Infinite draughts: an unsolved open game

Davide Leonessi

Graduate Center
City University of New York

Infinite-Games Workshop

September 21, 2023

References

In this talk I will present joint work with Joel David Hamkins.

J. D. Hamkins and D. Leonessi, Transfinite game values in infinite draughts (2022). *Integers* **22** (2022), Paper no. G5.

D. Leonessi, Transfinite game values in infinite games (2021). MSc dissertation, Mathematics and Foundations of Computer Science, University of Oxford. arXiv:2111.01630.

Introduction to games

We focus on 2-player games of perfect information: no randomness, both players have all the information, rules are known.

Introduction to games

We focus on 2-player games of perfect information: no randomness, both players have all the information, rules are known.

A player has a *winning strategy* for a game if he can win, regardless of how his opponent plays.

Introduction to games

We focus on 2-player games of perfect information: no randomness, both players have all the information, rules are known.

A player has a *winning strategy* for a game if he can win, regardless of how his opponent plays.

Fundamental Theorem of finite games (Zermelo, 1913)

In every finite two-player game of perfect information with no draws, one of the players has a winning strategy.

Introduction to games

We focus on 2-player games of perfect information: no randomness, both players have all the information, rules are known.

A player has a *winning strategy* for a game if he can win, regardless of how his opponent plays.

Fundamental Theorem of finite games (Zermelo, 1913)

In every finite two-player game of perfect information with no draws, one of the players has a winning strategy.

Proof. Consider the game tree of a finite game.

Label the leaves as a win for one player or the other.

Introduction to games

We focus on 2-player games of perfect information: no randomness, both players have all the information, rules are known.

A player has a *winning strategy* for a game if he can win, regardless of how his opponent plays.

Fundamental Theorem of finite games (Zermelo, 1913)

In every finite two-player game of perfect information with no draws, one of the players has a winning strategy.

Proof. Consider the game tree of a finite game.

Label the leaves as a win for one player or the other.

Back-propagation: from the bottom, label a node if a player can win from that node.

The root node will get one label or the other, and whoever it is can win–play to stay on your labels. □

Theorem

Chess is determined, i.e. exactly one of the following is true:

1. White has a winning strategy,
2. Black has a winning strategy,
3. Both White and Black have a strategy to force a draw.

Theorem

Chess is determined, i.e. exactly one of the following is true:

1. White has a winning strategy,
2. Black has a winning strategy,
3. Both White and Black have a strategy to force a draw.

Proof.

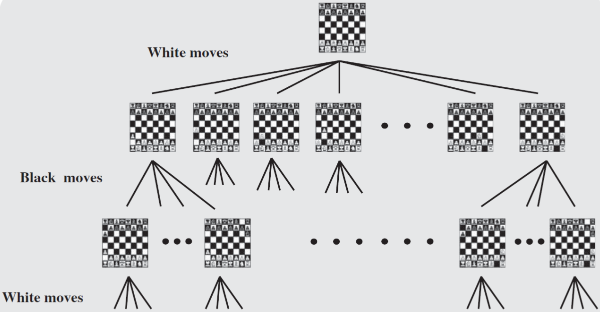


Figure 1.1 The game of chess presented in extensive form

From Maschler, Solan, Zamir, *Game Theory* (2013)

The Fundamental Theorem of infinite games

A game is *in nite* if its game tree is infinite—in particular, if it is possible for players to choose among infinitely many moves in a turn.

The Fundamental Theorem of infinite games

A game is *in nite* if its game tree is infinite—in particular, if it is possible for players to choose among infinitely many moves in a turn.

A game is *open* for some player, if that player can only win in finitely many moves. In other words, if that player's set of winning plays is open in the game tree with the product topology.

The Fundamental Theorem of infinite games

A game is *in nite* if its game tree is infinite—in particular, if it is possible for players to choose among infinitely many moves in a turn.

A game is *open* for some player, if that player can only win in finitely many moves. In other words, if that player's set of winning plays is open in the game tree with the product topology.

Theorem (Open determinacy, Gale & Stewart 1953)

In every infinite two-player open game of perfect information, one of the players has a winning strategy. (or both have drawing strategies, if draws are allowed)

The Fundamental Theorem of infinite games

A game is *in nite* if its game tree is infinite—in particular, if it is possible for players to choose among infinitely many moves in a turn.

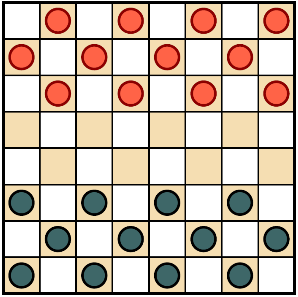
A game is *open* for some player, if that player can only win in finitely many moves. In other words, if that player's set of winning plays is open in the game tree with the product topology.

Theorem (Open determinacy, Gale & Stewart 1953)

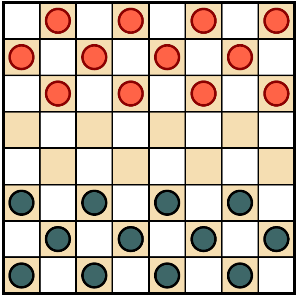
In every infinite two-player open game of perfect information, one of the players has a winning strategy. (or both have drawing strategies, if draws are allowed)

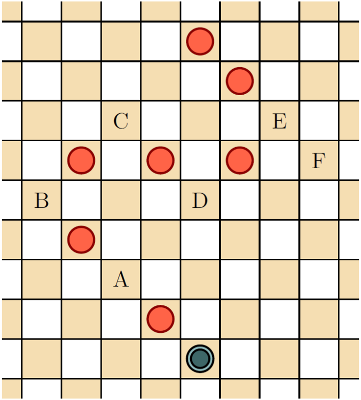
In open games, game values generalise the chess idea of mate-in-2 or mate-in-3. The game value of a position is an ordinal that measures the number of moves required for the open player to achieve a win.
We begin with some examples.

Finite and infinite draughts



Finite and infinite draughts



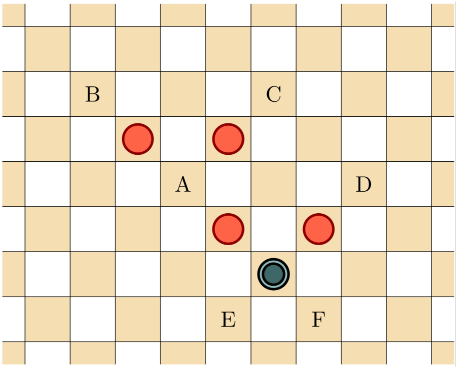


Rules of infinite draughts

Forced jump.

Forced iterated jump.

The first player who
has no legal move available loses.

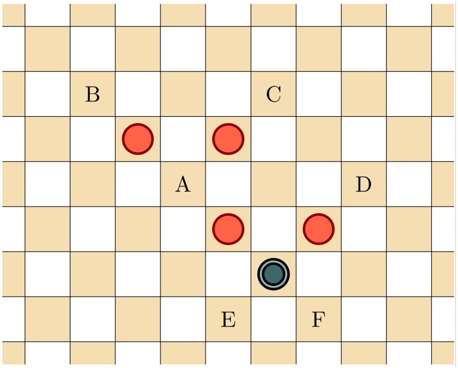


Rules of infinite draughts

Forced jump.

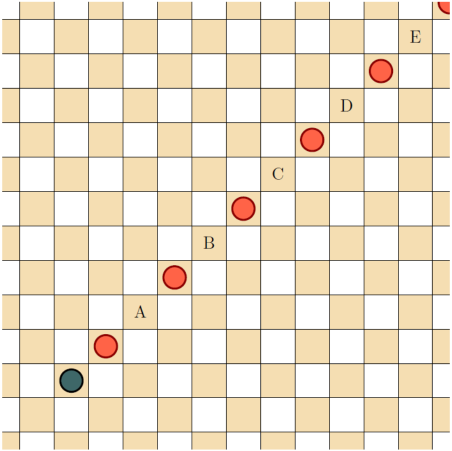
Forced iterated jump.

The first player who
has no legal move available loses.

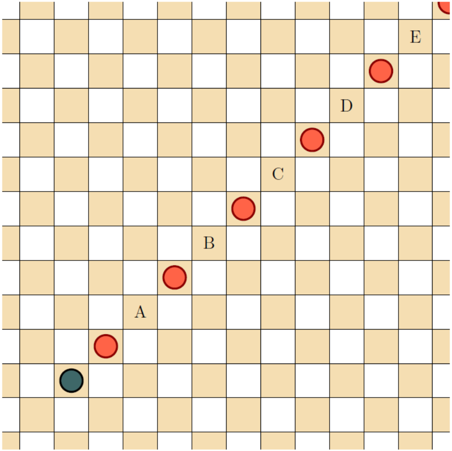


We consider only open games:
plays that last infinitely many moves are draws.

The infinite jump rule

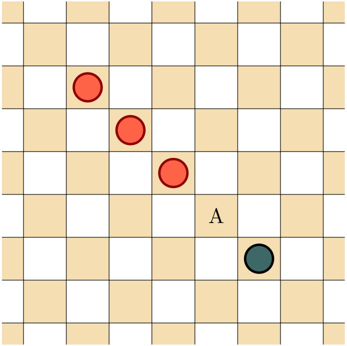


The infinite jump rule



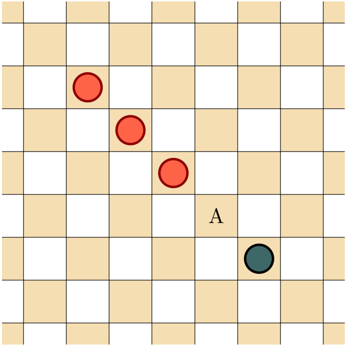
The black piece that makes an infinite iterated jump disappears from the board.

Finite game values: Red to move

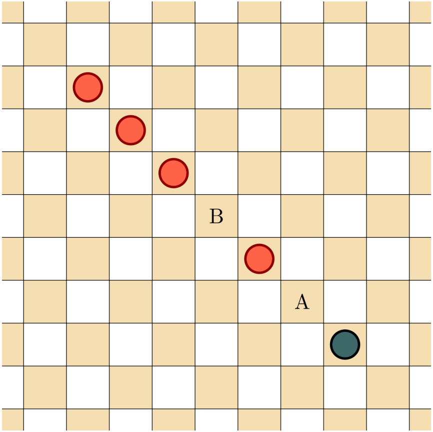


Game value 2

Finite game values: Red to move



Game value 2



Game value 3

Game value $!$

Game value *!*

Black to move, with obligation to jump—Black cannot make an infinite jump, which would lead to loss.

Game value $!$

Black to move, with obligation to jump—Black cannot make an infinite jump, which would lead to loss.

Black has to rest on some square n, reaching a position with game value n for Red: Black loses after n moves.

Game value *!*

Black to move, with obligation to jump—Black cannot make an infinite jump, which would lead to loss.

Black has to rest on some square n, reaching a position with game value n for Red: Black loses after n moves.

Playing this is alike to counting down from *!*.

Red can definitely win, in finitely many moves, but Black can choose how long it takes, by choosing a large n. Black makes such choice on the first move only.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.
2. If the game is not yet won and Red can move to a position with value , then for the smallest such ordinal , the value of the position is + 1.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.
2. If the game is not yet won and Red can move to a position with value , then for the smallest such ordinal , the value of the position is + 1 .
3. If it is Black's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet de ned.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.
2. If the game is not yet won and Red can move to a position with value , then for the smallest such ordinal , the value of the position is $+1$.
3. If it is Black's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet de ned.

If a position has value $+1$ for Red, then Red can make a move and reach a position with value , while Black cannot play to reach a position with higher or no value.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.
2. If the game is not yet won and Red can move to a position with value , then for the smallest such ordinal , the value of the position is $+1$.
3. If it is Black's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet de ned.

If a position has value $+1$ for Red, then Red can make a move and reach a position with value , while Black cannot play to reach a position with higher or no value.

If a position has value a limit ordinal for Red, then Black can only play to reach a position with value $<$.

[redacted]

In an game open for players Red and Black, the ordinal game value of a position for Red is de ned by trans nite recursion:

1. If the game is already won by Red, then the value of the position is 0.
2. If the game is not yet won and Red can move to a position with value , then for the smallest such ordinal , the value of the position is $+1$.
3. If it is Black's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet de ned.

If a position has value $+1$ for Red, then Red can make a move and reach a position with value , while Black cannot play to reach a position with higher or no value.

If a position has value a limit ordinal for Red, then Black can only play to reach a position with value $<$.

If a position has no value for Red, then Red cannot reach a position with value and Black can play so to reach another position without value.

Hence, starting from a position with some value for Red, Red can follow the *value-decreasing* strategy and win in finitely many moves.

Otherwise, from a position with no value for Red, Black can follow the *value-avoiding* strategy and prevent a Red win.

Hence, starting from a position with some value for Red, Red can follow the *value-decreasing* strategy and win in finitely many moves.

Otherwise, from a position with no value for Red, Black can follow the *value-avoiding* strategy and prevent a Red win.

Notice that in any open game, the initial position either has a game value for one player, the other player, or neither. Thus, either one player has a winning strategy, or they can both force a draw.

Hence, starting from a position with some value for Red, Red can follow the *value-decreasing* strategy and win in finitely many moves.

Otherwise, from a position with no value for Red, Black can follow the *value-avoiding* strategy and prevent a Red win.

Notice that in any open game, the initial position either has a game value for one player, the other player, or neither. Thus, either one player has a winning strategy, or they can both force a draw.

This is a proof of the Fundamental Theorem of infinite games:

Theorem (Open determinacy, Gale & Stewart 1953)

In every infinite two-player open game of perfect information, one of the players has a winning strategy. (or both have drawing strategies, if draws are allowed)

Theorem (Hamkins & L. 2022)

Every countable ordinal arises as the game value of a position in infinite draughts.

Proof idea, as in Evans & Hamkins 2014.
Embed well-founded trees, which do not have infinite branches, into positions of infinite draughts.

Theorem (Hamkins & L. 2022)

Every countable ordinal arises as the game value of a position in infinite draughts.

Proof idea, as in Evans & Hamkins 2014. Embed well-founded trees, which do not have infinite branches, into positions of infinite draughts.

The draughts play will proceed as though Black is climbing the tree, so that Black loses when reaching a leaf.

The game value will thus track the ordinal rank of the well-founded tree itself. □

Theorem (Hamkins & L. 2022)

Every countable ordinal arises as the game value of a position in infinite draughts.

Proof idea, as in Evans & Hamkins 2014.
Embed well-founded trees, which do not have infinite branches, into positions of infinite draughts.

The draughts play will proceed as though Black is climbing the tree, so that Black loses when reaching a leaf.

The game value will thus track the ordinal rank of the well-founded tree itself. □

Lemma.

The full binary tree can be embedded in the infinite draughtboard.

[redacted]

Every countable ordinal arises as the game value of a position in in nite draughts with the forced jump rule, but without the forced iterated jump rule.

[redacted]

Every countable ordinal arises as the game value of a position in in nite draughts with the forced jump rule, but without the forced iterated jump rule.

Proof. Induction on the game value.

Suppose that the game values $_1$ $_2$ $_3$ $\ldots$ have all been realised as well-founded trees embedded in the board.

[redacted]

Every countable ordinal arises as the game value of a position in in nite draughts with the forced jump rule, but without the forced iterated jump rule.

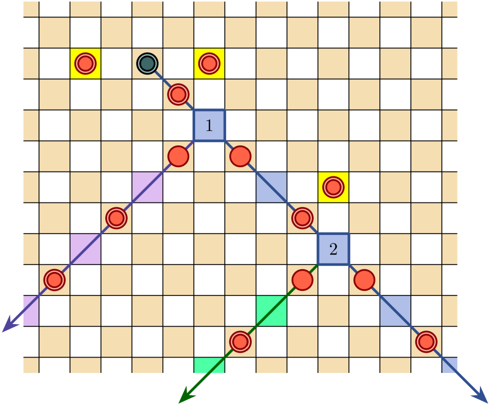
Proof. Induction on the game value.

Suppose that the game values $_1$ $_2$ $_3$ ::: have all been realised as well-founded trees embedded in the board.

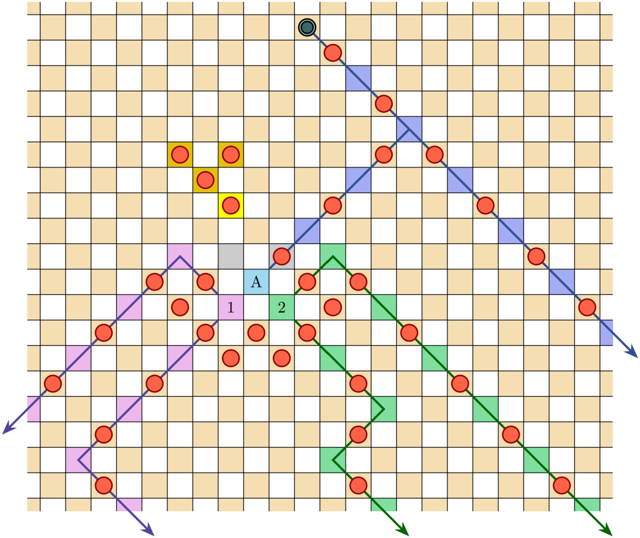
Construct this position, in which Black can access branch nodes with value $_n$ and rest on numbered squares.

This position realises the game value $\sup_n(\ _n + 1)$. □

Construction *without* forced iterated jump and forced jump



Construction *with* forced iterated jump and forced jump



The *omega one* of a game is the supremum of the values realisable in it.

Corollary (Hamkins & L. 2022)

The omega one of infinite draughts is at least true ω_1.

$$\omega_1^{\text{draughts}} \qquad \omega_1.$$

The *omega one* of a game is the supremum of the values realisable in it.

Corollary (Hamkins & L. 2022)

The omega one of infinite draughts is at least true ω_1.

$$\omega_1^{\text{draughts}} \qquad \omega_1.$$

Remark. (McCallum) There are positions in infinite draughts in which a player can choose among uncountably many moves.

The *omega one* of a game is the supremum of the values realisable in it.

Corollary (Hamkins & L. 2022)

The omega one of infinite draughts is at least true ω_1.

$$\omega_1^{\text{draughts}} \qquad \omega_1:$$

Remark. (McCallum) There are positions in infinite draughts in which a player can choose among uncountably many moves.

A black king at the root of a full binary tree can choose among uncountably many infinitely iterated jumps, one for each branch of the tree.

Is there a position of infinite draughts with uncountable game value? We don't know. In that case, we could have $\omega_1^{\text{draughts}} > \omega_1$.

All defensive strategies can be implemented

All defensive strategies can be implemented

Distinct trees can have the same rank. Each tree can be implemented as a draughts position uniquely, giving rise to a position with the corresponding game value.

Computable play

[redacted]

There is a computable position in in nite draughts, such that Red has a computable strategy that wins against any computable Black strategy, and forc[redacted] draw or better against any Black strategy.
Meanwhile, Black has a (noncomputable) drawing strategy.

Computable play

[redacted]

There is a computable position in infinite draughts, such that Red has a computable strategy that wins against any computable Black strategy, and forc[redacted] draw or better against any Black strategy.
Meanwhile, Black has a (noncomputable) drawing strategy.

Proof, as in Evans & Hamkins 2014. There is an infinite computable binary branching tree T with no computable infinite branch.

Computable play

[redacted]

There is a computable position in in nite draughts, such that Red has a computable strategy that wins against any computable Black strategy, and forc draw or better against any Black strategy.
Meanwhile, Black has a (noncomputable) drawing strategy.

Proof, as in Evans & Hamkins 2014There is an in nite computable binary branching treeT with no computable in nite branch.

We can construct a position so that play unfolds as though Black is climbing through T. SinceT has an in nite branch, there will be a strategy for Black to climb the tree without getting stuck in a terminal node, and this will be a draw b in nite play|such strategy must be not computable.

Computable play

Theorem (Hamkins & L. 2022)

There is a computable position in infinite draughts, such that Red has a computable strategy that wins against any computable Black strategy, and forces a draw or better against any Black strategy.
Meanwhile, Black has a (noncomputable) drawing strategy.

Proof, as in Evans & Hamkins 2014. There is an infinite computable binary branching tree T with no computable infinite branch.

We can construct a position so that play unfolds as though Black is climbing through T. Since T has an infinite branch, there will be a strategy for Black to climb the tree without getting stuck in a terminal node, and this will be a draw by infinite play—such strategy must be not computable.

But if Black plays according to a computable strategy, then he will find himself stuck at a terminal node, where he will lose.

The strategy for Red in either case is to play so to force Black to keep climbing the tree, as seen before. □

Thank you!

Davide Leonessi
Program in Mathematics
Graduate Center, City University of New York
http://leonessi.org