Parallels in universality between the universal algorithm and the universal finite set

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for insightful comments and helpful discussions.

This list includes several of my co-authors on papers relevant for this talk.

Introduction

A delightful theorem

The universal algorithm

and its generalization to set theory,

The universal finite set

with applications to set-theoretic potentialism.

Review of the universal algorithm

Theorem (Woodin)

There is a Turing machine program e such that:

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1. *Program e enumerates a finite sequence only, and* PA *proves this.*
2. *Program e enumerates the empty sequence in the standard model* $\mathbb{N}$.
3. *If* $M \models \mathrm{PA}$ *and e enumerates s in M, then for any larger t in M there is an end-extension* $N \models \mathrm{PA}$ *in which e enumerates t.*

In particular, every finite sequence $s \in \mathbb{N}^{<\omega}$ is enumerated by e in some model $M \models \mathrm{PA}$.

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- Visser points out affinity with the classical ‘exile’ argument in proof theory.
- Weak forms go back to Mostowski and Kripke 1960s.
- Current generalizations to set theory by Hamkins and Woodin 2017, and Hamkins, Williams, Welch 2018.

My simplified proof

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- Stage n succeeds, if there is a proof from fragment PA_{k_n}, with k_n smaller than all earlier k_i, of a statement of the form

  "it is not the case that e has exactly n stages and releases s at stage n,"

  where s is an explicitly listed sequence of numbers.

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- Proceed in stages, releasing the sequence in batches.
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  > *"it is not the case that e has exactly n stages and releases s at stage n,"*

  where s is an explicitly listed sequence of numbers.
- In this case, release s at stage n. Proceed to next stage.

Proof of universal algorithm

Succinctly:

The program e enumerates s at stage n, if it finds proof, in a strictly smaller fragment of PA each time, that it does not do so as its last stage.

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Note

Use Kleene recursion theorem to find e, solving the circular definition.

Finiteness

Observation

The universal sequence is finite.

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Proof.

The fragments k_n must descend, and so there can be at most finitely many successful stages.

Thus, the sequence enumerated by e will be finite.

And PA can undertake this argument. □

Empty in the standard model

Claim

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Proof.

Consider last stage n. The assertion

"e enumerates s with exactly n successful stages"

has complexity Σ^0_2. For standard k, the Mostowski reflection theorem shows $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{Tr}_k)$.

So M cannot have proof from PA_k of something contrary to actual behavior. So k_n must be nonstandard. □

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In particular, e enumerates empty sequence in $\mathbb{N}$.

Proving the extension property of e

Assume e enumerates s in M and $s \subseteq t$.

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Key Observation

Since stage n was not successful, M must think that

$$\mathrm{PA}_k + \text{“}e \text{ has exactly } n \text{ stages and enumerates } t\text{”}$$

is consistent.

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So N end-extends M and thinks e enumerates t. And $N \models \mathrm{PA}$ since k is nonstandard. This proves the theorem.□

Classical consequences

Several classical results in the model theory of arithmetic can be seen as immediate consequences of the universal algorithm.

Let us explore a few examples.

Maximal Σ_1 diagrams

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Proof.

If $M \models \mathrm{PA}$, then there is an unsuccessful stage n, which becomes successful in an end-extension N.

So the assertion "stage n is successful" is a new Σ_1 statement about n true in N, false in M. $\square$

For example, there is a diophantine equation, with coefficients in M, having no solution in M, but it has a solution in N.

Independent Π^0_1 sentences

Corollary (Kripke, Mostowski)

There are infinitely many independent Π^0_1 sentences

$$\eta_0, \ \eta_1, \ \eta_2, \ldots$$

Any desired true/false pattern is consistent with PA.

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Proof.

Let $\eta_k =$ "k does not appear on the universal sequence." $\square$

Independent buttons

Corollary "independent buttons"

There are Σ^0_1 sentences

$$\rho_0, \ \rho_1, \ \rho_2, \ \cdots$$

all false in $\mathbb{N}$ and for any $M \models \mathrm{PA}$, any pattern I coded in M, there is end-extension N with

1. Every ρ_k becomes true in N for $k \in I$.
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Independent Orey sentences

Corollary "Independent switches"

There is an infinite list of independent Orey sentences

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Proof.

Let $\sigma_k =$ "k is amongst the numbers added at the last stage." $\square$

Flexible formulas

Corollary (Kripke)

For $n \geq 2$, there is a Σ^0_n formula $\varphi(x)$ that can be made so as to agree with any desired Σ^0_n formula $\phi(x)$ in an end-extension.

That is, for any $M \models \mathrm{PA}$ and any such ϕ, there is an end-extension N satisfying

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Proof.

Let $\varphi(x) = \Phi(k, x)$ where k is the last element of the universal sequence and Φ is a universal Σ^0_n formula. $\square$

Flexible formula, uniform version

Theorem

There is a computable sequence $\sigma_n(x)$ of Σ_n formulas, for $n \geq 2$, such that for any $M \models \mathrm{PA}$ and any sequence of Σ_n formulas ϕ_n coded in M, there is end-extension N satisfying

$$\forall x \quad \sigma_n(x) \leftrightarrow \phi_n(x).$$

Set-theoretic analogue

What is the set-theoretic analogue of the universal algorithm?

One wants a version of the theorem for models of set theory.

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The locally verifiable sets have members confirmed in some V_α, as the set-theoretic universe grows.

$$\{\, x \mid \varphi(x) \,\}, \quad \text{where } \varphi(x) \leftrightarrow \exists \alpha \; V_\alpha \models \psi(x).$$

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Elementary Fact

Locally verifiable sets $=$ Σ_2 definable.

Question

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Question (Hamkins)

Is there a Σ_2 definable set $\{ x \mid \varphi(x) \}$ with the following?

- ZFC proves $\{ x \mid \varphi(x) \}$ is a set.
- For every countable model $M \models \mathrm{ZFC}$, if

$$M \models \{ x \mid \varphi(x) \} = y \subseteq z,$$

then there is top-extension N with

$$N \models \{ x \mid \varphi(x) \} = z.$$

Partial progress

I had initially made partial progress.

- I could do it with Π_3 definitions.
- I could do it with Σ_2, restricting to models of certain theories, such as eventual GCH or $V \neq \mathrm{HOD}$.
- I could do it with Σ_2, if we allow $\{ x \mid \varphi(x) \}$ sometimes to be a proper class.

Woodin and I began to work together on the problem in September 2017.

Universal finite set

Ultimately, we were able to answer the question.

Theorem (Hamkins + Woodin)

There is a Σ_2 definition φ such that

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There is a Σ_2 definition φ such that

1. ZFC *proves* $\{ x \mid \varphi(x) \}$ *is finite.*
2. *If M is transitive, then* $M \models \{ x \mid \varphi(x) \} = \emptyset$.
3. *If M is a countable model of* ZFC *with*

$$M \models \{ x \mid \varphi(x) \} = y \subseteq z,$$

where z is finite in M, then there is top-extension N with

$$N \models \{ x \mid \varphi(x) \} = z.$$

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And so on for other cardinals or other kinds of universal sets.

Also, there is a sequence version of the theorem.

Proof sketch

We break the proof into two pieces.

- Process A will work with ω-nonstandard models.
- Process B will work with ω-standard models.

In the end, we unify the two processes into a single Σ_2 definition.

Process A

We define $\varphi_A(x)$ as follows.

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Stage n succeeds, if there is a beth-fixed point β_n, above all previous β_i, and number k_n, below all previous k_i, such that

> $\langle V_{\beta_n}, \in \rangle$ *has no topped extension to* $N \models \mathrm{ZFC}_{k_n}$ *with* $N \models$ *"process A ends at stage n with* β_n *and this is preserved by* $P(\beta_n)$*-preserving forcing."*

In this case, let y_n be the finite set coded into the GCH pattern starting at β_n^+. Declare $\varphi_A(x)$ for all $x \in y_n$.

Use the Gödel-Carnap fixed-point lemma to find $\varphi_A(x)$ solving this circular definition.

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Since the k_n are descending, we have only finitely many successful stages. So the set is finite.

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Proof.

If $M \models \{ x \mid \varphi_A(x) \} = y$ (and maximal successful stages in any forcing extension), then for standard k find many θ with $V_\theta^M \models \mathrm{ZFC}_k + \{ x \mid \varphi_A(x) \} = y$. So $V_{\beta_{k_n}}^M$ has many top-extensions in M after all, but it wasn't supposed to since n was successful. $\square$

Process A extension property

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Go to an elementary top-extension $M \prec M^+$, and let $\beta = \beth_\beta > M$ in M^+.

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Stage n still not successful in M^+. In particular, not successful with β.

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So there is a topped extension $N \models \mathrm{ZFC}_k$ of $\langle V_\beta, \in \rangle^{M^+}$ with exactly stage n successful at β, and preserved by forcing. N top-extends M.

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Build $N[G]$ to code z at β^+. So $N[G] \models \{ x \mid \varphi_A(x) \} = z$, as desired. $\square$

Process B

Turn now to process B.

This will work with ω-standard models.

In order to ensure finiteness, we shall need to find a different way to descend.

Process B definition

Define $\varphi_B(x)$.

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Stage n succeeds, if there is $\beth$-fixed point γ_n, larger than previous γ_i, and ordinal λ_n, smaller than previous λ_i, such that after collapse forcing $\mathrm{Coll}(\omega, \gamma_n)$, the now-countable structure $\langle V_{\gamma_n}, \in \rangle$ has no topped extension to a model $N \models \mathrm{ZFC}$ satisfying "process B has exactly n stages, uses γ_n at stage n, and this is preserved by $P(\gamma_n)$-preserving forcing," and such that, furthermore, this true Π^1_1 assertion in $V[g]$ is witnessed by the well-foundedness of a tree of rank λ_n.

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In this case, let y_n be the finite set coded into the GCH pattern at γ_n^+ and declare $\varphi_B(x)$ for all $x \in y_n$.

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Since the λ_n are descending, there are at most finitely many successful stages.

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Claim

If stage n is successful, then λ_n is a nonstandard ordinal.

Proof.

The tree can't actually be well-founded, since the statement isn't true in the ambient universe. □

The extension property follows as with property A.

Unify A and B

Finally, we unify the two processes into one Σ_2 definition $\varphi(x)$.

Run the processes concurrently, but only accept the results of process B if no stages in A have yet been successful.

Process A can only succeed in an ω-nonstandard model, so once that happens, we get extension by the process A analysis.

Can verify the process A extension property without having new instances of process B in the extension.

So can unify into one definition $\{ x \mid \varphi(x) \}$. This proves the universal finite set theorem. $\square$.

Applications of the universal finite set

Let us similarly discuss some applications of the universal finite set theorem.

No model of set theory has maximal Σ_2 theory

Theorem

No model of set theory M has a maximal Σ_2 diagram. Indeed, there is a Σ_2 assertion $\sigma(n)$ with some natural-number parameter $n \in \omega^M$, which is not true in M but is consistent with the Σ_2 diagram of M.

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Proof.

Let $\sigma(n)$ be the assertion that stage n is successful. Some stage is not successful in M, but could become successful in a top-extension. So this is a Σ_2 assertion about n that is not yet true, but consistent with ZFC plus the Σ_2 diagram of M. $\square$

Maximal Σ_2 extensions of ZFC

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If $M \models T$, then M must have stage n successful for every standard n.

Corollary

No ω-standard model of ZFC has a maximal Σ_2 theory.

In particular, no transitive model of ZFC has a maximal Σ_2 theory.

This raises issues for certain arguments in the philosophy of set theory.

Σ_2 definability

Theorem

In any countable model of set theory M, every element becomes Σ_2 definable from a natural number parameter in some top-extension of M.

Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension N.

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Indeed, there is a single definition and single parameter $n \in \omega^M$, such that every $a \in M$ is defined by that definition with that parameter in some top-extension N.

Proof.

The definition is, "the unique set added at stage n," where n is the first unsuccessful stage in M. $\square$

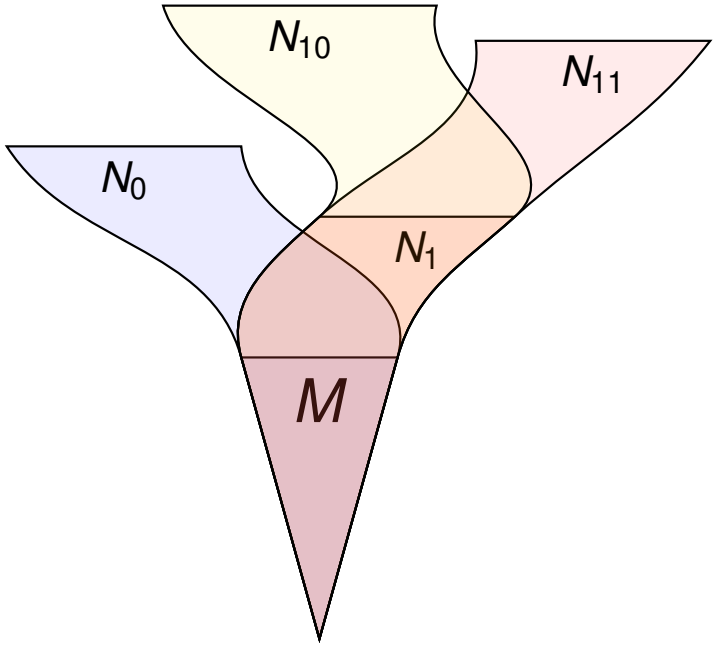
Parameter-free Σ_2 definability

Corollary

For any countable ω-standard model of set theory M, every $a \in M$ becomes Σ_2 definable without parameters in some top-extension N of M.

Since N is also ω-standard, the result can be iterated.

The tree of top-extensions



This picture leads into the topic of potentialism, with which I've been recently preoccupied with several papers and talks.

Theorem

In the potentialist system consisting of the countable models of ZFC *under top-extensions, considered as a Kripke model:*

1. *Exactly S4 is valid with respect to assertions in $\mathcal{L}_\in$ with parameters.*
2. *For any countable $M \models \mathrm{ZFC}$, there is parameter $n \in \omega^M$ such that exactly S4 is valid with respect to assertions in $\mathcal{L}_\in(n)$.*
3. *For sentences, the validities are between S4 and S5.*
4. *These bounds are sharp; both endpoints are realized.*

The modal analysis raises certain issues with regard to the philosophy of potentialism versus actualism in the philosophy of mathematics.

Universal Σ_1-definable finite sequence

In recent work with myself, Kameryn Williams and Philip Welch, we have found a version of the theorem using Σ_1 definitions in set theory, working with end-extensions of models of any theory extending ZF.

Σ_1-definable universal finite sequence

Theorem

Every c.e. extension $\overline{\mathrm{ZF}} \supseteq \mathrm{ZF}$ *defines a* Σ_1 *sequence*

$$a_0,\ a_1,\ \ldots,\ a_n$$

with the following properties:

1. ZF *proves that the sequence is finite.*
2. *In any transitive model* M *of* $\overline{\mathrm{ZF}}$*, the sequence is empty.*
3. *If sequence is* s *in countable* $M \models \overline{\mathrm{ZF}}$*, then for every extension* $s \subseteq t$ *there is covering end-extension* $M \subseteq N \models \overline{\mathrm{ZF}}$ *in which sequence is* t*.*
4. *It suffices merely that* $M \models \mathrm{ZF}$*, provided there is an inner model* $W \subseteq M$ *with* $W \models \overline{\mathrm{ZF}}$*.*

Infinitary-logic-free proof of Barwise theorem

The proof method of the universal algorithm provides a new proof of the Barwise extension theorem, without any use of Barwise compactness or indeed infinitary logic.

Corollary

Every countable model of ZF *has an end-extension to a model of* $\mathrm{ZFC} + V = L$.

Further work and questions

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- Find universal finite sequences for other extension concepts.
- View further classical results in the model theory of arithmetic and set theory as consequences of the universal finite set.
- Remove the countability requirement from the set-theoretic case of universal finite set.
- Further applications to potentialism.
- View the universal finite sequence as a generalized Laver sequence or $\diamondsuit$ sequence; use this perspective to find new applications in forcing and set-theoretic combinatorics.

References

- W. Hugh Woodin. “A potential subtlety concerning the distinction between determinism and nondeterminism.” Infinity. Cambridge UP, 2011, p. 119–129.
- Rasmus Blanck and Ali Enayat. “Marginalia on a theorem of Woodin.” J. Symb. Log. 82.1 (2017), p. 359–374.
- Rasmus Blanck. “Contributions to the Metamathematics of Arithmetic. Fixed points, Independence, and Flexibility.” PhD thesis. Univ. Gothenburg, 2017.
- Joel David Hamkins and W. Hugh Woodin. “The universal finite set.” under review, p. 1–16. arXiv:1711.07952[math.LO]. http://jdh.hamkins.org/the-universal-finite-set
- Joel David Hamkins. “The modal logic of arithmetic potentialism and the universal algorithm.” under review, p. 1–36. arXiv:1801.04599[math.LO]. http://jdh.hamkins.org/arithmetic-potentialism-and-the-universal-algorithm
- Joel David Hamkins, Philip Welch, Kameryn J. Williams. “The Σ_1-definable universal finite sequence.” in preparation.

Thank you.

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