INCONSISTENCY OF A SUPER-REINHARDT CARDINAL

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We present a proof that the following extension of ZF is inconsistent. Add a constant symbol **j** to the language and add Separation and Replacement axioms for formulas involving **j**, and add the axiom that j is an elementary embedding $V \prec V$ and also DC_λ where λ is the least fixed point of j above the critical point of j. We wish to claim that this theory is inconsistent, and it follows from this that a super-Reinhardt cardinal is inconsistent with ZF.

Lemma 0.1. Work in the theory described above. Then it is provable on the stated assumptions that V_κ is a model for the existence of a cardinal κ' with the following property. There exists a sequence $\langle \kappa_i : i \in \omega \rangle$ where $\kappa_0 = \kappa'$, such that if we let $\lambda := \sup\{\kappa_i : i \in \omega\}$, then for any $X \in V_{\lambda+2}$ there exists an $\alpha < \kappa_0$ and an $X' \in V_{\lambda+2}$ such that there is an elementary embedding $j : (V_{\lambda+1}, X') \prec (V_{\lambda+1}, X)$ with critical point α such that $j(\alpha) = \kappa_0$ and $j(\kappa_i) = \kappa_{i+1}$.

Proof. If we let j be the elementary embedding originally assumed to exist for κ itself then $\langle j^n(\kappa) : n \in \omega \rangle$ and $\lambda := \sup\{j^n(\kappa) : n \in \omega\}$ witness the stated reflection property for κ, and the existence of the elementary embedding j means that we can find a $\kappa' < \kappa$ such that the stated reflection property holds when we let $\kappa_0 := \kappa'$ and $\kappa_n := j^{n-1}(\kappa)$ for $n > 0$, κ_0 being reflected to κ' and each κ_n for $n > 0$ being reflected to κ_{n-1}. Let j_0 be an elementary embedding with domain $V_{\lambda+2}$ witnessing this instance of reflection. Then after the first reflection, κ_0 can be reflected to some ordinal between κ' and κ_0, and so on. At stage n for $n > 0$, let j_n be an elementary embedding with domain $V_{\lambda+2}$ witnessing the reflection. Thus for each $n > 0$ we can find a sequence $\langle \kappa'_i : i \in \omega \rangle$ such that $\kappa_0 = \kappa'$ and $\kappa'_i < \kappa$ for integers i such that $0 \leq i \leq n$ and $\kappa'_i = \kappa_{i-n-1}$ for integers $i > n$, and the stated reflection property holds for $\langle \kappa'_i : i \in \omega \rangle$ and λ as before. Then we can reflect the entire sequence $\langle \kappa_n : n \in \omega \rangle$ to a sequence $\langle \kappa'_n : n \in \omega \rangle$, with $\kappa'_n < \kappa$ for all n, and take $\lambda' := \sup\{\kappa'_n : n \in \omega\}$. We want to claim that these data witness our reflection principle for κ'_0. We can use the restrictions of j_n to V_λ to construct an embedding $j' : V_{\lambda'+1} \to V_{\lambda'+1}$. Consider an $X' \in V_{\lambda'+2}$, we wish to find an $X'' \in V_{\lambda'+2}$ for which

$j' : (V_{\lambda'+1}, X'') \prec (V_{\lambda'+1}, X')$ will hold. Suppose $Y' \in X' \subseteq V_{\lambda'+1}$, and let $Y'_n := Y \cap V_{\kappa'_n}$, $Y_n := (j_n^{-1}(j \mid V_\lambda))^{n+1}(Y'_n)$. We can glue the Y_n together to obtain a $Y \in V_{\lambda+1}$ for each $Y' \in X'$, denote by X the set of all such Y. We have just defined a mapping $V_{\lambda'+1} \to V_{\lambda+1}$ which we will denote by e. Now we have $X \in V_{\lambda+2}$, then let $X'' := (j(X))^U$ where $U := \{e(Y') : Y' \in V_{\lambda'+1}\}$ and $A^U := A \cap U$ for $A \in V_{\kappa_0+1}$ and $A^U := \{B^U : B \in A\}$ for $A \in V_{\lambda+1} \setminus V_{\kappa_0+1}$. This is the desired X'' with the property which we seek. Thus V_κ is a model for the existence of a cardinal κ' satisfying the stated reflection principle. □

Lemma 0.2. The existence of a cardinal with the stated reflection property is still inconsistent with choice.

Proof. The proof of this claim is given by the proof of Theorem 5, Section V of [2]. We reproduce the proof for convenience. Assume ZFC and let $g \in V_{\lambda+2}$ be an ω-Jonsson function over λ. Then let g' be such that there is an elementary embedding $j : (V_{\lambda+1}, g') \prec (V_{\lambda+1}, g)$ with critical point $\alpha < \kappa_0$ and $j(\alpha) = \kappa_0$ and $j(\kappa_n) = \kappa_{n+1}$. We have $g(x) = \alpha$ for some $x \in [j"\lambda]^\omega$. Let $\beta := g'(j^{-1}(x))$. We get $j(\beta) = \alpha$ which contradicts α being the critical point of j. □

Putting these lemmas together we obtain that the theory described in the opening paragraph is inconsistent.

References

[1] Akihiro Kanamori. The Higher Infinite: Large cardinals in set theory from their beginnings, 2nd edition. Springer Monographs in Mathematics, 2003.

[2] M. Victoria Marshall R. Higher order reflection principles, *Journal of Symbolic Logic,* vol. 54, no. 2, 1989, pp. 474–489.