Infinite draughts: an unsolved open game

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Infinite-Games Workshop

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Games	Draughts	Game values	Strategies	
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References				

In this talk I will present joint work with Joel David Hamkins.

- J. D. Hamkins and D. Leonessi, Transfinite game values in infinite draughts (2022). *Integers* **22** (2022), Paper no. G5.
- D. Leonessi, Transfinite game values in infinite games (2021). MSc dissertation, Mathematics and Foundations of Computer Science, University of Oxford. arXiv:2111.01630.

Introduction to games

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Proof. Consider the game tree of a finite game.

Label the leaves as a win for one player or the other.

Back-propagation: from the bottom, label a node if a player can win from that node.

The root node will get one label or the other, and whoever it is can win-play to stay on your labels.

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Theorem				
Chess is de	termined, i.e. exa	ctly one of the follov	ving is true:	
 White 	has a winning str	rategy,		
🕚 Black	has a winning str	ategy,		
Both \	White and Black	have a strategy to fo	rce a draw.	



Both White and Black have a strategy to force a draw.



From Maschler, Solan, Zamir, Game Theory (2013)

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The Fundamental Theorem of infinite games

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In open games, game values generalise the chess idea of mate-in-2 or mate-in-3. The game value of a position is an ordinal that measures the number of moves required for the open player to achieve a win. We begin with some examples.

Finite and infinite draughts



Finite and infinite draughts





Rules of infinite draughts

Forced jump.

Forced iterated jump.

The first player who has no legal move available loses.



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We consider only open games: plays that last infinitely many moves are draws.

The infinite jump rule



The infinite jump rule



The black piece that makes an infinite iterated jump disappears from the board.

Infinite draughts

Games	Draughts	Game values	Strategies
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Finite game values: Red to move



Game value 2

Games	Draughts	Game values	Strategies
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Game value 2



Game value 3



Game value ω

Black to move, with obligation to jump—Black cannot make an infinite jump, which would lead to loss.



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Black has to rest on some square n, reaching a position with game value n for Red: Black loses after n moves.

Playing this is alike to counting down from ω .

Red can definitely win, in finitely many moves, but Black can choose how long it takes, by choosing a large n. Black makes such choice on the first move only.



Games	Draughts	Game values	Strategies	
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- If it is Black's turn, then the value of the position is the supremum of the values of the positions to which a legal move can be made, if all such positions have a value, otherwise the value is not yet defined.

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If a position has value a limit ordinal γ for Red, then Black can only play to reach a position with value $\beta < \gamma.$

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Games	Draughts	Game values	Strategies	
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Hence, starting from a position with some value for Red, Red can follow the *value-decreasing* strategy and win in finitely many moves.

Otherwise, from a position with no value for Red, Black can follow the *value-avoiding* strategy and prevent a Red win.

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Notice that in any open game, the initial position either has a game value for one player, the other player, or neither. Thus, either one player has a winning strategy, or they can both force a draw.

This is a proof of the Fundamental Theorem of infinite games:

Theorem (Open determinacy, Gale & Stewart 1953)

In every infinite two-player open game of perfect information, one of the players has a winning strategy. (or both have drawing strategies, if draws are allowed)

Every countable ordinal arises as the game value of a position in infinite draughts.

Proof idea, as in Evans & Hamkins 2014. Embed well-founded trees, which do not have infinite branches, into positions of infinite draughts.

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The game value will thus track the ordinal rank of the well-founded tree itself.

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Lemma.

The full binary tree can be embedded in the infinite draughtboard.

Every countable ordinal arises as the game value of a position in infinite draughts with the forced jump rule, but *without the forced iterated jump rule*.

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Proof. Induction on the game value.

Suppose that the game values $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots$ have all been realised as well-founded trees embedded in the board.

Every countable ordinal arises as the game value of a position in infinite draughts with the forced jump rule, but *without the forced iterated jump rule*.

Proof. Induction on the game value.

Suppose that the game values $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots$ have all been realised as well-founded trees embedded in the board.

Construct this position, in which Black can access branch nodes with value α_n and rest on numbered squares.

This position realises the game value $\sup_n(\alpha_n + 1)$.



Draughts	Game values	Strategies	
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Construction without forced iterated jump and forced jump



Games	Draughts	Game values	Strategies
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Construction with forced iterated jump and forced jump



The omega one of a game is the supremum of the values realisable in it.

Corollary (Hamkins & L. 2022)

The omega one of infinite draughts is at least true ω_1 .

 $\omega_1^{\text{draughts}} \ge \omega_1.$

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A black king at the root of a full binary tree can choose among uncountably many infinitely iterated jumps, one for each branch of the tree.

Is there a position of infinite draughts with uncountable game value? We don't know. In that case, we could have $\omega_1^{\rm draughts} > \omega_1$.



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Distinct trees can have the same rank. Each tree can be implemented as a draughts position uniquely, giving rise to a position with the corresponding game value.



Theorem (Hamkins & L. 2022)

There is a computable position in infinite draughts, such that Red has a computable strategy that wins against any computable Black strategy, and forces a draw or better against any Black strategy. Meanwhile, Black has a (noncomputable) drawing strategy.

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But if Black plays according to a computable strategy, then he will find himself stuck at a terminal node, where he will lose.

The strategy for Red in either case is to play so to force Black to keep climbing the tree, as seen before. $\hfill\square$

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Thank you!

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