"COVERING REFLECTION" ① JOEL DAVID HAMKINS

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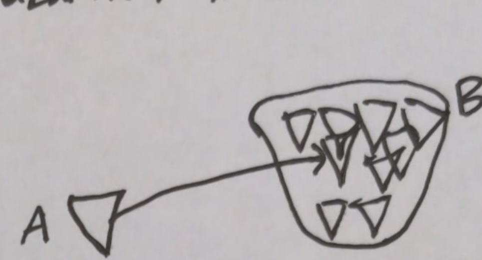
NOTRE DAME LOGIC SEMINAR

JOINT WORK IN PROGRESS
MYSELF, NAI-CHUNG HOU, ANDREAS LIETZ, FARMER SCHLUTZENBERG
ARISING ON MATHOVERFLOW

COVERING REFLECTION PRINCIPLE FOR CARDINAL δ HOLDS IFF
FOR EVERY STRUCTURE B IN A COUNTABLE LANGUAGE $\mathcal{L}$ ADMITS $\mathcal{L}$-STRUCTURE A SIZE $<\delta$ SUCH THAT B IS COVERED BY ELEMENTARY IMAGES OF A.

THAT IS, $\forall b\in B\ \exists j:A\xrightarrow[\text{ELE}]{}B \quad b\in \mathrm{RAN}(j)$.



"LOOKS LIKE MODEL THEORY, BUT IT IS ACTUALLY SET THEORY."
Q: IS IT TRUE? IS IT CONSISTENT? WHICH δ?

ELEMENTARY OBSERVATIONS

① COVERING REFLECTION PROPERTY IS CLOSED UPWARD.
THAT IS, $\delta<\delta' \Rightarrow (\mathrm{CRP}_\delta \rightarrow \mathrm{CRP}_{\delta'})$.

② CRP FAILS FOR $\delta=\aleph_0$. SO δ MUST BE UNCOUNTABLE.

③ CRP_δ IMPLIES $\delta>2^{\aleph_0}$. I.E. FALSE FOR $\delta=2^{\aleph_0}$.
CONSIDER $\langle \mathbb{R},+,\cdot\rangle$. IF $j:A\rightarrow\langle\mathbb{R},+,\cdot\rangle$ THEN j IS UNIQUE.

④ CRP_δ EQUIVALENT FOR B IN FINITE LANGUAGES.
PROOF: EQUIP B WITH $0, S$ TO MAKE DEFINABLE ELTS. + PAIRING FUNCTION.
CAN USE $S^n 0$ AS INDEX IN $R(x,y)$. $y=\langle y_0\cdots y_k\rangle$.

(4.5) CAN ALSO ALLOW MUCH LARGER $\mathcal{L}$.

⑤ CRP_δ EQUIVALENT WITH SUBMODEL (I.E.) EMBEDDINGS $j:A\rightarrow B$ INSTEAD OF ELE EMB.
PROOF: ADD SKOLEM FCNS TO LANG.

⑥ CRP_δ EQUIVALENT WHEN STATED ONLY FOR B SIZE $\leq 2^{<\delta}$.
PROOF: S'POSE B IS CTREX. TO CRP_δ. CONSIDER ALL POSSIBLE A.
ONLY $2^{<\delta}$ MANY ISO. TYPES. PICK A_α ALL $\alpha<2^{<\delta}$ REALIZING ALL ISO. TYPES.
PICK $x_\alpha\in B$ NOT COVERED BY A_α. FIND $B_0\prec B$ $x_\alpha\in B_0$ ALL α.
B_0 SIZE $2^{<\delta}$. B_0 ALSO NOT COVERED BY ANY A_α.

"COVERING REFLECTION" ②

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I0
I1
SUPERHUGE
HUGE
ALMOST HUGE
EXTENDIBLE
SUPERCOMPACT
STRONG
MSBLE
RAMSEY
INDESCRIBABLE
WEAKLY COMPACT
~~INDESCRIBABLE~~
MAHLO
INACC
WORLDLY

⑦ CRP_δ IS A Π^1_1 PROPERTY OF $(V_\delta, \in)$.

COROLLARY THE LEAST δ WITH CRP_δ IS NOT WEAKLY COMPACT.

PF: W.C. δ ARE Π^1_1-INDESCRIBABLE. $\boxtimes$

COROLLARY LEAST δ WITH CRP_δ IS BELOW LEAST Σ_2-CORRECT CARD.
IN PARTICULAR $\delta <$ LEAST STRONG CARDINAL.

$\rightarrow$ CRP_δ IS WEAK?

⑧

SUBCOVERING REFLECTION FOR δ. $\forall B\ \exists A$ SIZE $< \delta$

$\forall b \in B\ \exists \bar{A} \preccurlyeq A \quad j: \bar{A} \xrightarrow[\text{ELE}]{} B \quad b \in \text{RAN}(j)$.

THEOREM $SCRP_\delta$ IFF $\delta > 2^{\aleph_0}$.

PROOF: $(\rightarrow)$ $\langle \mathbb{R}, +, \cdot \rangle$.

$(\leftarrow)$ ASSUME $\delta > 2^{\aleph_0}$. FIX B. $\forall b \in B$ PICK $B_b \prec B$ WITH $b \in B_b$. (CTBLE SIZE)

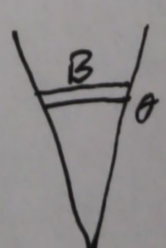
PICK FAMILY $\{B_b \mid b \in I\}$ REALIZING EVERY ISO. TYPE. $2^{\aleph_0}$ MANY ISO. TYPES. PICK $A \prec B$ WITH $B_b \preccurlyeq A$ ALL $b \in I$. SIZE $2^{\aleph_0} < \delta$.

EVERY b IS HIT BY SOME $B_{b'} \cong B_b$. $\boxtimes$

• CRP_δ IS STRONG.

ASSUME CRP_δ. WE OBTAIN STRONG CONSEQUENCES GRADUALLY.

LET $\theta \gg \delta$. LET $B = (V_{\theta+1}, \in)$.



BY CRP_δ, GET COVERING A SIZE $< \delta$.

A IS WF, SO WLOG TRANSITIVE.

NOTE EVERY $j: A \rightarrow B$ HAS $\theta \in \text{RAN}(j)$ SO $cp(j) =$ LEAST κ S.T. $\kappa < j(\kappa)$ EXISTS.

LET $\kappa_0 =$ LEAST $cp(j)$ FOR ANY $j: A \rightarrow B$.

CLAIM $P(\kappa_0) \subseteq A$.

PROOF: S'POZE $X \subseteq \kappa_0$. $X \in B$. SO $\exists j: A \rightarrow B$ WITH $j(X_0) = X$. NOTE $\kappa_0 \leq cp(j)$.

SO $X = X \cap \kappa_0 = X_0 \cap \kappa_0 \in A$. $\boxtimes$

CLAIM κ_0 IS A MSBLE CARDINAL.

PROOF: FIX $j : A \to B$ WITH $cp(j) = \kappa_0$. DEFINE $X \in \mu \longleftrightarrow \kappa_0 \in j(X)$ FOR $X \subseteq \kappa_0$.

"SEED THEORY".

ULTRAFILTER. κ_0-COMPLETE. NONPRINCIPAL. $\boxtimes$

CONCLUSION $CRP_\delta \to \exists$ MSBLE BELOW δ.

LET'S GET MORE...

LET κ_1 = LEAST $cp(j)$ $j : A \to B$ WITH $\kappa_0 \in RAN(j)$. SO $\kappa_0 < \kappa_1$.

CLAIM $P(\kappa_1) \subseteq A$.

PF: $X \subseteq \kappa_1$. FIX $j : A \to B$ WITH $\{\kappa_0, X\} \in RAN(j)$. SO $cp(j) > \kappa_0$. SO $\kappa_1 \leq cp(j)$.

$X = X \cap \kappa_1 = X_0 \cap \kappa_1 \in A$. $\boxtimes$

CLAIM κ_1 IS MSBLE.

PF: FIX $j : A \to B$ $cp(j) = \kappa_1$. μ: $X \in \mu \longleftrightarrow \kappa_1 \in j(X)$. $\boxtimes$

CONCLUSION $CRP_\delta \to \exists$ 2 MSBLE CARDINALS BELOW δ.

KEEP GOING...

BACK TO κ_0. SINCE $P(\kappa_0) \subseteq A$ IT FOLLOWS $V_{\kappa_0+1} \subseteq A$. SO $\exists j : V_{\kappa_0+1} \to (V_{j(\kappa_0)+1})^B$

$= V_{j(\kappa_0)+1}$.

SO κ_0 IS 1-EXTENDIBLE. MUCH STRONGER THAN MSBLE.

IMPLIES SOME PARTIAL SUPERCOMPACTNESS.

BUT SINCE $P(\kappa_1) \subseteq A$ WE ALSO GET $V_{\kappa_1+1} \subseteq A$. SO

κ_0 IS (κ_1+1)-EXTENDIBLE. VERY STRONG ALREADY.

KEEP GOING...

$\kappa_0 < \kappa_1 < \cdots$

DEFINE κ_β = LEAST $cp(j)$ WHERE $j : A \to B$ $\langle \kappa_\alpha \mid \alpha < \beta \rangle \in RAN(j)$.

(i) IF $\beta \leq \kappa_\beta$ (CERTAINLY TRUE FOR ALL β INITIALLY...) THEN $\beta < \kappa_\beta$.

THEN ALL INITIAL SEGS $\langle \kappa_\alpha \mid \alpha < \beta \rangle \restriction \beta'$ ALSO IN $RAN(j)$.

SO $\kappa_{\beta'} \leq \kappa_\beta$ SINCE j WORKS AT $\beta' < \beta$ ALSO $\kappa_{\beta'} < \kappa_\beta$.

SO STRICTLY INCREASING FOR A LONG WAY.

BUT SINCE A HAS FEWER THAN κ CARDINALS, IT CANNOT GO UP FOREVER.

(ii) SO $\exists\beta$ $\kappa_\beta < \beta$. FIRST TIME. $\langle \kappa_\alpha \mid \alpha<\beta\rangle \in RAN(j)$.

SO j WORKS FOR ALL $\alpha < \kappa_\beta$. SO $\alpha < \kappa_\beta \rightarrow \kappa_\alpha \le \kappa_\beta$ HENCE $\kappa_\alpha < \kappa_\beta$.

BUT $\alpha \le \kappa_\alpha$. SO $\kappa_\beta = \sup_{\alpha<\kappa_\beta} \kappa_\alpha$. LET $\lambda = \kappa_\beta$.

SO λ IS MSBLE LIMIT OF CARDS κ_α THAT ARE λ-EXTENDIBLE.

CLAIM $V_\lambda \models ZFC +$ ~~PROPER CLASS OF~~ λ MANY $<\lambda$- EXTENDIBLE CARDINALS.

PF: λ MSBLE SO $V_\lambda \models ZFC$.

CONSIDER κ_α. CONSIDER ARB $\kappa_\gamma > \kappa_\alpha$ $\quad j: V_{\lambda+1} \longrightarrow V_{j(\lambda)+1}$, $cp(j) = \kappa_\gamma$.

CAN ASSUME $j(\lambda)$ IS BIG ENOUGH THAT

$V_{j(\lambda)+1} \models \kappa_\alpha$ IS $<\kappa_\gamma$-EXTENDIBLE.

CONSEQUENTLY, $V_{\lambda+1} \models \kappa_\alpha$ IS $<\kappa_\gamma$-EXTENDIBLE.

SO $V_\lambda \models \kappa_\alpha$ IS $<\lambda$-EXTENDIBLE. I.E. EXTENDIBLE. $\boxtimes$

CONCLUSION SRP_δ IMPLIES $\exists\lambda<\delta$ $V_\lambda \models ZFC +$ PROPER CLASS OF EXTENDIBLE CARDS.

VERY STRONG INDEED.

- UPPER BOUNDS

THEOREM ASSUME κ IS HUGE. THEN SRP_κ HOLDS.

PROOF: ASSUME κ IS HUGE. SO $\exists\, j: V \to M$ $cp(j) = \kappa$, $M^{j(\kappa)} \subseteq M$.

CONSIDER B SIZE $\kappa = 2^{<\kappa}$ CTREXAMPLE TO SRP.

SO $M \models j(B)$ IS A CTREXAMPLE. IN PARTICULAR, $M \models B$ DOES NOT COVER $j(B)$
with $j(\kappa)$.

BUT M & V AGREE ON EMBEDDINGS SIZE κ. SO $V \models B$ DOES NOT COVER $j(B)$.

SO $\exists x^{\in j(B)}$ s.t. x IS NOT IN $RAN(h)$ FOR ANY $h: B \to j(B)$.

APPLY j. SO $M \models j(x)$ IS NOT IN $RAN(h)$ ANY $h: j(B) \to j(j(B))$.
SIZE $j(\kappa)$ $\in M$.

BUT $j(B) \in V$ SO $j \restriction j(B): j(B) \to j(j(B))$ IS ELE EMB. SIZE $j(\kappa)$. HENCE IN M BY HUGENESS.

& $j(x) \in RAN(j \restriction j(B))$. $\rightarrow\leftarrow$ $\boxtimes$

"COVERING REFLECTION"

- EXACT STRENGTH

<u>THEOREM</u> THE LEAST CARDINAL δ WITH CRP_δ IS EXACTLY THE LEAST <u>ANCHOR</u> CARDINAL.

κ IS <u>ANCHOR</u> IFF $\forall X \subseteq V_\kappa \ \exists \kappa_0 < \kappa_1 < \kappa \ \ j : (V_{\kappa_1}, \in, X \cap \kappa_1) \to (V_\kappa, \in, X)$

$$\kappa_0 = cp(j), \ \ j(\kappa_0) = \kappa_1 .$$

κ HUGE $\Rightarrow$ NORMAL MEAS 1 OF ANCHOR CARDS BELOW κ.