reitz) True in any model of the form $L[A]$, where A is a set.

Meanwhile, there were attempts to find a counterexample.

- Woodin proposed a candidate counterexample model, based on inner model considerations.
- S. Friedman proposed a candidate counterexample, built by forcing.

It turns out, however, that there is no counterexample.

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Usuba's breakthrough

This question is now answered by Toshimichi Usuba.

Theorem (Usuba)

The grounds of V are downward-directed. Any two grounds have a common deeper ground.

Indeed, any set-indexed family of grounds have a common deeper ground.

I should like to explain the proof.

To my way of thinking, this is the most important advance in set theory in recent years.

Ground model enumeration theorem

Theorem (Fuchs,JDH,Reitz)

There is a parameterized family $\{ W_r \mid r \in V \}$ *such that*

1. *Every* W_r *is a ground of* V *and* $r \in W_r$.
2. *Every ground of* V *is* W_r *for some* r.
3. *The relation "*$x \in W_r$*" is first order.*

This reduces second-order statements about grounds to first-order statements about parameters.

For example, the *ground axiom* (JDH,Reitz) asserts $\forall r\, V = W_r$.

The *mantle* is $M = \bigcap_r W_r$.

Downward directedness hypotheses

Definition

1. The Downward Directed Grounds Hypothesis DDG asserts that the grounds are downward directed.

   For every r and s there is t such that $W_t \subseteq W_r \cap W_s$.

2. The Strong DDG asserts that they are downward set-directed.

   For every set I there is t with $W_t \subseteq \bigcap_{r\in I} W_r$.

By the ground-model enumeration theorem, these are expressible in the first-order language of set theory.

Downward-directed Grounds hypothesis

Theorem (Usuba)

The strong DDG *is true.*

In other words, for any set I, there is a ground W that is contained in every ground W_r with $r \in I$.

Let me present a detailed proof.

Review of some tree combinatorics

König's Lemma

Every infinite finitely-branching tree has a cofinal branch.

In other words, every tree of height ω with all levels finite has a branch.

How does this generalize to higher cardinals?

Aronszajn trees, of course, block one approach.

But consider a tree of height ω_2 with all levels countable.

More generally, consider a *tall narrow* tree...

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Every tall narrow tree has a branch, but not too many

Lemma (Kurepa)

If T is a tree, height λ, all levels size $< \delta$, with δ regular, $\delta < \operatorname{cof}(\lambda)$, then T has a cofinal branch, but fewer than δ many.

Proof.

For levels β with cofinality δ, find $\nu_\beta < \beta$ such that level β nodes separate already by level ν_β. By Fodor, there is stationary $S \subseteq \lambda$ with constant value ν. So all nodes on level $\beta \in S$ separate by level ν. By pigeon-hole, one node on level ν has successors on cofinally many levels, and these must cohere to form a branch.

Meanwhile, any two cofinal branches have separated by level ν, so there are fewer than δ many branches. □

Tall narrow trees gain no new branches

Lemma

If $W \subseteq V$ is an inner model of ZFC *and T is a tall narrow tree in W, then all cofinal branches of T are already in W.*

Proof.

We know T has fewer than δ many branches in W. If b is a new branch through T in V, then there must be a level by which b is distinguished from those branches. So there is a node p on b, but not on any branch in W. But T_p is a tall narrow tree in W, and so has a branch in W, contradiction. □

Approximation and cover properties

Suppose that W is a transitive class and that δ is a cardinal.

Definition

- The extension $W \subseteq V$ satisfies the δ-*approximation property*, if whenever $A \subseteq W$, $A \in V$ and $A \cap a \in W$ for all $a \in W$ of size less than δ, then $A \in W$.
- The extension $W \subseteq V$ satisfies the δ-cover property, if for every $A \subseteq W$ with $A \in V$ and $|A| < \delta$, there is $B \in W$ with $A \subseteq B$ and $|B| < \delta$.

These concepts are central in set-theoretic geology, used in my proof of the ground-model definability theorem.

Uniform cover property

Consider a *uniform* version of covering: every sequence of small sets is uniformly covered.

The *uniform* δ-cover property for $W \subseteq V$

If $A_i \subseteq W$ with $|A_i| < \delta$ every $i \in I$, with $I \in W$, then there is a covering sequence in W:

- $\langle B_i \mid i \in I \rangle \in W$.
- $A_i \subseteq B_i$.
- $|B_i| < \delta$.

Using the axiom of choice, it suffices to consider $\langle A_\alpha \mid \alpha < \lambda \rangle$ for λ ordinal and $A_\alpha \subseteq \lambda$.

Uniform covering, alternative formulations

If $W \subseteq V$ and δ regular, $\delta \leq \lambda$, then the following are equivalent:

1. For every $\langle A_\alpha \mid \alpha < \lambda \rangle$ with $A_\alpha \subseteq \lambda$ size $< \delta$, there is $\langle B_\alpha \mid \alpha < \lambda \rangle \in W$ with $A_\alpha \subseteq B_\alpha$ and $|B_\alpha| < \delta$.
2. For every $A \subseteq \lambda \times \lambda$, with all vertical sections size $< \delta$, there is $B \in W$ with $A \subseteq B$ and all sections size $< \delta$.
3. For every function $f : \lambda \to \lambda$ there is $B \in W$ with $B \subseteq \lambda \times \lambda$, all vertical sections size $< \delta$ and $f \subseteq B$.

Let's call this: λ-*uniform* δ-*covering*.

Uniform covering implies approximation

Lemma

If $W \subseteq V$ has λ-uniform δ-cover property, with δ regular and λ strong limit, then it has δ^+-approximation property for subsets of λ.

Proof.

Assume $s \in 2^\lambda$ has all δ^+-small approximations in W; aim to show $s \in W$. Assume inductively that $s \upharpoonright \alpha \in W$ all $\alpha < \eta$. By uniform covering, can find a tree $T \in W$ height η, levels size $< \delta$, such that $s \upharpoonright \alpha \in T$ for $\alpha < \eta$. If $\delta < \mathrm{cof}(\eta)$, this is tall narrow tree, and so $s \in W$.

Otherwise, $\mathrm{cof}(\eta) \leq \delta$. By a simple closure argument, find cofinal $J \subseteq \eta$ size δ, such that distinct nodes p, q on level $\beta \in J$ have $p(\alpha) \neq q(\alpha)$ some $\alpha \in J$. By approximation assumption, $s \upharpoonright J \in W$, and this determines s. □

Bukovskỳ ground model characterization

Theorem (Bukovskỳ, 1973)

Suppose that $W \subseteq V$ is an inner model of ZFC. Then the following are equivalent:

1. *W is a ground of V.*
2. *For some cardinal δ, the extension $W \subseteq V$ exhibits the uniform δ-cover property.*

The direction $1 \to 2$ is immediate, using the δ-chain condition.

The direction $2 \to 1$ is reminiscent of Vopěnka's proof that every set is HOD-generic, uses an infinitary logic.

Downward-directed Grounds hypothesis

Let us now put it together to prove the main result.

Theorem (Usuba)

The downward-directed grounds hypothesis is true.

Fix any set I and consider the grounds W_r for $r \in I$. We seek to find a ground W with $W \subseteq W_r$ for all $r \in I$.

For each ground W_r, we may realize V as a forcing extension $V = W_r[G_r]$, where $G_r \subseteq \mathbb{Q}_r \in W_r$ is W_r-generic.

Let κ be regular, larger than every $|\mathbb{Q}_r|^{W_r}$ and larger than $|I|$.

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Let κ be regular, larger than every $|\mathbb{Q}_r|^{W_r}$ and larger than $|I|$.

Consider large $\theta = \beth_\theta > \kappa$.

Lemma

There is $A \subseteq \theta$ with $L[A] \subseteq W_r$ all $r \in I$, such that $L[A] \subseteq V$ has $<\theta$-uniform κ^+-covering property.

Proof.

Let $h : \theta \to \theta$ be universal, in that every $t : \lambda \to \lambda$ for $\lambda < \theta$ occurs as a block in h. Since $W_r \subseteq V$ has uniform κ-covering, there is $H_{0,r} \subseteq \theta \times \theta$ with all vertical sections size $< \kappa$ and $h \subseteq H_{0,r} \in W_r$. Continuing, for $\eta < \kappa$ find $H_{\eta,r} \in W_r$ with all sections size $< \kappa$ and $H_{\nu,s} \subseteq H_{\eta,r}$ for $\nu < \eta$, $s \in I$. Let H be the union of all $H_{\eta,r}$, so $H \subseteq \theta \times \theta$, with all sections size $\leq \kappa$. Note that $H \in W_r$ all $r \in I$ by approximation property, since $H \cap a = H_{\eta,r} \cap a$ for κ-small $a \in W$. Let $A \subseteq \theta$ code $H \subseteq \theta \times \theta$. So $L[A] \subseteq W_r$ all $r \in I$. Also, $L[A] \subseteq V$ has $<\theta$-uniform κ^+-covering, since $h \subseteq H$. □

So for each $\theta = \beth_\theta$ above κ, we have $L[A_\theta] \subseteq \bigcap_r W_r$ and $L[A_\theta] \subseteq V$ has $<\theta$-uniform κ^+-covering, hence κ^{++}-approximation for bounded subsets of θ.

It follows by the ground-model definability theorem that $V_\theta^{L[A]}$ is definable in V_θ from parameter $p = (2^{<\kappa^{++}})^{L[A_\theta]}$.

Some such p must be used for unboundedly many θ, and the $V_\theta^{L[A_\theta]}$ cohere by the ground-model definability theorem. Let $W = \bigcup_\theta V_\theta^{L[A_\theta]}$ be the union of these.

Note that W is closed under Gödel operations, is almost universal and has well-orders. So $W \models \mathrm{ZFC}$.

Also, $W \subseteq W_r$ for all $r \in I$, since $W_\theta \subseteq L[A_\theta] \subseteq W_r$.

Finally, $W \subseteq V$ has uniform κ^{++}-cover property, since $V_\theta^{L[A_\theta]} \subseteq V_\theta$ had $<\theta$-uniform κ^{++}-covering. So by Bukovskỳ, W is a ground of V, contained in every W_r for $r \in I$, as desired. QED

Conclusion: in any model of ZFC, any set-indexed family of grounds have a common deeper ground.

$$W \subseteq \bigcap_r W_r \subseteq V$$

Jonas mentioned several consequences of the DDG in his tutorial, so let us briefly mention some of them.

Consequences

Usuba has proved that the strong DDG holds. This settles many prominent open questions of set-theoretic geology.

Corollaries

1. Bedrock models are unique when they exist.
2. The mantle is absolute by forcing.
3. The mantle is a model of ZFC.
4. The mantle is the same as the generic mantle.
5. The mantle is the largest forcing-invariant class, and equal to the intersection of the generic multiverse.

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Inclusion in the generic multiverse

The *generic multiverse* of M consists of all models obtainable by successively passing to a forcing extension or a ground.

Question

If M, N are in the same generic multiverse and $M \subseteq N$, must M be a ground of N?

Theorem

Yes. The inclusion relation agrees with the ground-of relation in the generic multiverse.

The point is that the DDG implies that the generic multiverse of M consists of all the forcing extensions of grounds of M, since this collection is closed under forcing extensions and grounds.

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Modal logic of forcing

I had introduced the forcing modalities.

$\Box\varphi$ if φ holds in all forcing extensions.

$\Diamond\varphi$ if φ holds in some forcing extension.

This modal language can express diverse forcing principles.

Maximality principle (Stavi+Väänänen, indep. JDH) asserts

$$\Diamond\Box\varphi \to \varphi$$

Question (JDH)

What exactly are the ZFC-provable forcing validities?

Theorem (JDH,Löwe)

If ZFC is consistent, then the ZFC-provably valid principles of forcing are precisely those in the modal theory S4.2.

Up and down forcing modalities

In set-theoretic geology, we naturally have *two* pairs of forcing modalities:

- $\overline{\Box}\,\varphi$ if φ holds in all forcing extensions.
- $\overline{\Diamond}\,\varphi$ if φ holds in some forcing extension.
- $\underline{\Box}\,\varphi$ if φ holds in all grounds.
- $\underline{\Diamond}\,\varphi$ if φ holds in some grounds.

Question (JDH,Löwe)

What exactly are the mixed-modality principles of forcing?

For example, these temporal-logic-like principles are valid:

$$\varphi \to \overline{\Box}\,\underline{\Diamond}\,\varphi \qquad\qquad \varphi \to \underline{\Box}\,\overline{\Diamond}\,\varphi$$

Modal logic of grounds

The DDG implies (and is nearly equivalent to) the validity of axiom .2 for the downward-logic.

$$\Diamond\Box\varphi \to \Box\Diamond\varphi$$

Corollary

If ZFC is consistent, then the ZFC-provably valid downward principles of forcing are exactly S4.2.

The point is that Benedikt Löwe and I had already proven S4.2 as an upper bound, but our previously best lower bound was S4. With the DDG, we get S4.2, so this theory hits it exactly.

Large cardinal $\to$ few grounds

Theorem (Usuba)

If there is a hyper-huge cardinal, then the universe has a bedrock.

In other words, the mantle is a ground.

This is an amazing connection between large cardinal existence and the structure of grounds!

A cardinal κ is *hyper-huge*, if for every ordinal λ there is $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$ and $M^{j(\lambda)} \subseteq M$.

supercompact $<$ super-huge $<$ hyper-huge $<$ super almost 2-huge

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Lemma

If κ is hyper-huge and W is a ground of V, then V is a forcing extension of W by forcing of size $< \kappa$.

Proof.

$V = W[G]$ for some $G \subseteq \mathbb{P} \in W$. Pick inaccessible $\lambda > |\mathbb{P}|$ and let $j : V \to M$ witness hyper-hugeness. Fix stationary partition $\langle S_\alpha \mid \alpha < j(\lambda)\rangle$ of $\mathrm{Cof}_\omega \cap \lambda$ in W, and argue that $\beta \in j\,"\,j(\lambda) \iff j(\vec{S})_\beta$ is stationary in $\sup j\,"\,j(\lambda)$ in $j(W)$. Conclusion: $j\,"\,j(\lambda) \in j(W)$. It follows that

$$W_{j(\lambda)} \subseteq j(W)_{j(\lambda)} \subseteq M_{j(\lambda)} \subseteq V_{j(\lambda)} = W[G]_{j(\lambda)}.$$

So $M_{j(\lambda)}$ is extension of $W_{j(\lambda)}$ by forcing size $< j(\kappa)$. By elementarity, V_λ is a forcing extension of W_λ by forcing of size $< \kappa$. □

Thus, V has a bedrock, by the strong DDG.

The generic multiverse

Let us turn now to other topics.

The *generic multiverse* of a model of set theory M is the collection of models arising successively as forcing extensions or grounds of models already in the class.

We'd like to understand basic structural features of the generic multiverse.

Amalgamation

Question

If M is a model of set theory with forcing extensions $M[G]$ and $M[H]$, is there a common further extension?

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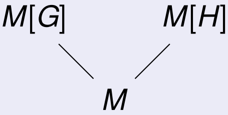

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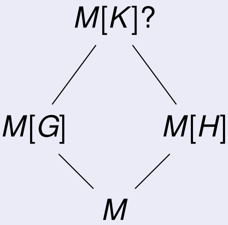

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Non-amalgamation

Theorem (Woodin)

Every countable transitive model of set theory W has non-amalgamable forcing extensions $W[c]$ and $W[d]$, meaning that there is no common extension $\bar{W} \models \mathrm{ZFC}$ with the same ordinals.

Build M-generic Cohen reals c and d in stages, so as to meet all the dense sets of M, while also coding some "bad" real z. Individually, $M[c]$ and $M[d]$ are fine, but any model with both c and d will be able to decode z.

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Extending to large finite families

Similarly, we can have three extensions $M[c]$, $M[d]$ and $M[e]$, such that any two of them can be amalgamated, as in $M[c, d]$, but there is no common extension of all three.

And similarly for any finite number.

Upper bound in the generic multiverse?

Question

If we successively add Cohen reals

$$M[c_0] \quad \subseteq \quad M[c_1] \quad \subseteq \quad M[c_2] \quad \subseteq \quad \cdots$$

Is there an upper bound, an extension $M[d]$ with

$$M[c_n] \subseteq M[d]$$

for all n?

Countable closure

Theorem (JDH, Venturi)

If M is a countable model of ZFC *and*

$$M[c_0] \subseteq M[c_0][c_1] \subseteq M[c_0][c_1][c_2] \cdots$$

are generic extensions by adding Cohen reals, then there is an M-generic Cohen d with M[d] an upper bound of the chain.

Proof.

The idea is to construct an M-generic $d \subseteq \mathbb{N} \times \mathbb{N}$, whose n^{th} slice d_n agrees with c_n except on a finite set. □

Upward closure in the generic multiverse

Theorem (Fuchs,JDH,Reitz)

If W is a countable model of ZFC and

$$W \subseteq W[G_0] \subseteq W[G_1] \subseteq W[G_2] \subseteq \cdots$$

is a countable tower of forcing extensions, with forcing of bounded size in W, then there is a common forcing extension $W[H]$ above them all.

Thus, any finitely amalgamable family of forcing extensions is fully amalgamable.

Isomorphic to a forcing extension?

Question

Can a model of set theory be isomorphic to one of its nontrivial forcing extensions?

$$M \cong M[G]?$$

A model of set theory V can easily be *elementarily equivalent* to a forcing extension $V[G]$, that is, have the same theory

$$V \equiv V[G].$$

For example, if you add a Cohen real c, and then add another d, then

$$V[c] \equiv V[c][d],$$

because we may merge the two Cohen reals into one, and so $V[c]$ and $V[c * d]$ are two extensions by adding a Cohen real, which is homogeneous and so all extensions have the same theory.

But of course, $V[c]$ will not be isomorphic to $V[c * d]$ in $V[c * d]$.

Isomorphic to forcing extension?

No transitive $\in$-standard model is isomorphic to a nontrivial forcing extension.

$$M \ncong M[G]$$

This is because distinct transitive sets are never isomorphic: if $\pi : M \cong N$ is an $\in$-isomorphism, then $\pi(x) = x$ by $\in$-induction, and so $M = N$.

Nevertheless

Theorem

There is a model M of set theory (if consistent) isomorphic to all its forcing extensions by a Cohen real $M \cong M[c]$.

Lemma (Smoryński)

If M, N are countable, computably saturated, same theory and same standard system, then $M \cong N$.

Lemma (JDH)

If M is countable computably saturated, then so also are all forcing extensions $M[G]$.

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Theorem

There is a model M of set theory (if consistent) that is isomorphic to all its forcing extensions $M[c]$ by a Cohen real.

Proof.

Let N be any countable computably saturated model. Let $M = N[d]$ be a forcing extension by a Cohen real. Consider any $M[c]$.

- $M[c] = N[d][c] = N[d * c]$.
- Consequently, $M \equiv M[c]$, since the forcing is homogeneous.
- M and $M[c]$ have the same arithmetic, hence same standard system.
- M and $M[c]$ are computably saturated.
- Hence, $M \cong M[c]$.

□

Theorem

There is a model M of set theory (if consistent) that is isomorphic to all its forcing extensions $M[c]$ by a Cohen real.

Proof.

Let N be any countable computably saturated model. Let $M = N[d]$ be a forcing extension by a Cohen real. Consider any $M[c]$.

- $M[c] = N[d][c] = N[d * c]$.
- Consequently, $M \equiv M[c]$, since the forcing is homogeneous.
- M and $M[c]$ have the same arithmetic, hence same standard system.
- M and $M[c]$ are computably saturated.
- Hence, $M \cong M[c]$.

□

My favorite situation

A philosophical concern...

leads to interesting mathematical questions...

whose answers illuminate the philosophical issue.

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The philosophical question

Question

To what extent does definiteness of mathematical objects lead to definiteness of our theory of mathematical truth about those objects?

Many mathematicians express a commitment to the definiteness of the natural numbers $0, 1, 2, \ldots$ and the structure $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Does this also commit us to the definiteness of arithmetic truth?

Definiteness of truth

For example, Feferman and others have defended a view whereby arithmetic truth has a definite character, while higher-order truth, such as set-theoretic assertions at the level of $P(\mathbb{N})$ and above, are less definite.

For example, on such a view you might view the continuum hypothesis as a vague mathematical assertion, not capable of genuine resolution.

From structure to truth

Solomon Feferman (EFI 2013):

> In my view, the conception [of the bare structure of the natural numbers] is completely clear, and thence all arithmetical statements are definite.

It is Feferman's 'thence' to which I call attention.

Donald Martin (EFI 2012):

> What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers.

Structure to truth

So we are interested in the philosophical question concerning the extent one gets definiteness of the theory of truth for a structure merely from the definiteness of the objects and relations of the structure itself.

Or is the definiteness of the theory of truth for a structure a kind of higher-order ontological commitment requiring its own justification?

The mathematical question

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The mathematical question

Question (Yang)

Can a mathematical structure exist inside two different models of set theory, which disagree on the theory of that structure?

The answer is yes, and indeed, this is pervasive.

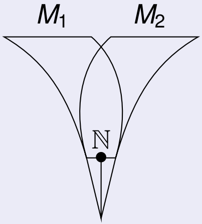
Satisfaction is not absolute

Theorem (JDH, Yang)

If ZFC *is consistent, then there are* $M_1, M_2 \models \text{ZFC}$ *which have the same natural numbers and arithmetic structure*

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2},$$

but which disagree on arithmetic truth.



There is a sentence σ *in* M_1 *and* M_2

M_1 *believes* $\mathbb{N} \models \sigma$

M_2 *believes* $\mathbb{N} \models \neg\sigma$

Proof

Fix any countable $M_1 \models \mathrm{ZFC}$ with $\langle\mathbb{N}, \mathrm{TA}\rangle^{M_1}$ computably saturated.

Claim there are sentences σ, τ with same 1-type in $\mathbb{N}^{M_1} = \langle\mathbb{N}, +, \cdot, 0, 1, <\rangle^{M_1}$, but M_1 thinks σ is true and τ false in $\mathbb{N}^{M_1}$. (Proof: consider the type $p(s,t)$ containing $\varphi(s) \iff \varphi(t)$ and $s \in \mathrm{TA}$ and $t \notin \mathrm{TA}$; this is finitely realized, since TA is not definable.)

By back-and-forth, there is automorphism $\pi : \mathbb{N}^{M_1} \to \mathbb{N}^{M_1}$ with $\pi(\tau) = \sigma$.

Build a copy M_2 of M_1 so that π extends to an isomorphism $\pi^* : M_1 \to M_2$. So $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$. But M_1 thinks σ is true, yet M_2 thinks $\sigma = \pi^*(\tau)$ is false. $\square$

A generalization

Theorem

For any countable $M \models \mathrm{ZFC}$, any structure $\mathcal{N} \in M$ finite language, any $S \subseteq N$ in M not definable in $\mathcal{N}$. Then there are $M \prec M_1$ and $M \prec M_2$ with $\mathcal{N}^{M_1} = \mathcal{N}^{M_2}$, yet $S^{M_1} \neq S^{M_2}$.

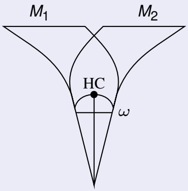
Note S^{M_1} and S^{M_2} share all properties of S in M.

Proof.

Fix $M \prec M_1$ countable computably saturated. So $\langle \mathcal{N}, S \rangle^{M_1}$ is computably saturated. Since S not definable, there are $a, b \in \mathcal{N}^{M_1}$ with same 1-type in $\mathcal{N}^{M_1}$, but $a \in S$, $b \notin S$. So $\exists \pi : \mathcal{N}^{M_1} \cong \mathcal{N}^{M_1}$ with $\pi(b) = a$. Extend π to $\pi^* : M_1 \cong M_2$. So $a \in S^{M_1}$ but $a = \pi(b) \notin S^{M_2}$. □

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their natural numbers $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$, their reals $\mathbb{R}^{M_1} = \mathbb{R}^{M_2}$ and their hereditarily countable sets $\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$, but which disagree on their theories of projective truth.



$M_1, M_2 \models \mathrm{ZFC}$

$\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ $\quad \mathbb{R}^{M_1} = \mathbb{R}^{M_2}$

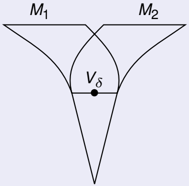
$\langle \mathrm{HC}, \in \rangle^{M_1} = \langle \mathrm{HC}, \in \rangle^{M_2}$

M_1 *believes* $\mathrm{HC} \models \sigma$

M_2 *believes* $\mathrm{HC} \models \neg\sigma$

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which have a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, but which disagree on truth in this structure.



$M_1, M_2 \models \mathrm{ZFC}$

$V_\delta^{M_1} = V_\delta^{M_2} \models \mathrm{ZFC}$

M_1 *believes* $V_\delta \models \sigma$

M_2 *believes* $V_\delta \models \neg\sigma$

Iterated truth predicates

Begin with the standard model $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$.

Add a truth predicate $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0 \rangle$, where Tr_0 is a truth predicate for arithmetic assertions.

Add a truth predicate for that structure, $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \mathrm{Tr}_1 \rangle$, where Tr_1 is a truth predicate for assertions in the language with Tr_0.

And so on $\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$...

Truth about truth is not absolute

Corollary

For every countable model of set theory M and any natural number n, there are $M \prec M_1$ and $M \prec M_2$ with $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and same iterated truths up to n

$$\langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, <, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle^{M_2}$$

but which disagree on the next order of truth Tr_{n+1}.

The point: Tr_{n+1} is not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, \mathrm{Tr}_0, \ldots, \mathrm{Tr}_n \rangle$.

Disagreement on the Church-Kleene ordinal

Corollary

Every countable model of set theory M has elementary extensions $M \prec M_1$ and $M \prec M_2$, which agree on their standard model of arithmetic $\mathbb{N}^{M_1} = \mathbb{N}^{M_2}$ and have a computable linear order $\lhd$ on $\mathbb{N}$ in common, yet M_1 thinks $\langle \mathbb{N}, \lhd \rangle$ is a well-order and M_2 does not.

Proof.

Being the computable index of a well-order is Π^1_1-complete and hence not definable in $\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle$. □

Disagreement on definability

Theorem

Every countable model of set theory M has $M \prec M_1$ and $M \prec M_2$, which agree on

$$\langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_1} = \langle \mathbb{N}, +, \cdot, 0, 1, < \rangle^{M_2}$$

and which have a set $A \subseteq \mathbb{N}$ in common, yet M_1 thinks A is first-order definable in $\mathbb{N}$ and M_2 thinks it is not.

The proof relies on the non-absoluteness theorem, plus:

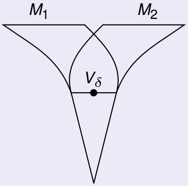
Lemma (Andrew Marks)

There is $B \subseteq \mathbb{N} \times \mathbb{N}$, such that $\{\, n \in \mathbb{N} \mid B_n \text{ is arithmetic} \,\}$ is not definable in the structure $\langle \mathbb{N}, +, \cdot, 0, 1, <, B \rangle$.

Precise violation of ZFC

Theorem

Every countable model of set theory $M \models \mathrm{ZFC}$ has elementary extensions M_1 and M_2, with a transitive rank-initial segment $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$ in common, such that M_1 thinks that the least natural number n for which V_δ violates Σ_n-collection is even, but M_2 thinks it is odd.



$M_1, M_2 \models \mathrm{ZFC}$

$V_\delta^{M_1} = V_\delta^{M_2}$

n is least with $\neg\Sigma_n$-collection in V_δ

M_1 believes n is even

M_2 believes n is odd

Proof

Suppose that $M \models \mathrm{ZFC}$, and let

$$T_1 = \Delta(M) + V_\delta \prec V + \{\, m \in V_\delta \mid m \in M \,\} + \text{the least } n \text{ such that } V_\delta \not\models \Sigma_n\text{-collection is even},$$

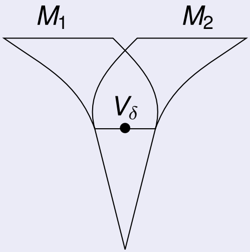
Consistent via the reflection theorem. Similar with theory T_2, where we assert n is odd.

Let $\langle M_1, M_2\rangle$ be a computably saturated model pair, with $M_1 \models T_1$ and $M_2 \models T_2$. It follows that $\langle V_\delta^{M_1}, V_\delta^{M_2}\rangle$ is a computably saturated model pair of elementary extensions of $\langle M, \in^M\rangle$, which are therefore elementarily equivalent in the language of set theory with constants for elements of M, and hence isomorphic by an isomorphism respecting those constants. So without loss $\langle M, \in^M\rangle \prec \langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$. Meanwhile, M_1 thinks that this V_δ violates Σ_n-collection first at an even n and M_2 thinks it does so first for an odd n, as desired. □

Disagreement about whether $V_\delta \models \text{ZFC}$

Theorem

If M is a countable model of set theory in which the worldly cardinals form a stationary proper class, then there are $M \prec M_1$ and $M \prec M_2$ with $\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$, but M_1 thinks $V_\delta \models \text{ZFC}$ and M_2 thinks $V_\delta \not\models \text{ZFC}$.



$M_1, M_2 \models \text{ZFC}$

$\langle V_\delta, \in\rangle^{M_1} = \langle V_\delta, \in\rangle^{M_2}$

M_1 *believes* $V_\delta \models \text{ZFC}$

M_2 *believes* $V_\delta \not\models \text{ZFC}$

Models inside models

Question

If M is a model of set theory ZFC, when do we expect to find a structure inside M that is, externally, a model of ZFC?

If $\langle M, \in^M \rangle$ thinks m is a set with binary relation E, we can extract this to an actual structure $\langle \bar{m}, \bar{E} \rangle$

- Domain of $\bar{m}$ consists of a for which $M \models a \in m$.
- $a \, \bar{E} \, b$ iff $M \models a \, E \, b$.

Models inside models

Question

If M is a model of set theory ZFC, when do we expect to find a structure inside M that is, externally, a model of ZFC?

If $\langle M, \in^M \rangle$ thinks m is a set with binary relation E, we can extract this to an actual structure $\langle \bar{m}, \bar{E} \rangle$

- Domain of $\bar{m}$ consists of a for which $M \models a \in m$.
- $a \mathrel{\bar{E}} b \qquad$ iff $\qquad M \models a \mathrel{E} b$.

From Brice Halimi:

Theorem

Every model of ZFC *has an element that is a model of* ZFC*. Specifically, if* $\langle M, \in^M\rangle \models \mathrm{ZFC}$*, then there is* $\langle m, E\rangle$ *in* M*, which when extracted as an actual structure, satisfies* ZFC*.*

Obviously wrong? After all, perhaps M satisfies $\neg\,\mathrm{Con}(\mathrm{ZFC})$.

No, that objection is wrong, since it conflates the object theory ZFC with the actual ZFC. So let's give a proof.

Every model has a model inside

Theorem

Every model of ZFC *has an element that is, externally, a model of* ZFC*. Specifically, if* $\langle M, \in^M\rangle \models \mathrm{ZFC}$*, then there is* $\langle m, E\rangle$ *in* M*, which when extracted as an actual structure, satisfies* ZFC*.*

Proof.

If M is ω-nonstandard, then by the reflection theorem plus overspill, there is some $\langle V_\delta^M, \in\rangle^{\mathcal{M}}$ satisfying a nonstandard fragment of ZFC, and hence satisfies the actual ZFC externally.

If M is ω-standard, then it must satisfy Con(ZFC) and so one has the Henkin model inside M. □

Universal algorithm

There is a *universal algorithm*, capable of computing any function, in the right universe.

Theorem

There is a Turing machine program p, such that for any function $f : \mathbb{N} \to \mathbb{N}$, there is a model $M \models \mathrm{ZFC}$ (assuming consistency), such that program p inside M computes f on standard input.

Inside M, on standard input n, the program p with output $f(n)$.

Related to old theorem of Mostowsk and Kripke: independent Π^0_1 assertions.

Rosser tree

Theorem

There is a universal program p, which can compute any $f : \mathbb{N} \to \mathbb{N}$ on standard input, in the right universe.

Proof 1. Use the *Rosser tree*: place desired theory at root, and assign to successor nodes either $T + \rho$ or $T + \neg\rho$, using Rosser sentence over previous theory T.

Every branch through the theory is consistent.

If initial theory proves $\neg\,\mathrm{Con}(T)$, then the difference between ρ and $\neg\rho$ is which proof comes first. Computable!

So the universal algorithm is: search for proofs of the next Rosser assertion. You can thereby compute the path the theory followed.

Theorem

There is a universal program p, which can compute any $f : \mathbb{N} \to \mathbb{N}$ on standard input, in the right universe.

Proof 2 (Vadim Kosoy)

There is a program p that searches for a proof that program p disagrees with a certain finite list of function input/output values. For the first such proof that is found, output the correspond output.

The theory cannot prove that the function disagrees with any particular value, since then it wouldn't. So for any function f, it is consistent with the theory that p computes exactly $f(n)$ on input n.

Thanks to Toshimichi Usuba.

Various other parts of this work were joint variously with Giorgio Venturi, Gunter Fuchs, Jonas Reitz, Ruizhi Yang.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

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