Self-reference in computability theory and the universal algorithm

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Ouroboros: Formal Criteria of Self-Reference
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Introduction

A delightful theorem

The universal algorithm

and its generalization to set theory,

The universal finite set.

Both rely fundamentally on self-reference.

Self-reference in computability

Definition

A *Quine* is a program that gives as output its own source code.

Imagine a Turing machine program that prints out its own source code on the tape.

Since this output can be used in further computation, a Quine is a program able to handle itself computationally.

Quine examples

Programmers compete to produce short Quines.

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C Quine

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int main(){char*s="int main(){char*s=%c%s%c;printf(s,34,s,34);return 0;}";printf(s,34,s,34);re
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SQL Quine

```
SELECT REPLACE(REPLACE('SELECT REPLACE(REPLACE("$",CHAR(34),CHAR(39)),CHAR(36),"$") AS Quine',
```

From the Quine page and StackOverflow.

Quines are not a computability concept

The Quine concept is not actually computability theoretic.

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Reason:

- It does not respect the equivalence of programs.
- It is sensitive to the computational model.

If you replace a program with another that computes the same function, or with an equivalent program in a different computational language, then it is no longer a Quine.

Nevertheless, Quines are fascinating!

Quines exist

Fact

Every Turing-complete programming language admits Quines.

Let me explain how this follows from the Kleene recursion theorem.

Recursion theorem: self-reference in computability

Kleene recursion theorem

For every computable transformation of the space of programs,

$$e \mapsto f(e)$$

there is a program e such that e and $f(e)$ compute the same function.

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Kleene recursion theorem

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there is a program e such that e and $f(e)$ compute the same function.

The program $f(e)$ can make explicit reference to e, and so the fixed point e in effect makes reference to itself.

The recursion theorem is a computability analogue of the Gödel-Carnap fixed-point lemma: for every $\varphi(x)$ in language of arithmetic, there is a sentence σ with

$$\sigma \leftrightarrow \varphi(\ulcorner \sigma \urcorner).$$

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Theorem

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Proof.

Let $f(e)$ be a program that gives e as output. By the recursion theorem, there is a program e such that e and $f(e)$ give the same output. So program e gives e as output. □

Since the proof of the recursion theorem is constructive, this argument can produce explicit Quines.

TEX is Turing-complete and therefore admits a Quine

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TEX source code:

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\def\T{
\tt \hsize 32.5em\parindent 0pt\def \S {\def \S ##1>{}}\S \string
\def \string \T \string {\par \expandafter \S \meaning \T \string
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Output:

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Ouroboros program: a Quine cycle

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Java program

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    "    for(int i = 20; i <= 25; i++)",
    "        cout << l[i] << endl;",
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    "        cout << l[0] + q + l[i] + q + ',' << endl;",
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    "    for(int i = 2; i <= 9; i++)",
    "        System.out.println(l[i]);",
    "    for(int i = 0; i < l.length; i++)",
    "        System.out.println( l[0] + q + l[i] + q + ',' );",
    "    for(int i = 10; i <= 18; i++))",
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Diverse longer Ouroboros chains have been constructed.

Universality via self-reference

Let me now explain how the capacity for self-reference in computability leads to universality.

Universal program (warm-up): the petulant child

Consider program e

It searches for a proof in PA of a statement:

"program e does not give output n."

When found, it gives output n and halts.

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Key Observation: you cannot refute any particular output for this program, since then indeed it would have that output.

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Consequently, for any *n* it is consistent with PA that this program gives output *n*.

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Conclusion

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- When run in the standard model $\mathbb{N}$, the program never halts.
- For any number n, there is a nonstandard model of arithmetic in which the program eventually gives output n.

Universal algorithm: sequence version

Consider program p

Searches for a proof of:

> *"program p does not enumerate the finite sequence $a_0, a_1, \ldots, a_n$ and then halt."*

When found, enumerate that sequence immediately.

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You cannot refute any particular sequence being enumerated by p.

So it is consistent with PA that p enumerates any desired sequence.

Universal program: function version

Theorem

There is a program p, such that for every function $f : \mathbb{N} \to \mathbb{N}$, there is a nonstandard model of arithmetic N, such that in N, program p on input $n \in \mathbb{N}$ gives output $f(n)$.

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Proof.

The assertions

> *"the n^{th} number enumerated by p is $f(n)$"*

are finitely consistent with PA. So by compactness they are all true in some model. □

The universal algorithm: full extension version

Theorem (Woodin)

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3. *In any model of arithmetic, if e enumerates sequence s and t is an extension of s, then there is an end-extension of the model in which e enumerates t.*

In particular, every finite sequence $s \in \mathbb{N}^{<\omega}$ is enumerated by e in some model $M \models \mathrm{PA}$.

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- Weaker forms go back to early 1960s, Mostowski and Kripke.

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- Proceed in stages, releasing the sequence in batches.
- Stage n succeeds, if there is a proof from fragment PA_k, smaller than fragments at earlier stages, of a statement

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  where s is an explicitly listed sequence of numbers.
- In this case, release s at stage n. Proceed to next stage.

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Self-reference

Use Kleene recursion theorem to find e, solving the circular definition.

Finiteness

Observation

The universal sequence is finite.

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Proof.

The successful fragments PA_k are getting smaller, so this can happen at most finitely many times.

Thus, the sequence enumerated by e will be finite.

And PA can undertake this argument. □

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Claim

If stage n is successful, then fragment PA_{k_n} is nonstandard.

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Consider last stage n. The assertion

"e enumerates s with exactly n successful stages"

has complexity Σ^0_2. For standard k, the Mostowski reflection theorem shows $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{Tr}_k)$.

So M cannot have proof from PA_k of something contrary to actual behavior. So k_n must be nonstandard. □

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In particular, e enumerates empty sequence in $\mathbb{N}$.

Proving the extension property of e

Assume e enumerates s in M and t is an extension of s.

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So M can build a Henkin model of this theory, N. Note $N \models \mathrm{PA}$ since k is nonstandard.

So N end-extends M and thinks e enumerates t, as desired. □

It turns out that a number of classical results can be seen as immediate consequences of the universal algorithm.

Let us give a few of these examples.

Maximal Σ_1 diagrams

Corollary

The Σ_1 diagram of a model of arithmetic is never maximal.

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Proof.

In any model of arithmetic M, there is some stage n that is not successful, but can become successful in an end-extension.

So the extension has new Σ_1 facts: "stage n is successful." □

Independent Π^0_1 sentences

Corollary (Kripke, Mostowski)

There are infinitely many independent Π^0_1 sentences

$$\eta_0,\ \eta_1,\ \eta_2, \ldots$$

Any desired true/false pattern is consistent with PA.

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Proof.

Let $\eta_k =$"k does not appear on the universal sequence." □

Independent buttons

Corollary "independent buttons"

There are Σ^0_1 sentences, all false in $\mathbb{N}$.

$$\rho_0,\ \rho_1,\ \rho_2,\ \cdots$$

But over any model of arithmetic M, any addition set of them I, coded in M, can become true in an end-extension N.

1. $N \models \rho_k$ for all $k \in I$.
2. For $k \notin I$, truth of ρ_k is not changed.

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Independent Orey sentences

Corollary "independent switches"

There is an infinite list of independent Orey sentences

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Proof.

Let $\sigma_k =$"k^{th} binary bit of the last number on the universal sequence is 1." □

Flexible formulas

Corollary (Kripke)

There is a Σ_n formula $\varphi(x)$, for any $n \geq 2$, that can be made equivalent to any desired Σ_n formula $\phi(x)$ in an end-extension.

That is, for any $M \models \mathrm{PA}$ and ϕ, there is an end-extension satisfying

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Proof.

Let $\varphi(x) = \Phi(k, x)$ where k is the last element of the universal sequence and Φ is a universal Σ_n formula. □

Set-theoretic analogue

What is the set-theoretic analogue of the universal algorithm?

One wants a version of the theorem for models of set theory.

Arithmetic vs. set theory

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The computably enumerable sets are gradually revealed as time proceeds. Elements are confirmed at some stage of time.

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$$\{ x \mid \varphi(x) \} \qquad \varphi(x) \leftrightarrow \exists \alpha \; V_\alpha \models \psi(x).$$

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Elementary Fact

Locally verifiable sets $= \Sigma_2$ definable.

Question

So the set-theoretic analogue of c.e. is Σ_2 definable.

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Question(Hamkins)

Is there a Σ_2 definable set $\{ x \mid \varphi(x) \}$ with the following?

- ZFC proves $\{ x \mid \varphi(x) \}$ is a set.
- For every countable model $M \models \mathrm{ZFC}$, if

$$M \models \{ x \mid \varphi(x) \} = y \subseteq z,$$

then there is top-extension N with

$$N \models \{ x \mid \varphi(x) \} = z.$$

Partial progress

I had initially made partial progress.

- I could do it with Π_3 definitions.
- Or with Σ_2, if you work only in models of certain theories, such as eventual GCH or $V \neq \mathrm{HOD}$.
- Or with Σ_2, if you allow $\{ x \mid \varphi(x) \}$ to sometimes be a proper class.

Woodin and I began to work together on the problem in September.

Universal finite set

The question is answered in joint work with W. Hugh Woodin.

Theorem (Hamkins + Woodin)

There is a Σ_2 definition φ such that

1. ZFC *proves* $\{ x \mid \varphi(x) \}$ *is finite.*

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Theorem (Hamkins + Woodin)

There is a Σ_2 definition φ such that

1. ZFC *proves* $\{ x \mid \varphi(x) \}$ *is finite.*
2. *If M is transitive, then $M \models \{ x \mid \varphi(x) \} = \emptyset$.*
3. *If M is a countable model of* ZFC *with*

$$M \models \{ x \mid \varphi(x) \} = y \subseteq z,$$

where z is finite in M, then there is top-extension N with

$$N \models \{ x \mid \varphi(x) \} = z.$$

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And so on for other cardinals or other kinds of universal sets.

Also, there is a sequence version of the theorem.

Ideas from the proof

- The set is defined in stages.

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- The set is defined in stages.
- Look for absence of top-extensions of $\langle V_\beta, \in \rangle$, rather than for proofs.
- Sets are added at a stage if there is no suitable model in which they are added as the last elements.
- Self-reference via Gödel-Carnap fixed-point rather than Kleene recursion.
- Handling the ω-standard model case involves forcing and rank of well-founded trees.

For a detailed argument, see the paper or my slides from the Prague Gathering, Beauty of Logic 2018 conference last month.

Applications of the universal finite set

Let me mention a few applications of the universal finite set theorem.

No model of set theory has maximal Σ_2 diagram

Theorem

No model of set theory M has a maximal Σ_2 diagram.

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Theorem

No model of set theory M has a maximal Σ_2 diagram.

Proof.

If n is the first unsuccessful stage in M, then the assertion

"stage n is successful"

is Σ_2 assertion not yet true, but true in a top-extension of M. □

Maximal Σ_2 extensions of ZFC

Meanwhile, we can easily construct maximal consistent Σ_2 extensions of ZFC.

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Implications for Maddy's MAXIMIZE in philosophy of set theory.

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Implications for Maddy's MAXIMIZE in philosophy of set theory.

- Should maximize the sentences verifiable in some V_θ.
- But this never happens in well-founded models.
- Her principle therefore pushes us to ill-foundedness.

Σ_2 definability

Theorem

Every individual in a countable model of set theory M can become Σ_2 definable from a natural number parameter in a top-extension of M.

Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension N.

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Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension N.

Proof.

The definition is, "the unique set added at stage n," where n is the first unsuccessful stage in M. □

Everybody is possibly necessarily special

Corollary

Every individual in a countable ω-standard model of set theory can become Σ_2 definable, without parameters, in a top-extension.

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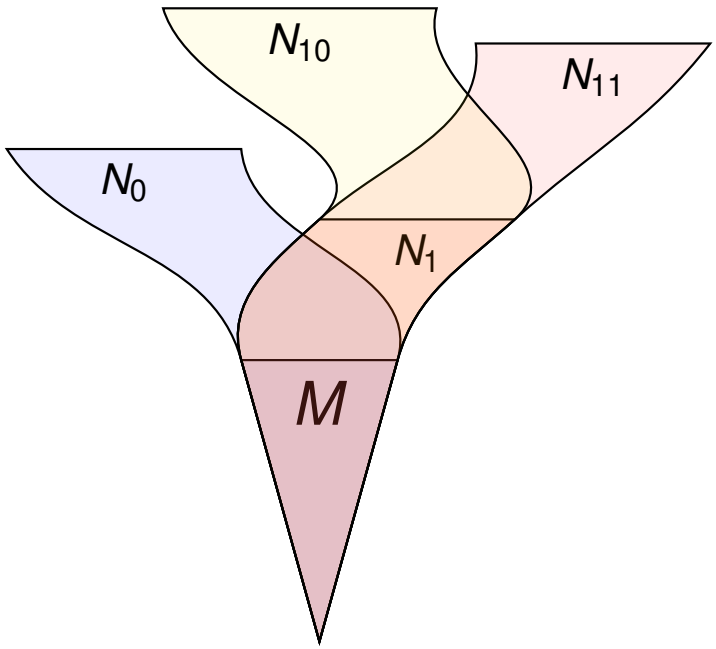
More precisely, in top-extensional potentialism, everybody is possibly necessarily special.

Since N is also ω-standard, the result can be iterated.

The tree of top-extensions

This picture leads into the topic of potentialism, which I recently spoke about at Oxford and at the Winter School in Hejnice.



We analyze the precise modal validities of set-theoretic top-extensional potentialism and other concepts of potentialism.

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Thank you.

Slides and articles available on http://jdh.hamkins.org.

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