The hierarchy of second-order set theories between GBC and KM and beyond

Joel David Hamkins

City University of New York
CUNY Graduate Center
Mathematics, Philosophy, Computer Science

College of Staten Island
Mathematics

MathOverflow

14th International Workshop in Set Theory
Centre International de Rencontres Mathématiques
Luminy campus, Marseille, France, October 9-13, 2017

Joint work in various projects (three papers) with various co-authors.

Victoria Gitman (CUNY)
Joel David Hamkins (CUNY)
Peter Holy (Bonn)
Thomas Johnstone (CUNY)
Philipp Schlicht (Bonn)
Kameryn Williams (CUNY)

Part of the work was a very enjoyable collaboration between Bonn and New York.

A new subject is emerging

A rich hierarchy of theories is emerging between Gödel-Bernays GBC set theory and Kelley-Morse KM.

Natural principles of second-order set theory fit neatly into the hierarchy.

Many principles turn out to be equivalent over the base theory.

A new subject is emerging:

The reverse mathematics of second-order set theory.

Traditional second-order theories: GBC and KM

Axiomatized with two-sorts: sets + classes

Usual axioms: extensionality, foundation, union, pairing, power set, infinity, replacement of sets by class functions.

Class comprehension: $\{ x \mid \varphi(x, a, A) \}$ is a class.

- Gödel-Bernays GBC has class comprehension for first-order φ.
- Kelley-Morse KM has class comprehension for second-order φ.

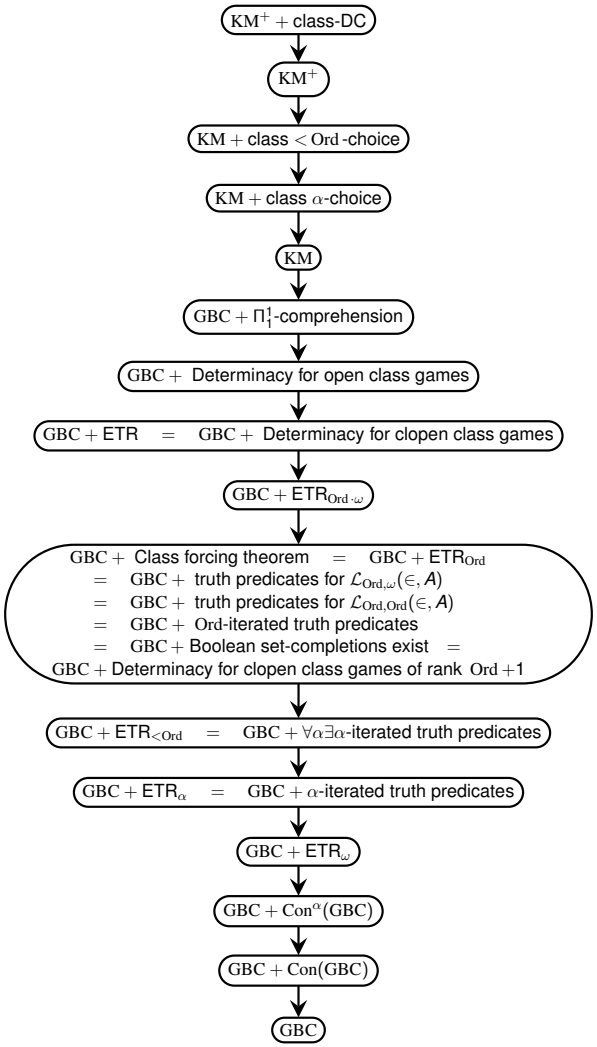
Both GBC and KM have the global choice principle.

Between GBC and KM

Recent work places some natural assertions of second-order set theory strictly between GBC and KM or beyond.

- The class forcing theorem.
- Determinacy of clopen class games.
- The class-choice principle.

Let me tell you how these assertions fit into the hierarchy of second-order theories.


$\mathrm{KM}^+ +$ class-DC
KM^+
$\mathrm{KM} +$ class $<\mathrm{Ord}$-choice
$\mathrm{KM} +$ class α-choice
KM
$\mathrm{GBC} + \Pi^1_1$-comprehension
$\mathrm{GBC} +$ Determinacy for open class games
$\mathrm{GBC} + \mathrm{ETR} = \mathrm{GBC} +$ Determinacy for clopen class games
$\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$
$\mathrm{GBC} +$ Class forcing theorem $= \mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$
$= \mathrm{GBC} +$ truth predicates for $\mathcal{L}_{\mathrm{Ord},\omega}(\in, A)$
$= \mathrm{GBC} +$ truth predicates for $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in, A)$
$= \mathrm{GBC} +$ Ord-iterated truth predicates
$= \mathrm{GBC} +$ Boolean set-completions exist $=$
$\mathrm{GBC} +$ Determinacy for clopen class games of rank $\mathrm{Ord}+1$
$\mathrm{GBC} + \mathrm{ETR}_{<\mathrm{Ord}} = \mathrm{GBC} + \forall\alpha\exists\alpha$-iterated truth predicates
$\mathrm{GBC} + \mathrm{ETR}_\alpha = \mathrm{GBC} + \alpha$-iterated truth predicates
$\mathrm{GBC} + \mathrm{ETR}_\omega$
$\mathrm{GBC} + \mathrm{Con}^\alpha(\mathrm{GBC})$
$\mathrm{GBC} + \mathrm{Con}(\mathrm{GBC})$
GBC

Hierarchy is robust for first-order strengthening

The theories in the hierarchy have low large cardinal strength.

Nevertheless, the hierarchy respects first-order increases in strength.

Hierarchy remains same over GBC as GBC + large cardinals.

So it is orthogonal to the large-cardinal consistency strength.

Two different ways to strengthen a second-order set theory:

1. Strengthen the first-order theory, e.g. large cardinals
2. Strengthen the second-order theory, e.g. ETR, class-choice, etc.

Backbone of the hierarchy: ETR

Elementary transfinite recursion (ETR) asserts: every first-order recursion φ along a class well-order $\Gamma = \langle A, \leq_\Gamma \rangle$ has a solution. A solution is $S \subseteq A \times V$ with every section defined by φ

$$S_\alpha = \{\, x \mid \varphi(x, S \restriction \alpha, Z) \,\},$$

where $S \restriction \alpha = \{\, (\beta, x) \in S \mid \beta <_\Gamma \alpha \,\}$.

Stratified by ETR_Γ, asserting solutions for recursions length Γ.

ETR_ω implies $\exists$ truth predicate. Srictly stronger than GBC.

$\mathrm{ETR}_{\mathrm{Ord}}$ asserts every class recursion length Ord has a solution.

The class forcing theorem

The class forcing theorem

Consider class forcing notion $\mathbb{P}$.

Perhaps it has a forcing relation $p \Vdash \varphi(\tau)$.

The class forcing theorem is the assertion that every $\mathbb{P}$ has forcing relations.

Goal

Analyze the strength of the class forcing theorem.

Class forcing theorem $\leftrightarrow$ $\mathrm{ETR}_{\mathrm{Ord}}$

Theorem (Gitman, Hamkins, Holy, Schlicht, Williams)

The following are equivalent over GBC.

1. *The class forcing theorem. Every class forcing notion* $\mathbb{P}$ *has forcing relations* $\Vdash_{\mathbb{P}}$.
2. *The principle* $\mathrm{ETR}_{\mathrm{Ord}}$. *Every elementary transfinite class recursion of length* Ord *has a solution.*

Should clarify: what does it mean exactly to say $\mathbb{P}$ admits forcing relations?

Should be expressed in second-order set theory for the theorem to be sensible.

Usual meta-mathematical approach to forcing relation

Common to define forcing relation for $\mathbb{P}$ over $M \models \mathrm{GBC}$ by:

$p \Vdash \varphi(\tau)$, if whenever $p \in G \subseteq \mathbb{P}$ generic, then $M[G] \models \varphi(\tau_G)$.

This works fine for model construction, when M is countable.

But problematic for reverse-mathematical strength.

- Takes place in the meta-theory, not in M.
- Does M recognize its forcing relations?
- Not expressed in language of second-order set theory.
- Doesn't work when there are no M-generic filters.

We want an *internal* account of the forcing relation that is expressible in any model of GBC.

A forcing relation is a solution to a certain recursion

An internal account is provided by the familiar forcing relation recursion.

Definition

$\mathbb{P}$ admits atomic forcing relations, if there are relations

$$p \Vdash \sigma \in \tau, \qquad p \Vdash \sigma \subseteq \tau, \qquad p \Vdash \sigma = \tau$$

respecting the recursive properties:

(a) $p \Vdash \sigma \in \tau$ iff densely many $q \leq p$ have some $\langle \rho, r \rangle \in \tau$ with $q \leq r$ and $q \Vdash \sigma = \rho$.

(b) $p \Vdash \sigma = \tau$ iff $p \Vdash \sigma \subseteq \tau$ and $p \Vdash \tau \subseteq \sigma$.

(c) $p \Vdash \sigma \subseteq \tau$ iff $q \Vdash \rho \in \tau$ whenever $\langle \rho, r \rangle \in \sigma$ and $q \leq p, r$.

A forcing relation is a solution to a certain recursion

Definition

A forcing relation $p \Vdash \varphi(\tau)$ is a relation (on a set of formulas) obeying the recursion:

(a) Obeys the atomic forcing-relation recursion.

(b) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$;

(c) $p \Vdash \neg\varphi$ iff no $q \leq p$ with $q \Vdash \varphi$; and

(d) $p \Vdash \forall x\, \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for every $\mathbb{P}$-name τ.

Entirely internal. No reference to generic filters or to extensions $M[G]$.

Resembles the Tarskian recursion for truth predicates.

Consequences of forcing relations

When there are forcing relations and an M-generic generic filter $G \subseteq \mathbb{P}$, then everything works as expected.

- Forced statements are true:
  If $p \in G$ and $p \Vdash \varphi(\tau)$, then $M[G] \models \varphi(\tau_G)$.
- True statements are forced:
  If $M[G] \models \varphi(\tau_G)$, then there is some $p \in G$ with $p \Vdash \varphi(\tau)$.

These are model-theoretic *consequences* of the forcing relation, rather than the definition.

$\mathrm{ETR}_{\mathrm{Ord}} \to$ forcing relations

For set forcing, the forcing-relation recursion is set-like and therefore has a solution in ZFC.

With class forcing, the recursion is not generally set-like, even in the atomic case.

But $\mathrm{ETR}_{\mathrm{Ord}}$ implies there is a solution.

Indeed, $\mathrm{ETR}_{\mathrm{Ord}}$ implies there is a *uniform* forcing relation $p \Vdash \varphi(\tau)$, handling all φ at once.

In ZFC, we are used to having forcing relations $p \Vdash \varphi(\tau)$ only as a scheme. With $\mathrm{ETR}_{\mathrm{Ord}}$, we get uniform forcing relations.

Forcing relations $\leftrightarrow$ $\mathrm{ETR}_{\mathrm{Ord}}$

From $\mathrm{ETR}_{\mathrm{Ord}}$, we constructed forcing relations.

The surprise is that we were able to reverse this implication.

If every class forcing notion $\mathbb{P}$ has atomic forcing relations, then $\mathrm{ETR}_{\mathrm{Ord}}$.

We actually find a long list of equivalent statements.

Equivalents of the class forcing theorem

Theorem (Gitman, Hamkins, Holy, Schlicht, Williams)

The following are equivalent over GBC.

(1) *Every class forcing* $\mathbb{P}$ *admits atomic forcing relations:*
$p \Vdash \sigma = \tau, \qquad p \Vdash \sigma \in \tau.$

(2) *Every* $\mathbb{P}$ *admits scheme of forcing relations* $p \Vdash \varphi(\tau)$.

(3) *Every* $\mathbb{P}$ *has uniform forcing relation* $p \Vdash \varphi(\tau)$.

(4) *Every* $\mathbb{P}$ *has uniform forcing relation for* $\varphi \in \mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in)$.

(5) *Every* $\mathbb{P}$ *has* $\mathbb{P}$*-name class* $\dot{\mathrm{Tr}}$ *with* $\mathbb{1} \Vdash \dot{\mathbb{T}}$ *is a truth-predicate.*

(6) *Every separative class order has a Boolean completion.*

(7) *The class-join separation principle plus* $\mathrm{ETR}_{\mathrm{Ord}}$*-foundation.*

More equivalents of the class-forcing theorem

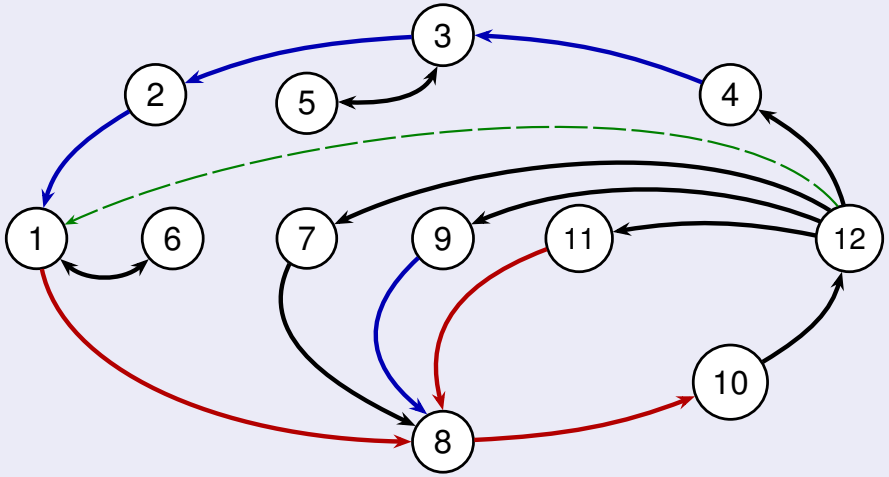
(8) For every class A, there is a truth predicate for $\mathcal{L}_{\mathrm{Ord},\omega}(\in, A)$.

(9) For every class A, there is truth predicate for $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in, A)$.

(10) For every A, there is Ord-iterated truth predicate for $\mathcal{L}_{\omega,\omega}(\in, A)$.

(11) Determinacy of clopen class games of rank at most $\mathrm{Ord}+1$.

(12) The principle $\mathrm{ETR}_{\mathrm{Ord}}$.



Let me provide a taste of the proof.

Several steps proceed via subtle syntactic translations into various infinitary languages.

Note first that if $\mathbb{P}$ admits atomic forcing relation, then it admits a scheme of forcing relations $p \Vdash \varphi(\tau)$ for any particular first-order φ.

That recursion is set-like and can therefore be undertaken in GBC.

Atomic $\to$ uniform quantifier-free infinitary

Theorem

If $\mathbb{P}$ admits atomic forcing relation, then it has a uniform forcing relation $p \Vdash \varphi(\tau)$ for φ in the quantifier-free infinitary forcing language $\mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{P}}, \dot{G})$.

Proof sketch.

Nontrivial construction due to Holy, Krapf, Lücke, Njegomir and Schlicht. To each infinitary φ carefully assign names $\dot{a}_\varphi$ and $\dot{b}_\varphi$, specifically designed to track the truth of φ.

Logical complexity of φ hidden in name structure of $\dot{a}_\varphi$ and $\dot{b}_\varphi$.

Ultimately define $p \Vdash \varphi$ if $p \Vdash \dot{a}_\varphi = \dot{b}_\varphi$.

Prove that this satisfies the desired forcing recursion. $\square$

Forcing theorem $\to$ truth predicate for $\mathcal{L}_{\mathrm{Ord},\omega}$

Theorem

The class forcing theorem implies there is a truth predicate for $\mathcal{L}_{\mathrm{Ord},\omega}(\in)$.

Uses the forcing $\mathbb{F} = \mathrm{Coll}(\omega, V) \sqcup \{ e_{n,m} \mid n, m \in \omega \}$, where for $f \in \mathrm{Coll}(\omega, V)$ we define

$$f \leq e_{n,m} \quad \longleftrightarrow \quad f(n) \in f(m)$$

Note that $\mathrm{Coll}(\omega, V)$ is dense in $\mathbb{F}$, but with class forcing, this doesn't mean they are forcing equivalent.

Define name $\dot{\varepsilon}$ such that $\langle V, \in \rangle \cong \langle \check{\omega}, \dot{\varepsilon} \rangle$ by generic map.

Forcing relation for $\mathbb{F} \to$ truth predicate for $\mathcal{L}_{\mathrm{Ord},\omega}$

Assume $\mathbb{F}$ has atomic forcing relations. Get forcing relation for $\mathcal{L}_{\mathrm{Ord},0}$.

Build translation $\varphi \mapsto \varphi^\star$, where $\varphi \in \mathcal{L}_{\mathrm{Ord},\omega}$ and $\varphi^\star \in \mathcal{L}_{\mathrm{Ord},0}$ as follows:

$$\begin{aligned}
(x \in y)^\star &= x \mathrel{\dot{\varepsilon}} y \\
(x = y)^\star &= x = y \\
(\varphi \wedge \psi)^\star &= \varphi^\star \wedge \psi^\star \\
(\neg\varphi)^\star &= \neg\varphi^\star \\
\Big(\bigwedge_i \varphi_i\Big)^\star &= \bigwedge_i \varphi_i^\star \\
(\forall x\, \varphi)^\star &= \bigwedge_{m\in\omega} \varphi^\star(\check{m}).
\end{aligned}$$

The idea: $\langle V, \in\rangle \models \varphi(a) \quad \longleftrightarrow \quad \langle \omega, \varepsilon\rangle \models \varphi^\star(\dot{n}_a)$.
By consulting $\mathbb{1} \Vdash \varphi^\star(\dot{n}_a)$, get a truth predicate on V.

Truth on $\mathcal{L}_{\mathrm{Ord},\omega} \to$ iterated truth predicate

From a truth predicate on the infinitary language $\mathcal{L}_{\mathrm{Ord},\omega}$, one can construct an Ord-iterated first-order truth predicate.

This proceeds via a technical translation $(\beta, \varphi) \mapsto \varphi^*_\beta$ for φ in the language of first-order set theory with an Ord-iterated truth predicate, with φ^*_β in $\mathcal{L}_{\mathrm{Ord},\omega}$.

This reduced infinitary truth to first-order iterated truth assertions.

Iterated truth predicate $\to$ $\mathrm{ETR}_{\mathrm{Ord}}$

Finally, from an Ord-iterated truth predicate for first-order set theory, one can derive $\mathrm{ETR}_{\mathrm{Ord}}$, since the solution of a recursion can be extracted from the iterated truth predicate. (details suppressed) This is the $\mathrm{ETR}_{\mathrm{Ord}}$ analogue of a result due to Fujimoto.

Executive summary: $\mathrm{ETR}_{\mathrm{Ord}}$ gives atomic forcing relations, and atomic forcing relations give $\mathrm{ETR}_{\mathrm{Ord}}$.

Surprising consequence: if every class forcing notion $\mathbb{P}$ has atomic forcing relations, then they all have fully uniform forcing relations for infinitary assertions $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}$.

Proper class games

Proper class games

Consider two-player games of perfect information on a class X, such as $X = \mathrm{Ord}$.

Player I	α_0		α_2		α_4		$\cdots$		
Player II		α_1		α_3		α_5		$\cdots$	

Player I wins if the resulting play $\vec{\alpha}$ is in fixed payoff class $A \subseteq X^\omega$.

The usual notions of open game, strategy, winning strategy are all expressible for class games in Gödel-Bernays GBC set theory.

Clopen determinacy has strength

Theorem (Gitman,Hamkins)

There is a definable clopen proper-class game, whose determinacy is equivalent in GBC *to the existence of a truth predicate for first-order set-theoretic truth.*

In particular, in ZFC there is a definable clopen proper-class game with no definable winning strategy.

Clopen determinacy for class games has strength over GBC: it implies $\mathrm{Con}(\mathrm{ZFC})$, as well as iterated consistency assertions $\mathrm{Con}^{\alpha}(\mathrm{ZFC})$ and much more.

The game is: the truth-telling game.

The truth-telling game

Two players, in a court of law

- The *truth-teller*, in the witness box, answering questions
- The *interrogator*, posing the tricky questions

On each turn, interrogator asks: $\varphi(\vec{a})$?

Truth-teller answers: *true* or *false*.

Existential proviso: if $\exists x\, \varphi(x, \vec{a})$ is declared true, then truth-teller must also provide witness $\varphi(b, \vec{a})$.

A play of the game consists of a sequence of inquiries and truth pronouncements.

Winning conditions for the truth-telling game

The truth-teller wins, if she does not violate the recursive Tarskian truth conditions.

- Atomic truth assertions must be truthful
- Truth assertions must respect Boolean connectives
- Truth assertions must respect quantifiers.

This is an open game for the interrogator, since any violation will occur at a finite stage.

Truth-teller wins $\leftrightarrow$ truth predicate

Lemma

The truth-teller has a winning strategy in the truth-telling game if and only if there is a truth predicate for first-order truth.

Proof.

($\leftarrow$) If there is a truth predicate, then truth-teller can win by playing in accordance with it. Use global well-order (GBC) to pick witnesses.

($\rightarrow$) Suppose that the truth-teller has a winning strategy τ in the truth-telling game. I claim that the truth pronouncements made by τ are independent of the play in which they occur. Prove by induction on formulas. This provides a truth predicate. $\square$

By Tarski's non-definability of truth, there is no definable winning strategy for the truth-teller.

Interrogator cannot win

Lemma

The interrogator has no winning strategy in the truth-telling game.

Proof.

Consider any strategy σ for interrogator, directing him to issue certain challenges $\varphi(\vec{a})$. By reflection, there is θ with V_θ closed under σ: if all challenges and witnesses come from V_θ, then σ replies in V_θ. Let truth-teller answer with theory of $\langle V_\theta, \in \rangle$. This will survive against σ, and so σ is not winning for interrogator. $\square$

Clopen determinacy $\to$ truth predicate

So open determinacy implies that there is a truth predicate for first-order truth.

One can modify the truth-telling game by requiring the interrogator to count down in the ordinals during play.

This results in a clopen game, whose strategy still gives a truth predicate.

Conclusion

If clopen determinacy holds for class games, then there is a truth predicate for first-order truth.

Clopen determinacy $\leftrightarrow$ ETR $\leftrightarrow$ iterated truth

In fact, we find a precise equivalence.

Theorem

In Gödel-Bernays set theory GBC*, the following are equivalent.*

1. *Clopen determinacy for class games.*
2. *The principle ETR: every well-founded class recursion has a solution.*
3. *Every class well-order $\langle I, \lhd \rangle$ admits an iterated truth predicate.*

Equivalence of 2 and 3 was previously established by Fujimoto.

Easy direction: ETR implies clopen determinacy

Assume ETR and consider the game tree of any clopen game. This is well-founded, because the game is clopen. Consider the back-propagation labeling of positions in the game tree, recursively labeled with the winner from that position. By ETR, there is such a labeling.

Whichever player gets their label on the initial position has a winning strategy: stay on positions with their label.

Clopen determinacy implies ETR

Assume clopen determinacy for class games.

To prove ETR, suppose we are faced with a recursion $\varphi(x, b, F)$ along a well-founded class partial-order $\lhd$ on I.

Play the (counting down) *recursion game*. Like the truth-telling game, but the truth-teller reveals information about the solution of the recursion. Use a winning strategy for the truth-teller to construct an actual solution.

Analogue with second-order arithmetic

Steel proved (in his dissertation) that open determinacy and clopen determinacy for games on ω are both equivalent to ATR_0, and hence to each other.

But in second-order set theory, the proof of clopen determinacy from ETR does not generalize to open determinacy.

But open determinacy for class games is provable in stronger theories, such as $\mathrm{GBC} + \Pi^1_1$-comprehension.

Separating open from clopen determinacy

Sherwood Hachtman separated clopen from open determinacy.

Theorem (Hachtman)

If there is a transitive model of $\mathrm{ZF}^- + \kappa$ is inaccessible, then there is a model of GBC, in which clopen determinacy for class games holds, but open determinacy fails.

The Borel class games obtained by σ-algebra generated by open classes $B \subseteq \mathrm{Ord}^\omega$. (Note: not the same as Δ^1_1.)

Theorem (Hachtman)

Kelley-Morse set theory, if consistent, does not prove the determinacy of Borel class games.

Unexpected weakness in Kelley-Morse set theory KM

Class choice

Suppose that for every natural number n there is a class X with $\varphi(n, X)$.

Question

Must there be a class $X \subseteq \omega \times V$ such that $\forall n \in \omega\ \varphi(n, X_n)$?

This would be an instance of the class ω-choice principle.

Class choice

The *class choice* principle is the assertion that if for every set a there is a class X with $\varphi(a, X, Z)$, then there is a class $X \subseteq V \times V$ such that $\forall a\, \varphi(a, X_a, Z)$.

This principle is used in many set-theoretic constructions.

For example, one uses it to prove the Łoś theorem for ultrapowers of models of second-order set theory.

Question

Are the models of KM closed under the (internal) ultrapower constrution?

Answer: no.

Some failures in KM

Theorem (Gitman, Hamkins)

Under suitable large cardinal assumptions

- *There is a model of* KM *whose internal ultrapower by an ultrafilter on* ω *is not a model of* KM.
- *The theory* KM *fails to prove that* Σ^1_1 *is closed under first-order quantifiers: a formula of the form* $\forall x\, \varphi(x)$*, where* φ *is* Σ^1_1*, can fail to be equivalent to a* Σ^1_1*-formula.*

Hierarchy of class choice

- There is a model of KM in which an instance of the class ω-choice principle fails for some first-order formula $\varphi(x, X)$:

$$\forall n \in \omega\, \exists X\, \varphi(n, X) \to \exists Z\, \forall n \in \omega\, \varphi(n, Z_n)$$

- There is a model of KM in which the class set-choice principle holds, but the class Ord-choice principle fails in the case of a parameter-free first-order formula.
- There is a model of KM in which the parameter-free class Ord-choice principle holds, but the class Ord-choice scheme fails for a Π^1_1-formula.

$\mathrm{KM}^+ +$ class-DC

$\downarrow$

KM^+

$\downarrow$

$\mathrm{KM} +$ class $<\mathrm{Ord}$-choice

$\downarrow$

$\mathrm{KM} +$ class α-choice

$\downarrow$

KM

$\downarrow$

$\mathrm{GBC} + \Pi^1_1$-comprehension

$\downarrow$

$\mathrm{GBC} +$ Determinacy for open class games

$\downarrow$

$\mathrm{GBC} + \mathrm{ETR} \quad = \quad \mathrm{GBC} +$ Determinacy for clopen class games

$\downarrow$

$\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$

$\downarrow$

$\mathrm{GBC} +$ Class forcing theorem $\quad = \quad \mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$
$= \quad \mathrm{GBC} +$ truth predicates for $\mathcal{L}_{\mathrm{Ord},\omega}(\in, A)$
$= \quad \mathrm{GBC} +$ truth predicates for $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in, A)$
$= \quad \mathrm{GBC} +$ Ord-iterated truth predicates
$= \quad \mathrm{GBC} +$ Boolean set-completions exist $\quad =$
$\mathrm{GBC} +$ Determinacy for clopen class games of rank $\mathrm{Ord}+1$

$\downarrow$

$\mathrm{GBC} + \mathrm{ETR}_{<\mathrm{Ord}} \quad = \quad \mathrm{GBC} + \forall\alpha\exists\alpha$-iterated truth predicates

$\downarrow$

$\mathrm{GBC} + \mathrm{ETR}_{\alpha} \quad = \quad \mathrm{GBC} + \alpha$-iterated truth predicates

$\downarrow$

$\mathrm{GBC} + \mathrm{ETR}_{\omega}$

$\downarrow$

$\mathrm{GBC} + \mathrm{Con}^{\alpha}(\mathrm{GBC})$

$\downarrow$

$\mathrm{GBC} + \mathrm{Con}(\mathrm{GBC})$

$\downarrow$

GBC

Abundance of open questions

Of paramount importance is to understand the interaction of the second-order principles with forcing.

Question

Which of the second-order principles in the hierarchy are preserved by set forcing?

Question

Does set forcing preserve ETR? Can forcing create new class well-ordered order-types?

Set forcing preserves ETR_Γ, but the issue is whether new order-types Γ can be created.

Role of global choice

Question

Is $\mathrm{GBC} +$ ETR conservative over $\mathrm{GB} + \mathrm{AC} +$ ETR?

One would want, of course, simply to force global choice over a model of $\mathrm{GB} + \mathrm{AC} +$ ETR, and then argue that ETR was preserved. A key issue again is whether this forcing creates new class well-ordered order types.

Unrolling

The process of unrolling a model of second-order set theory to a much-taller first-order model is now standard (see Carolin Antos dissertation). One builds a model of set theory on top of the universe using class codes of well-founded extensional relations.

KM^+ is bi-interpretable with $\mathrm{ZFC}_{\mathrm{I}}^-$ by this method.

Question

Which second-order theories can implement the unrolling construction? What theory does one obtain in the unrolling of a model of $\mathrm{GBC} +$ ETR?

One seems to need Δ^1_1-comprehension even to get extensionality in the unrolled superstructure. Is ETR enough?

Reflection

The second-order reflection principle asserts that every second-order assertion reflects to an encoded class model.

Question

What is the reverse-mathematical strength of the second-order reflection principle?

Related to question: does ZFC^- prove that every true first-order statement is true in some transitive set? (see work of Gitman & Friedman)

Reflection is provable in strongest second-order set theories, using class-DC.

Can we separate it from the weaker theories?

Well-order comparability

Question

What is the reverse-mathematical strength over GBC of the class well-order comparability principle?

The theory GBC + ETR is able to prove that any two class well-orders are comparable.

Is comparability of class well-orders equivalent to ETR?

Class Fodor theorem

Question

What is the reverse-mathematical strength of the class Fodor theorem?

The class Fodor theorem asserts that every regressive function $f : S \to \mathrm{Ord}$ on a stationary class $S \subseteq \mathrm{Ord}$ is constant on a stationary set.

This is provable in KM^+.

Can we show that the use of class-choice cannot be eliminated?

Class stationary partition

Question

What is the reverse-mathematical strength over GBC of the assertion that every stationary class can be partitioned into Ord many stationary classes?

This is provable in KM^+ by an analogue of the classical argument, using class-choice.

Can we prove this in weaker theories or separate it from weaker theories?

Related to the issue of class stationary reflection.

Class forcing axiom

Corey Switzer and I are investigating the *Class forcing axiom* (CFA), which asserts that for any class forcing notion $\mathbb{P}$ having forcing relations and not adding sets and for any Ord-sequence $\langle D_\alpha \mid \alpha < \mathrm{Ord}\rangle$ of dense classes $D_\alpha \subseteq \mathbb{P}$, there is a filter $F \subseteq \mathbb{P}$ meeting every D_α.

This is like a class-forcing analogue of MA or PFA, but for distributive forcing. Although inconsistent at ω_1, the CFA is conservative over GBC for first-order assertions.

CFA implies Ord is not Mahlo; no Ord-Suslin trees; $\diamondsuit_{\mathrm{Ord}}$ fails; every fat-stationary class contains a class club.

Class forcing maximality principle

The *class forcing maximality principle* asserts that any second-order statement $\varphi(X)$ with arbitrary class parameters that is forceably necessary by class forcing with forcing relations and not adding sets is already true.

This implies the class forcing axiom. Conservative over GBC.

Question

Which models of second-order set theory can be extended to a model of the class forcing maximality principle or the class forcing axiom? Which second-order theories are these axioms conservative for first-order assertions?

We can construct models of $\mathrm{GBC} + \mathrm{ETR} + \mathrm{CMP} + \mathrm{CFA}$.

References

Victoria Gitman and Joel David Hamkins "Open determinacy for class games," in *Foundations of Mathematics, Logic at Harvard, Essays in Honor of Hugh Woodins 60th Birthday*, AMS Contemporary Mathematics series, 2016.

Victoria Gitman, Joel David Hamkins, Peter Holy, Philipp Schlicht, and Kameryn Williams. "The exact strength of the class forcing theorem," manuscript under review. arXiv: 1707.03700[math.LO].

Victoria Gitman, Joel David Hamkins, Thomas Johnstone. "Kelley-Morse set theory and choice principles for classes," manuscript in preparation.

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
City University of New York