"COVERING REFLECTION" 1

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NOTRE DAME LOGIC SEMINAR

JOINT WORK IN PROGRESS
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ARISING ON STACKOVERFLOW

COVERING REFLECTION PRINCIPLE FOR CARDINAL $\delta$ HOLD S IFF

FOR EVERY STRUCTURE $B$ IN A COUNTABLE LANGUAGE $L$ ADMITS $\delta$-structure $A$
SIZE $< \delta$ SUCH THAT $B$ IS COVERED BY ELEMENTARY IMAGED OF $A$.

THAT IS, $\forall \delta \in \mathbb{B} \exists f : A \rightarrow B$ IN $\text{ELE} (j)$.

"LOOKS LIKE MODEL THEORY, BUT IT IS ACTUALLY SET THEORY."

Q: IS IT TRUE? IS IT CONSISTENT? WHICH $\delta$?

ELEMENTARY OBSERVATIONS

1. COVERING REFLECTION PROPERTY IS CLOSED UPWARD.

THAT IS, $\delta < \delta' \Rightarrow (\text{CRP}_\delta \rightarrow \text{CRP}_{\delta'})$.

2. CRP FAILS FOR $\delta = \aleph_0$. SO $\delta$ MUST BE UNCOUNTABLE.

3. $\text{CRP}_\delta$ IMPLIES $\delta > 2^{\aleph_0}$. I.E. FALSE FOR $\delta = 2^{\aleph_0}$.

CONSIDER $\langle \mathbb{R}, +, \cdot \rangle$. IF $j : A \rightarrow \langle \mathbb{R}, +, \cdot \rangle$ THEN $j$ IS UNIQUE.

4. $\text{CRP}_\delta$ EQUIVALENT FOR $B$ IN FINITE LANGUAGES.

PROOF: EQUIP $B$ WITH $0, \leq$ TO MAKE DEFINABLE ELTS. +PAIRING FUNCTION.

CAN USE $S^0$ AS INDEX IN $R(x, y)$. $\gamma = \langle y_0, \ldots, y_n \rangle$.

5. CAN ALSO ALLOW MUCH LARGER $\delta$. I.E.

$\text{CRP}_\delta$ EQUIVALENT WITH $\delta$-MODEL EMBEDDINGS $j : A \rightarrow B$ INSTEAD OF ELE EMB.

PROOF: ADD SKOLEM FNS TO LANG.

6. $\text{CRP}_\delta$ EQUIVALENT WHEN STATED ONLY FOR $B$ SIZE $\leq 2^{\aleph_0}$.

PROOF: $S' \rightarrow \exists B$ IS CTNLX. TO $\text{CRP}_\delta$. CONSIDER ALL POSSIBLE $A$.

ONLY $2^{\aleph_0}$ MANY ISO. TYPES. PICK $A_0$ ALL $\in 2^{\aleph_0}$ REALIZING ALL ISO.

PICK $x_0 \in B$ NOT COVERED BY $A_0$. FIND $B_0 < B^\delta$ x\in B_0 ALL $\alpha$.

$B_0$ SIZE $\leq 2^{\aleph_0}$. $B_0$ ALSO NOT COVERED BY ANY $A$. 

7. $\text{CRP}_\delta$ EQUIVALENT WITH $\delta$-MODEL EMBEDDINGS $j : A \rightarrow B$ INSTEAD OF ELE EMB.

PROOF: ADD SKOLEM FNS TO LANG.
CRP₈ is a Π₁¹ property of (V₀, ∈).

**Corollary** THE LEAST δ WITH CRP₈ IS NOT WEAKLY COMPACT.

**PF:** W.C. δ AND Π₁¹-INDESCRIBABLE. ☐

**Corollary** LEAST δ WITH CRP₈ IS BELOW LEAST Σ₁²-COMPACT CARDINAL.

IN PARTICULAR δ ≤ LEAST STRONG CARDINAL.

→ CRP₈ IS WEAK?

**Subcovering Reflection** For δ, ∀ B ∈ A SIZE < δ

∀ b ∈ B ∃ A ∈ A : j: A → B be ran(j).

**Theorem** SCRP₈ IFF δ ≥ 2^|BA|.

**Proof:** (→) (IR, +, ·).

(←) ASSUME δ ≥ 2^|BA|.

Fix B. ∀ b ∈ B pick b₀ ∈ B with b₀ ∈ B₀.

Pick family \{Bₜ \mid t ∈ I\} realizing every iso-type.

2^|BA| MANY ISO-TYPES. Pick A ∈ B with Bₜ ∈ A all b ∈ I. SIZE 2^|BA| < δ.

Every b is hit by some Bₜ ∋ Bₜ.

CRP₈ IS STRONG.

Assume CRP₈. We obtain strong consequences gradually.

Let δ ≥ δ. Let B = (V₀, ∈).

By CRP₈, get covering A SIZE < δ.

A is wf, so wloc transitive.

Note every j: A → B has θ ∈ ran(j) so cp(j) = LEAST k s.t. k < j(k) EXISTS.

Let k₀ = LEAST cp(j) for any j: A → B.

Claim p(k₀) ≤ A.

Proof: Suppose x ∈ k₀. x ∈ B. So ∃ j: A → B with j(x₀) = x. Note k₀ ≤ cp(j).

So x = x₀k₀ = x₀n₁k₀ e A. ☐
Claim $K_0$ is a msble cardinal.

Proof: Fix $j: A \to B$ with $\text{cp}(j) = K_0$. Define $X \in M \leftrightarrow K_0 \in j^{-1}(X)$ for $X \subseteq K_0$.


Conclusion $\exists \text{msble below } \delta$.

Let's get more.

Let $K_1 = \text{least } \text{cp}(j) : j: A \to B$ with $K_0 \in \text{ran}(j)$. So $K_0 < K_1$.

Claim $P(K_1) \subseteq A$.

Proof: $X \subseteq K_1$. Fix $j: A \to B$ with $\{K_0, X\} \in \text{ran}(j)$. So $\text{cp}(j) > K_0$. So $K_1 \subseteq \text{cp}(j)$.

$x = x \cap K_1 = x \cap K_1 \in A$.

Claim $K_1$ is msble.

Proof: Fix $j: A \to B$ with $\text{cp}(j) = K_1$. $M: X \in M \leftrightarrow K_1 \in j^{-1}(X)$.

Conclusion $\exists \exists 2$ msble cardinals below $\delta$.

Keep going.

Back to $K_0$. Since $P(K_0) \subseteq A$ it follows $\forall K_{0+1} \subseteq A$. So $\exists j: V_{K_0} \to (V_{j(K_0)+1})^B$.

So $K_0$ is $1$-extendible. Much stronger than msble.

Implies some partial supercompactness.

But since $P(K_1) \subseteq A$ we also get $\forall K_{1+1} \subseteq A$. So $K_0$ is $(K_1+1)$-extendible. Very strong already.

Keep going.

$K_0 < K_1 < \cdots$

Define $K_\beta = \text{least } \text{cp}(j)$ where $j: A \to B$ with $\text{ran}(j)$. $\langle K_\beta \cap \beta \rangle \in \text{ran}(j)$.

1. If $\beta \leq K_\alpha$ (certainly true for all $\beta$ initially...) then $\beta < K_\alpha$.

Then all initial sets $\langle K_\alpha \cap \beta \rangle_{\beta \in \beta}^\beta$ also in $\text{ran}(j)$.

So $K_{\beta+1} \subseteq K_{\beta}$ since $j$ works at $\beta \cap \beta$ also $K_{\beta} \cup K_{\beta}$.

So strictly increasing for a long way.
But since $A$ has fewer than $\kappa$ cardinals, it cannot go up forever.

(1) So $j^\beta: k_\alpha < \beta$. First time. $< k_\alpha, k_\beta > \in \text{ran}(j)$.

So $j$ works for all $\alpha < k_\beta$. So $\alpha < k_\beta \rightarrow k_\alpha \leq k_\beta$ hence $k_\alpha < k_\beta$.

But $\alpha \leq k_\alpha$. So $k_\beta = \sup k_\alpha$. Let $\lambda = k_\beta$.

So $\lambda$ is m.s.b.e. limit of cardinals $k_\alpha$ that are $\lambda$-extendible.

Claim $V_\lambda \models \text{ZFC + proper class of extendible cardinals}.$

Proof: $\lambda$ m.s.b.e. so $V_\lambda \models \text{ZFC}.

Consider $k_\alpha$. Consider all $\alpha < k_\beta, \ j: V_{\lambda^+} \rightarrow V_{\lambda^+}, \ cp(j) = k_\alpha$.

Can assume $j(\lambda)$ is big enough that $V_{j(\lambda)^+} \models k_\alpha < k_\beta$-extendible.

Consequently, $V_{\lambda^+} \models k_\beta$ is $\lambda$-extendible.

So $V_\lambda \models k_\beta$ is $\lambda$-extendible. I.e. extendible. \(\Box\)

Conclusion: $\text{SRP}_\delta$ implies $\exists \lambda < \delta \ V_\lambda \models \text{ZFC + proper class of extendible cardinals}.$

Very strong indeed.

*Upper bounds*

Theorem Assume $k$ is huge. Then $\text{SRP}_k$ holds.

Proof: Assume $k$ is huge. So $j: V \rightarrow M, \ cp(j) = k, \ M^{j(k)} \subseteq M$.

Consider $B$ size $k = 2^{|E|}$ ctexample to $\text{SRP}$.

So $M, j(B)$ is a ctexample. In particular $M \models B$ does not cover $j(B)$.

But $M \models V$ agree on embeddings size $k$. So $V \models B$ does not cover $j(B)$.

So $\exists x \in j(B) \text{ s.t. } x \text{ is not in ran}(h)$ for any $h: B \rightarrow j(B)$.

Apply $j$, so $M \models j(x)$ is not in ran($h$) for any $h: j(B) \rightarrow j(j(B))$.

So $j(B) \cap V = j(j(B))$ is ele emb. size $j(k)$. Hence in $M$ by hugeness.

$\forall x \in \text{ran}(j)(j(B)), \rightarrow \Box$
**Exact Strength**

**Theorem.** The least cardinal $\kappa$ with CRPs is exactly the least anchor cardinal.

$k$ is anchor iff $\forall x \in V_κ \exists k_0 < k_1 < k \ j : (V_{k_1}, \in \times V_{k_1}) \to (V_k, \in, x)$

$k_0 = \text{cp}(j), \ j(k_0) = k_1$.

$k$ huge $\Rightarrow$ normal meas 1 of anchor cans below $k$. 