Strategic thinking in infinite games

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Recursive chess

Etching by Django Pinter, Oxford.

The Chocolatier's game

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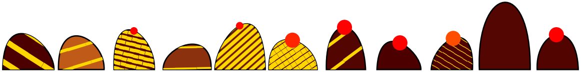

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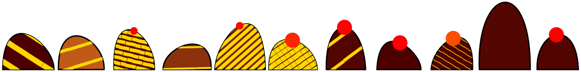

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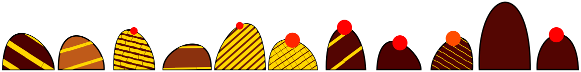

Each round, the *Glutton* selects one and eats it.

Glutton wins, after infinite play, if all chocolates are consumed.

Can the Glutton win?

Chocolates are served, the Glutton eats one.

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More chocolates are served, the Glutton eats just one.

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Chocolates steadily accumulate on the platter, and the Glutton falls further and further behind.

Nevertheless, with strategic choices, I claim that the Glutton CAN win. He can eat all the chocolate.

How the Glutton wins

Winning strategy

To win, the Glutton imagines the serving platter as a queue.

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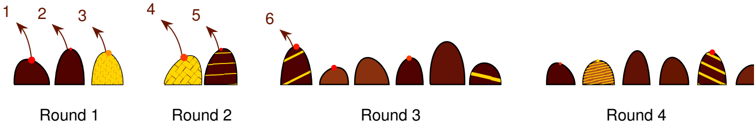

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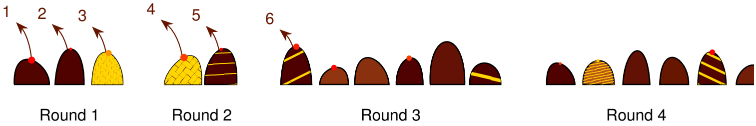

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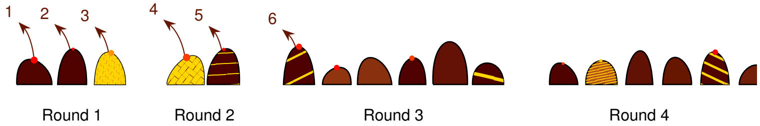

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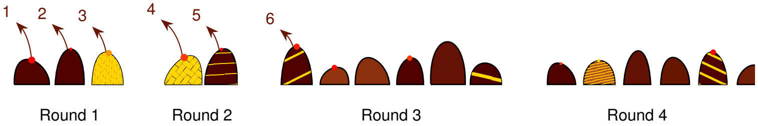


Newly served chocolates are placed at the back of the queue.

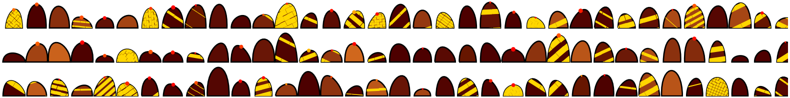
But the Glutton eats always from the front.

Every chocolate is finite distance from the front.

So after infinitely many turns, every chocolate will be eaten!

Infinite chocolates served each round

What if the Chocolatier serves up not merely finitely many chocolates each round, but a countable infinity of new chocolates each round?



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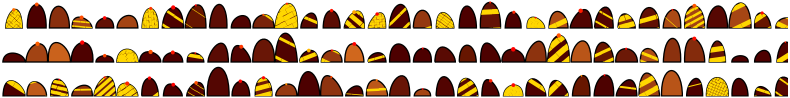

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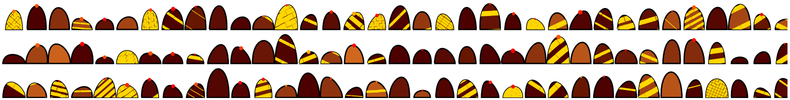

This breaks the previous strategy, since if the new chocolates are placed at the back of the queue, the Glutton will never get to the second-round chocolates.

Nevertheless, the Glutton can still win!

Winning with infinitely many chocolates each round

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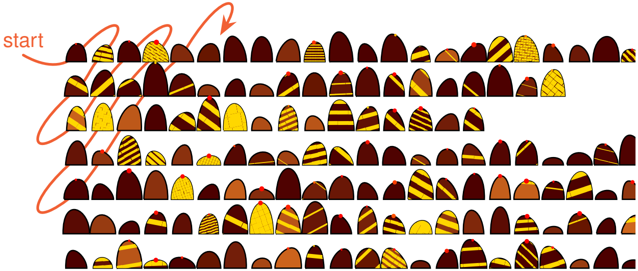

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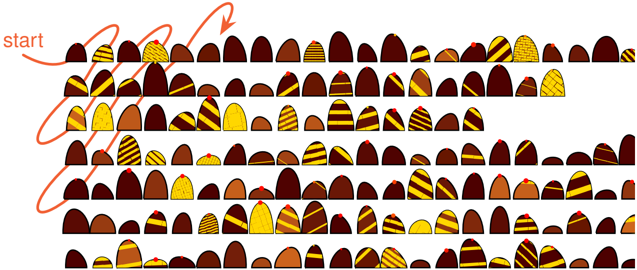

Glutton can win by eating along the winding path.

Every chocolate will be eaten!

Deeper issues

Further considerations lead to deeper mathematical issues.

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- Is there a memory-free strategy?
- If all chocolates are distinct exquisite morsels, but only countably many possible, then yes—eat the lowest on offer.
- What if the Chocolatier is uncountably creative?
- In this case, there can be no memory-free winning strategy for the Glutton.
- Nevertheless, assuming the axiom of choice, the Glutton can win while knowing only chocolates on offer + the previously eaten chocolate.

The Sudoku game

5			4			1		
3					5			7
	9				3	5		
2			7					
		4			8			
6								9
		6				4		
		1			9	2		
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- Consider Sudoku as a two-player game
- Start from empty board, perhaps very large
- Players take turns placing numbers
- Plays must obey the Sudoku condition—no repeats on any row, column, or subboard
- A player loses when there is no legal move
- So this version of the game is not about a global solution

Sudoku game, even size board

Second player has a winning strategy.

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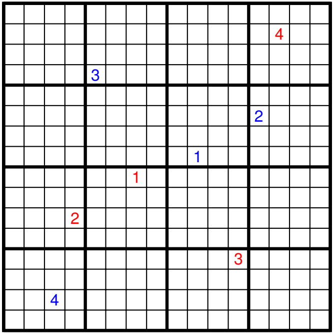

Strategy: Second player copies, reflected through the center.

Key observation: if previous move was legal, so is reflection.

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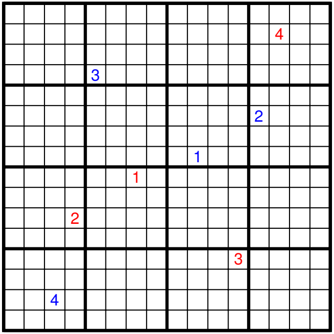

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So the second player wins.

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Now, it is a first-player win.

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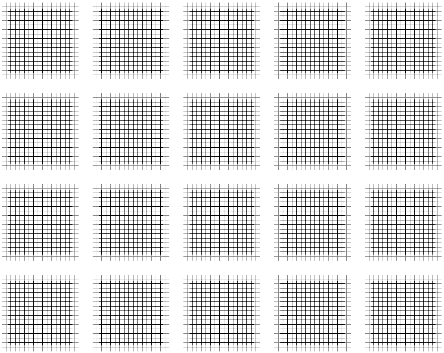

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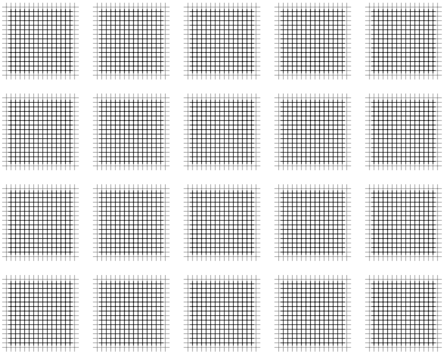

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Each subboard is like the integer grid.

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Can the Spoiler set traps and create obstacles so as to prevent the Solver from succeeding in the limit?

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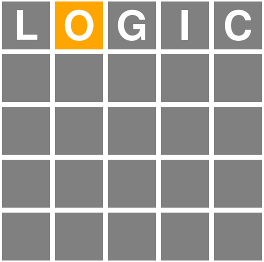
Thus, the Solver has a winning strategy.

Wordle



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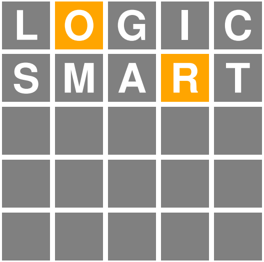

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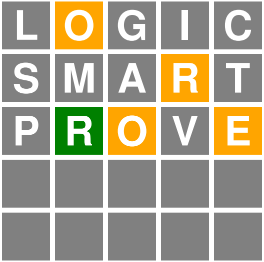

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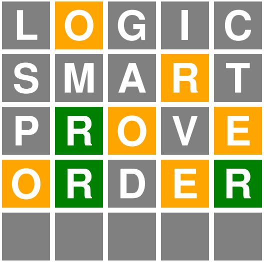

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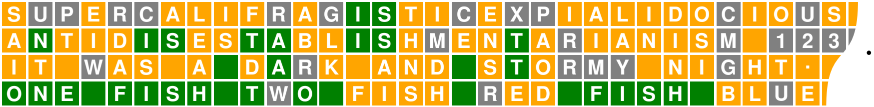

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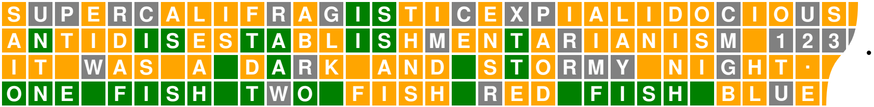

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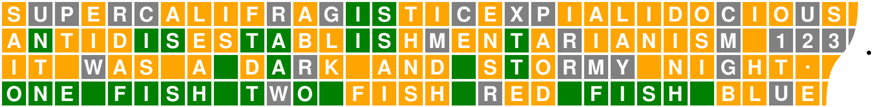

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Furthermore, let us consider any crazy fixed dictionary of allowed phrases, perhaps including many nonsense sequences of letters.

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Suppose we are playing Long Wordle, with phrases having millions of letters each.

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With 26 letters, there are $26^{1000000}$ many distinct "words" of length one million.

If we wrote down one million of them every second since the beginning of time at the Big Bang, we will have hardly started!

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Iterate. Since 26 letters, this can go on at most 26 steps. $\square$

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Related to Infinite Mastermind. Things get very interesting and complicated.

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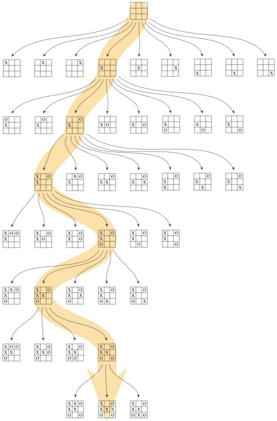
Some of these questions begin to engage with deep mathematical and sometimes philosophical questions.

Game tree

Every game gives rise to a *game tree*, the tree of all possible positions of the game. Plays of the game are paths through the game tree.



Fundamental theorem of finite games

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I know five distinct proofs of this beautiful, classical theorem.

When teaching my logic classes at the university, I would usually spend several lectures giving all five proofs.

Let me describe just one of them here.

Fundamental theorem—proof by back-propagation

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Terminal nodes labeled as a win for one player or the other.

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Whoever gets their label on the root will have a winning strategy—stay on your own label. $\square$

Generalization to games with draws

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Fundamental theorem of finite games (with draws)

In any finite game of perfect information, allowing draws, either one of the players has a winning strategy or both players have draw-or-better strategies.

Follows from the win/loss version.

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Chess

For the game of chess, either one of the players white or black has a winning strategy or both players have draw-or-better strategies.

Open determinacy

Similar ideas work for *open* games, which are games for which all winning plays for one of the players are known as wins at a finite stage of play.

Open determinacy theorem (Gale-Stewart 1953)

In every open game, one of the players has a winning strategy (or both have drawing strategies).

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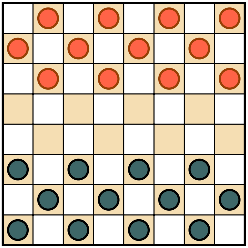
In every open game, one of the players has a winning strategy (or both have drawing strategies).

One way to prove this uses the concept of *transfinite ordinal game values*.

These generalize the chess idea of mate-in-2 or mate-in-3, but work with infinite games.

Finite draughts

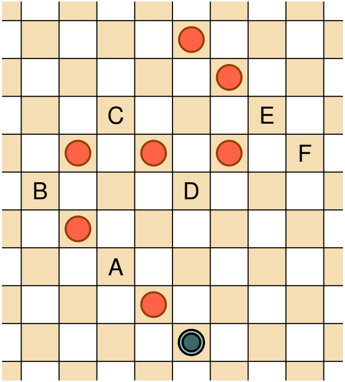
The familiar game of draughts, also known as checkers.



Although often played by children, draughts admits serious advanced play, with draughts grandmasters competing in international tournaments.

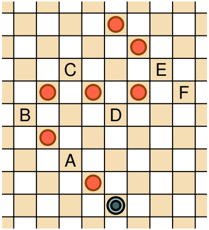
Infinite draughts

Let us endeavor instead to play *infinite* draughts.



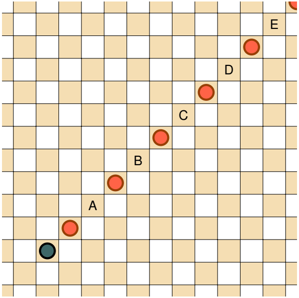
The checkerboard extends without end in every direction.

The rules



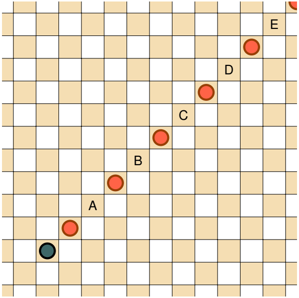
- No standard starting configuration.
- Analyze the game from any given position.
- Pawns and kings.
- Obligatory jumping rule.
- Obligatory iterated jumping.
- Winning condition: you lose when you have no legal move.

Infinite iterations



During an infinite iterated jump, the red pieces are all removed.

Infinite iterations



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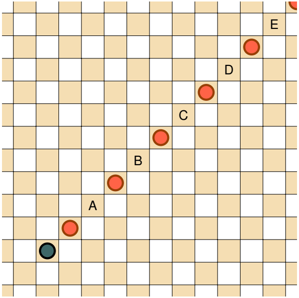
A conundrum: what about the black piece?

Infinite iterations

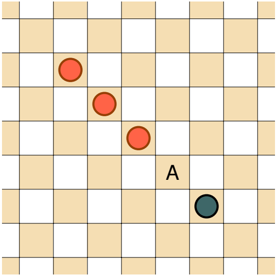


During an infinite iterated jump, the red pieces are all removed.

A conundrum: what about the black piece?

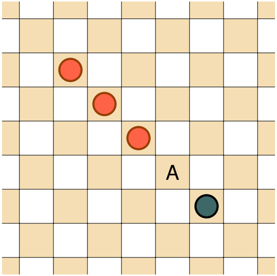
Infinite iterated-jump rule: the jumping piece also is removed.

Game value 2



This position, with Red to play, has game value 2 for Red.

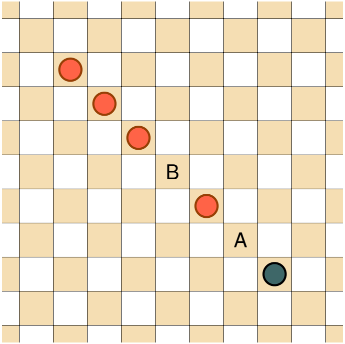
Game value 2



This position, with Red to play, has game value 2 for Red.

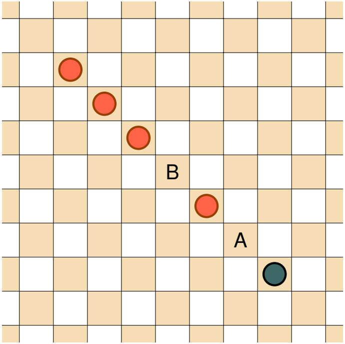
Red can advance leading pawn to A as bait, obligating Black to jump, after which Red recaptures.

Game value 3



Red to play, with game value 3.

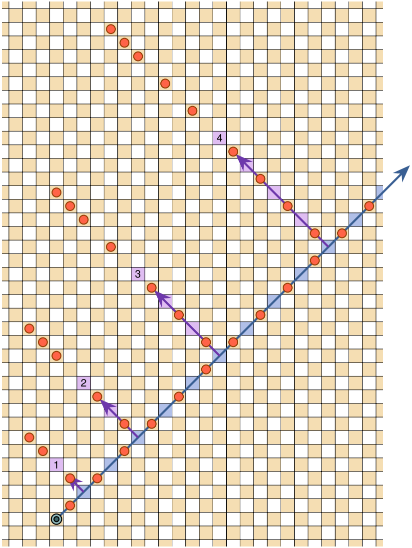
Game value 3



Red to play, with game value 3.

Red advances isolated pawn to A, obligating Black to jump, then advance to B, Black jumps, then Red captures.

Game value ω



Black is obligated to jump.

Can jump infinitely,
but that is an immediate loss.

So will come
to rest at some square n.

And then lose in n moves.

This is game value ω.

Infinite draughts exhibits high game values

Theorem (Hamkins & Leonessi 2022)

Every countable ordinal arises as the game value of a position in infinite draughts.

This result is optimal—infinite draughts exhibits the most robust spectrum of game values that is possible in a countable game.

Infinite draughts exhibits high game values

Theorem (Hamkins & Leonessi 2022)

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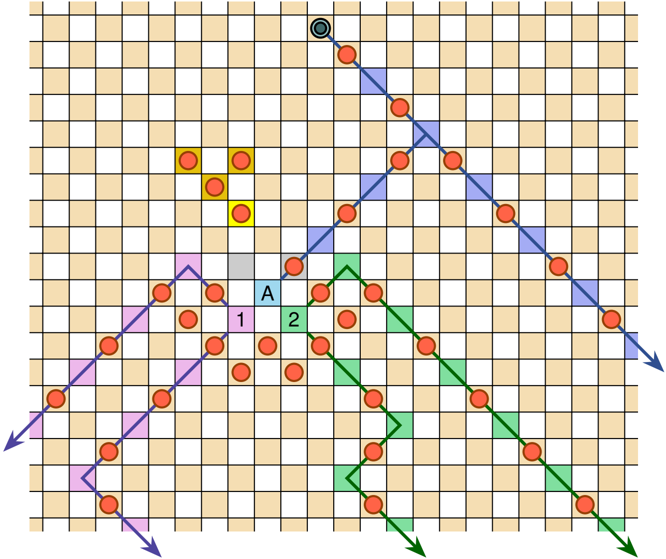
This result is optimal—infinite draughts exhibits the most robust spectrum of game values that is possible in a countable game.

Main proof idea: embed certain well-founded trees into positions of infinite draughts.

The draughts play will unfold in a manner as though Black is climbing the tree, and consequently the game value will track the ordinal rank of the well-founded tree itself.


A
1
2

Infinite chess

Let us turn now to infinite chess.

Infinite chess

Let us turn now to infinite chess.

Infinite chess is a game of the mind—we do not sit down in a café and play infinite chess over espresso.

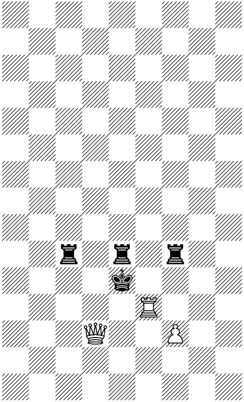
Infinite chess

Let us turn now to infinite chess.

Infinite chess is a game of the mind—we do not sit down in a café and play infinite chess over espresso.

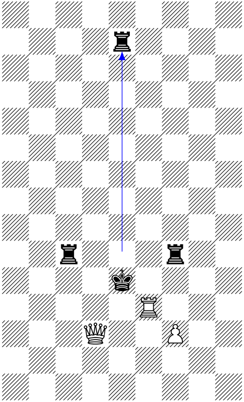
But rather, we sit down in that café and wonder what it would be like to play from this position or that one.

A finite position with value ω



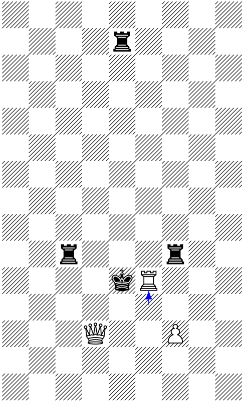
Black to move.

A finite position with value ω



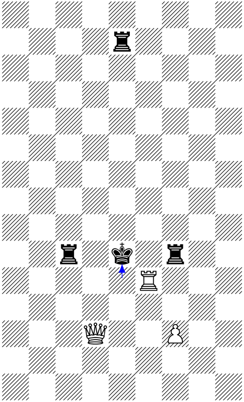
Black moves up arbitrary height

A finite position with value ω

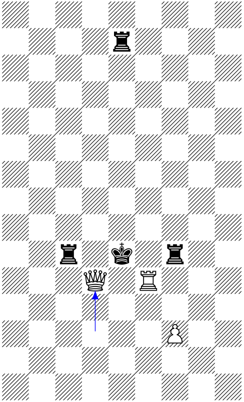


Check

A finite position with value ω



A finite position with value ω



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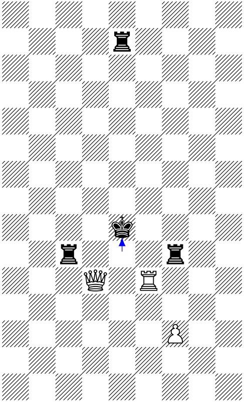

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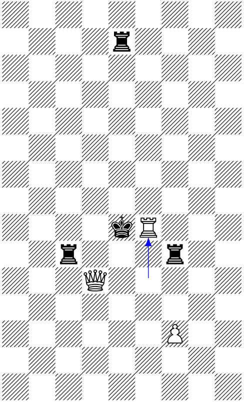

Check

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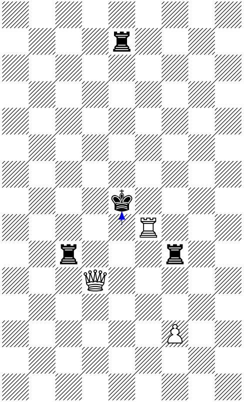

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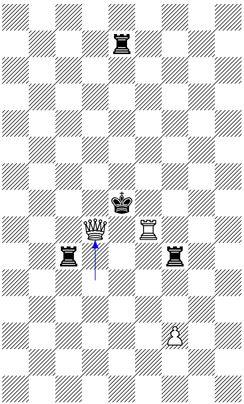

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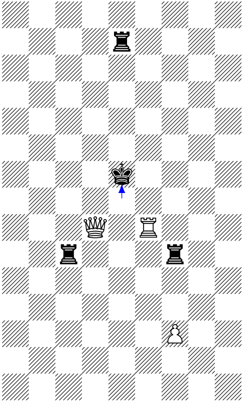

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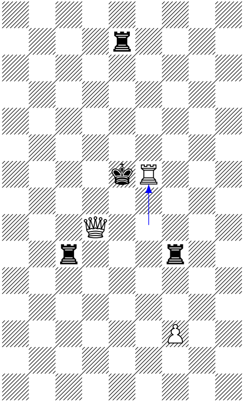

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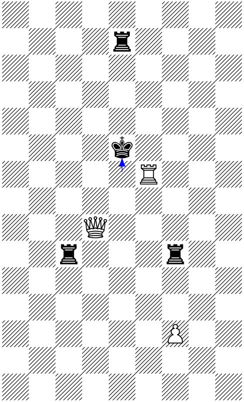

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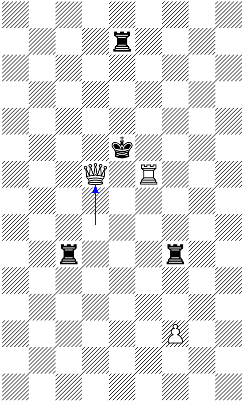

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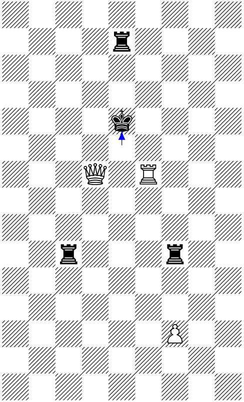

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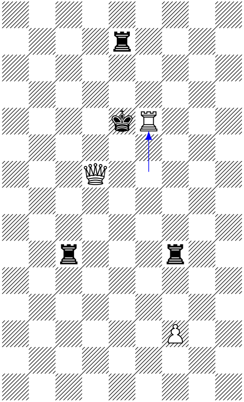

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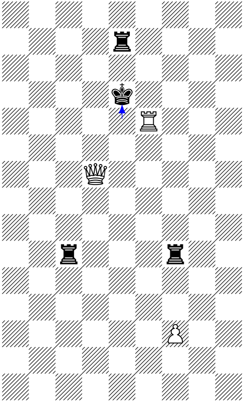

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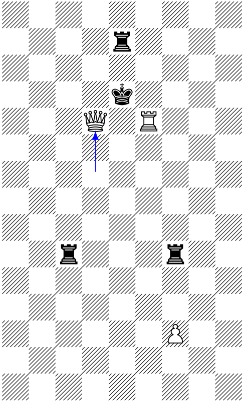

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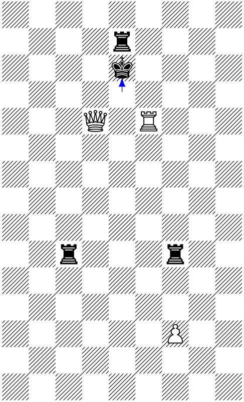

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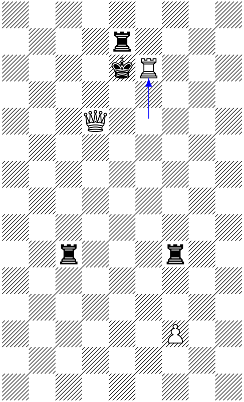
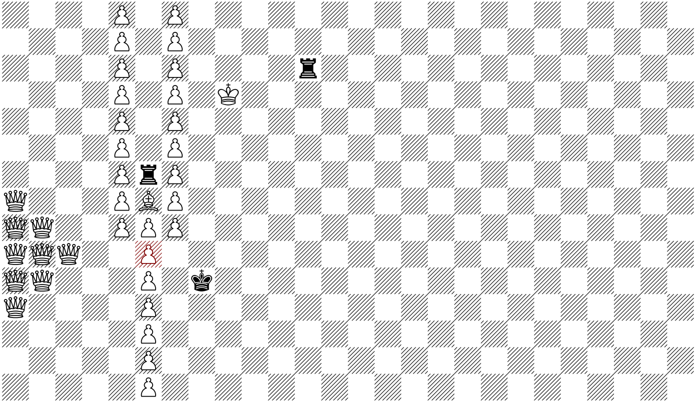


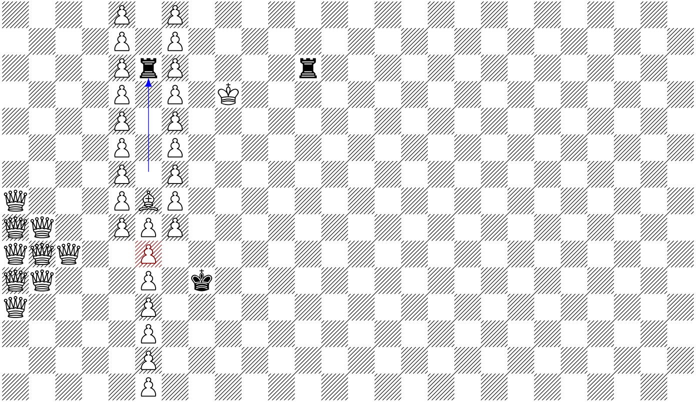
Checkmate. Black can cause arbitrary delay, but the only choice is on first move, so the initial value is ω.

Releasing the Hordes, with value ω^2



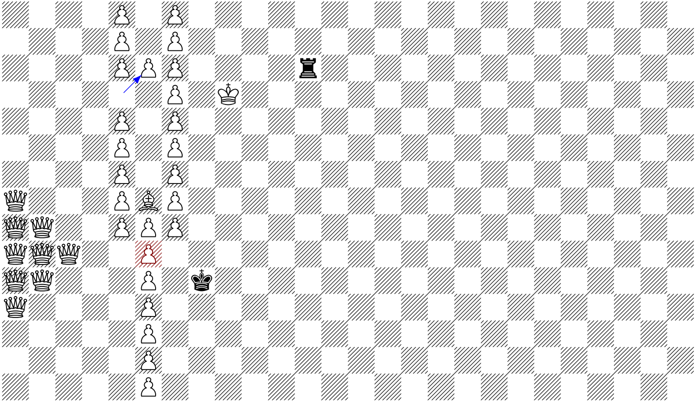
Black to move.

Releasing the Hordes, with value ω^2



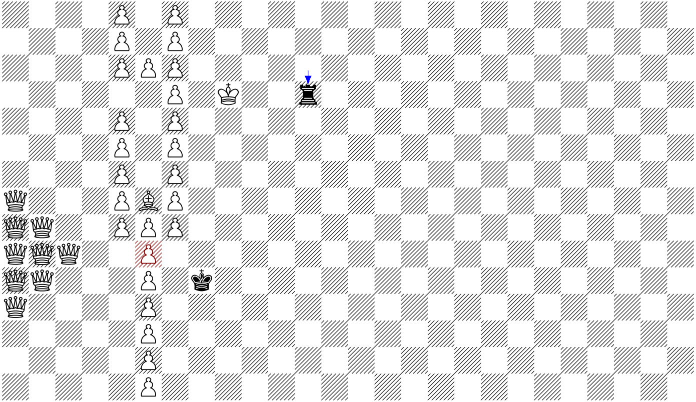
He moves trapped rook up arbitrary height.

Releasing the Hordes, with value ω^2



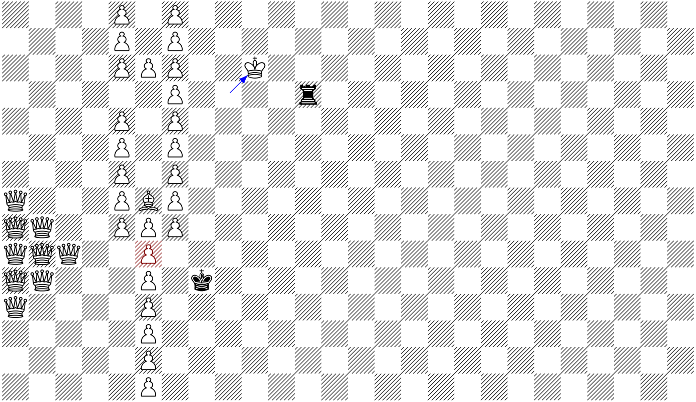
White should capture from left side.

Releasing the Hordes, with value ω^2



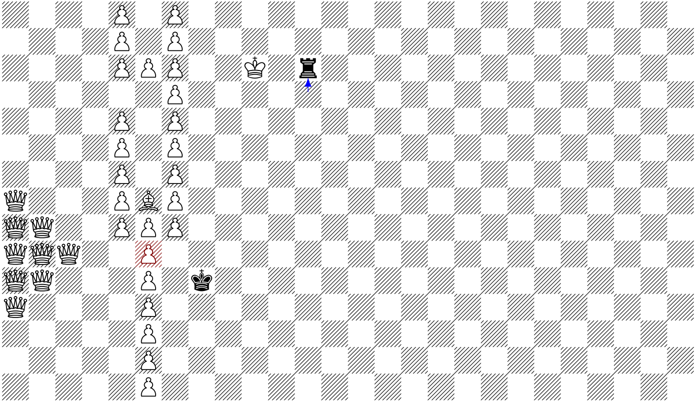
Now black begins to harass white king.

Releasing the Hordes, with value ω^2



White must chase down the rook to avoid perpetual check.

Releasing the Hordes, with value ω^2



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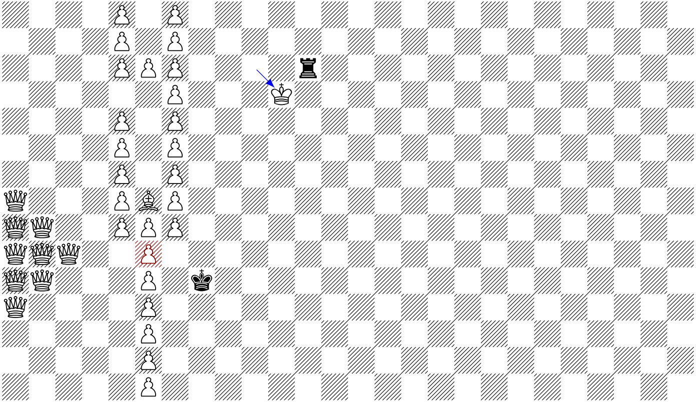

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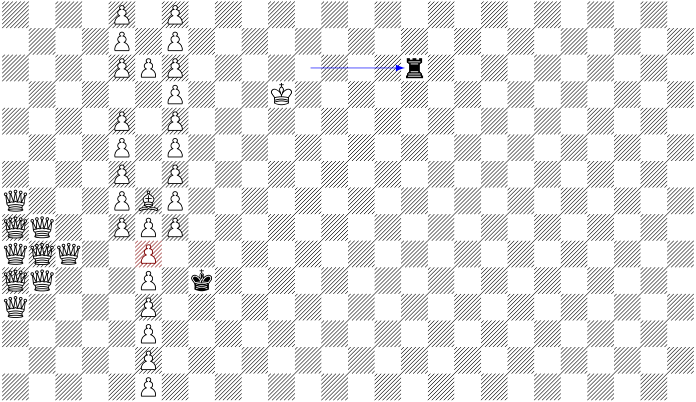
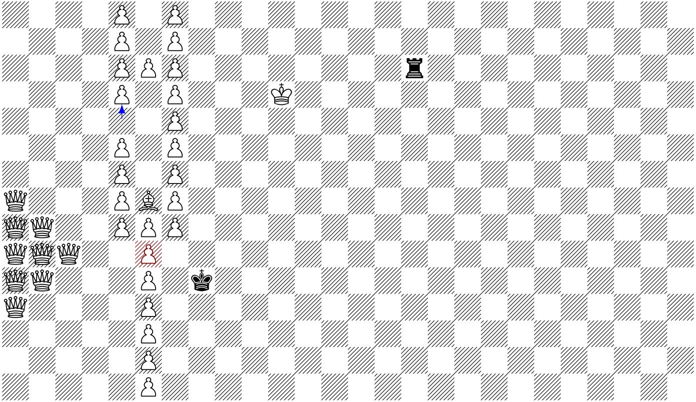


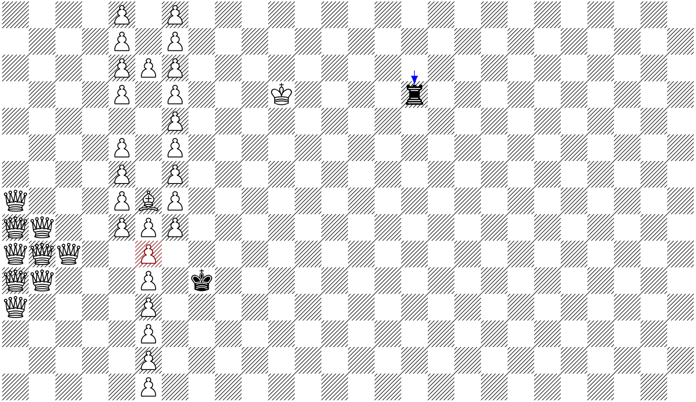
Black must move away to save rook.

Releasing the Hordes, with value ω^2



Now is white's chance to advance a pawn.

Releasing the Hordes, with value ω^2



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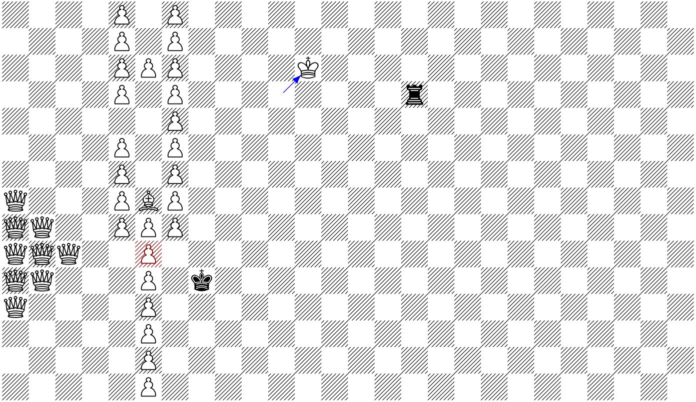

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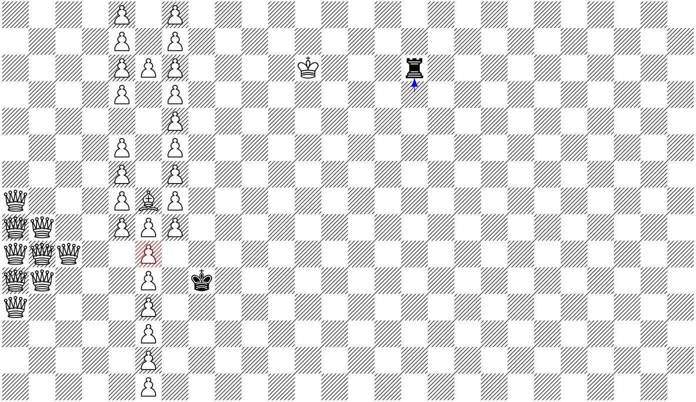

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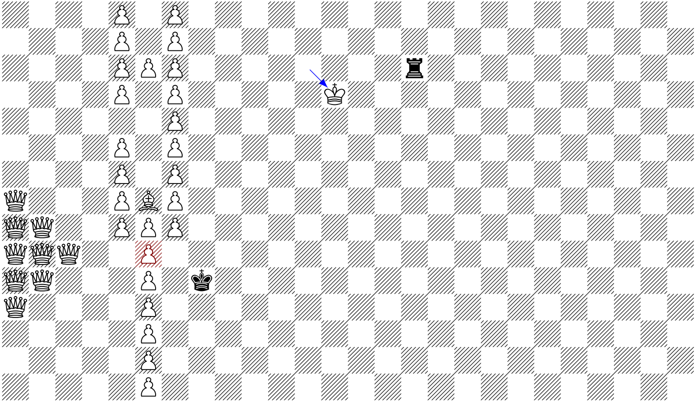

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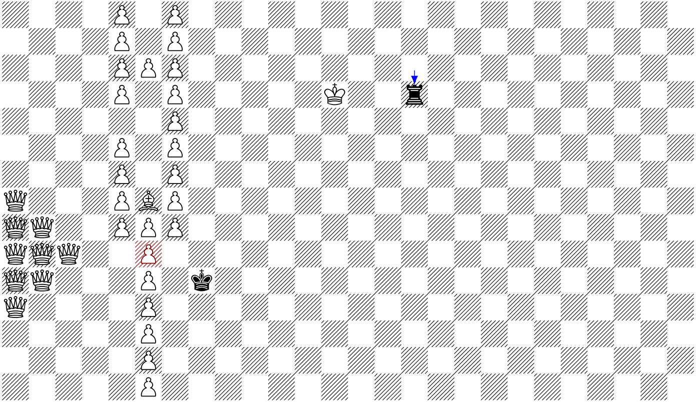

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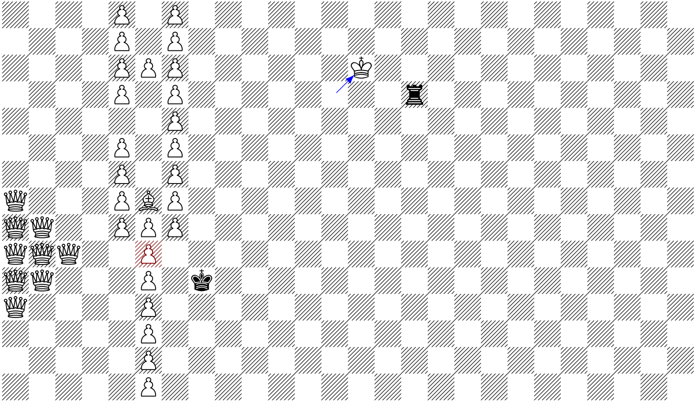

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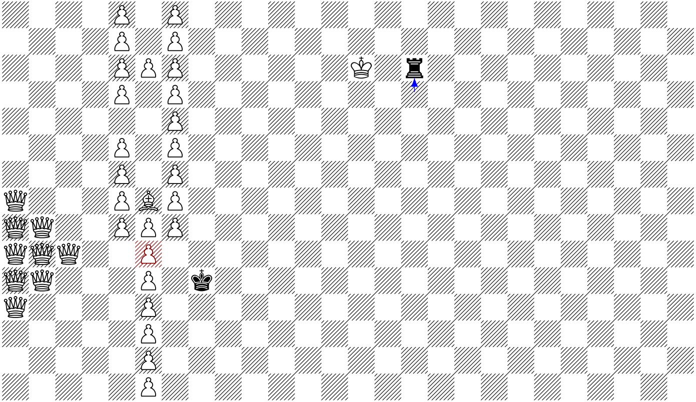

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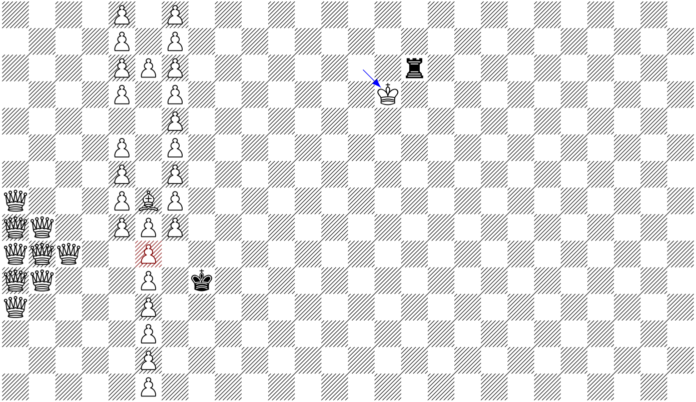

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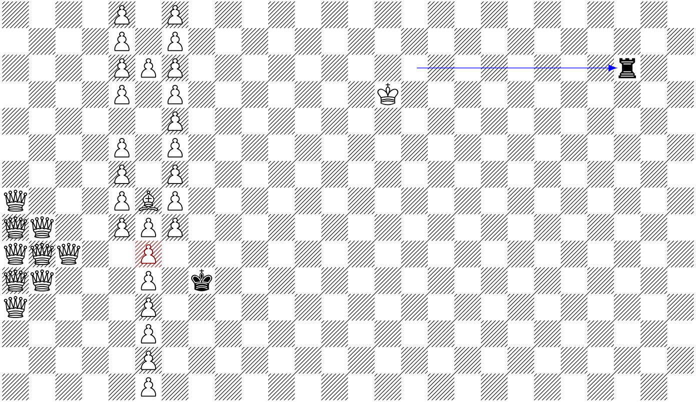
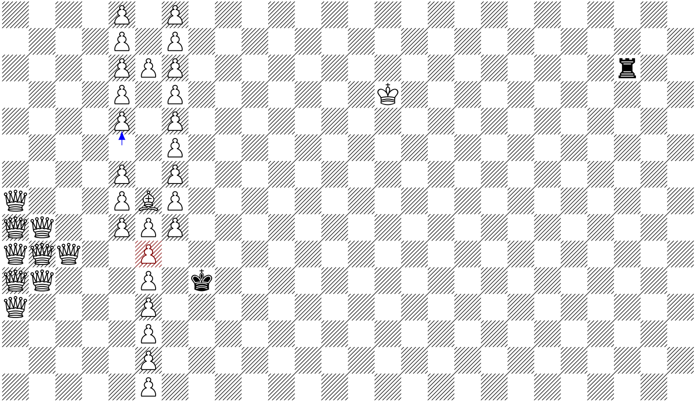


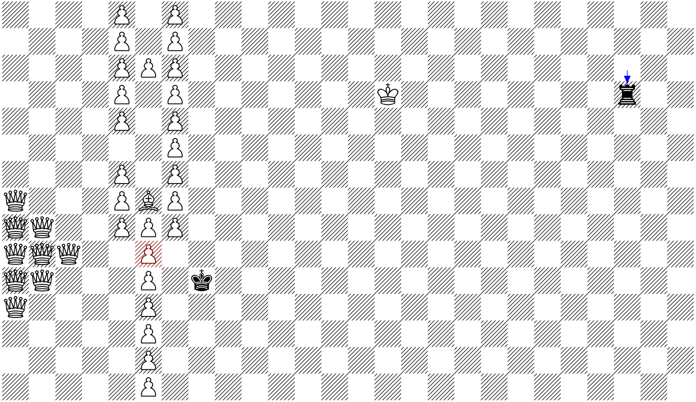
Black moves arbitrary distance out.

Releasing the Hordes, with value ω^2



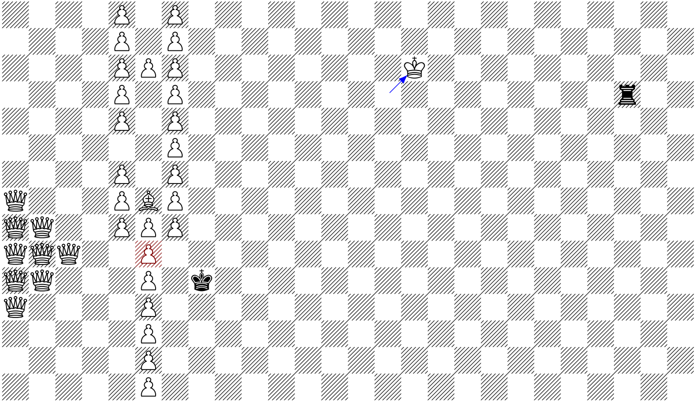
Another chance to advance a pawn.

Releasing the Hordes, with value ω^2



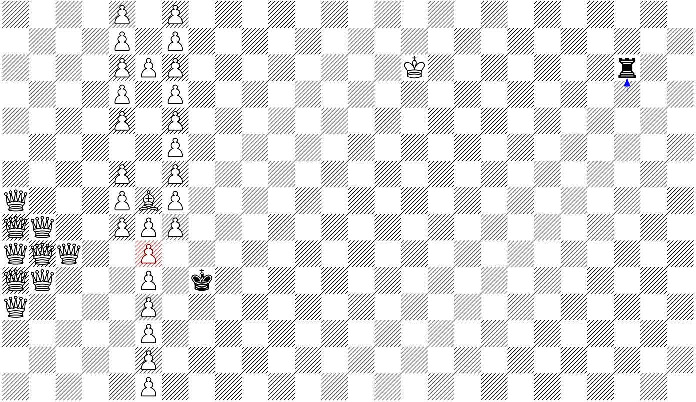
Black harasses the white king.

Releasing the Hordes, with value ω^2



White must chase him down.

Releasing the Hordes, with value ω^2



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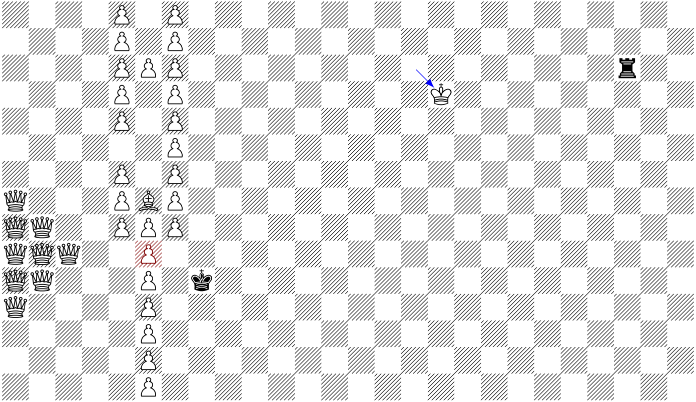

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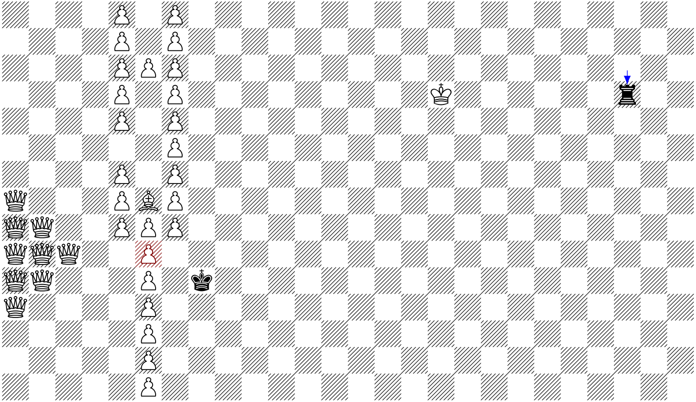

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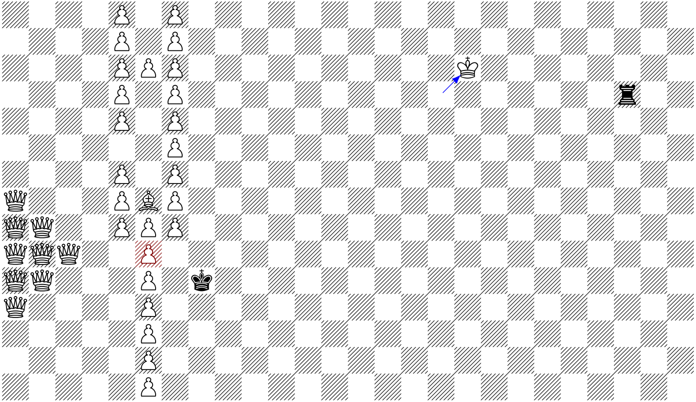

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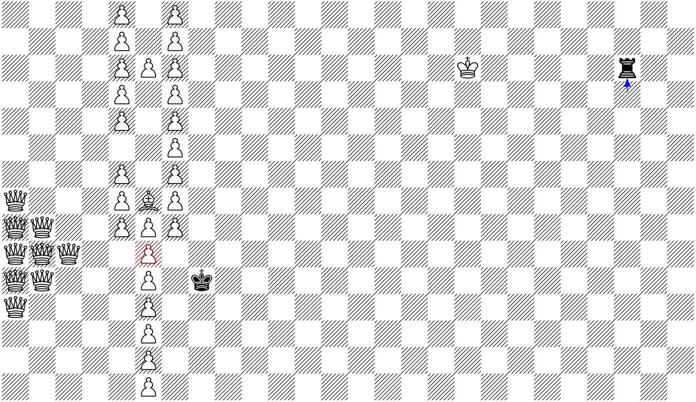

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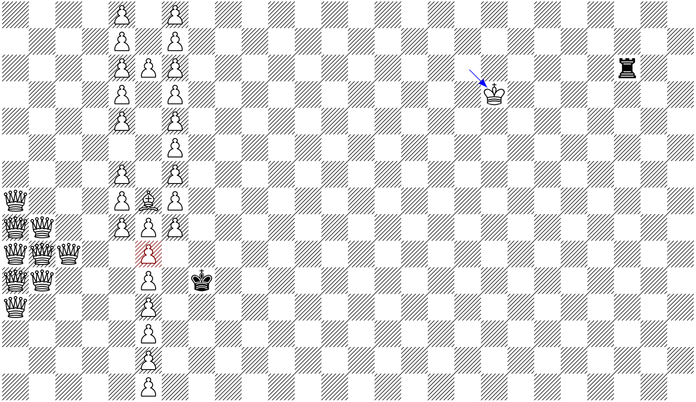

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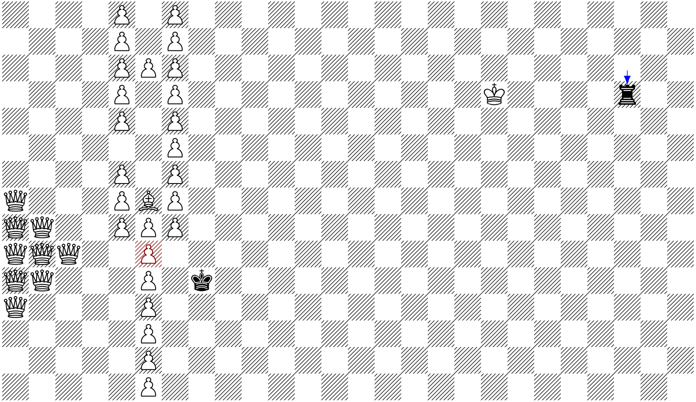

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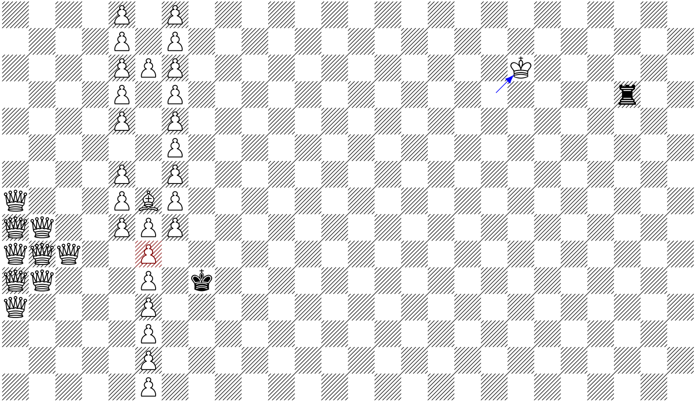
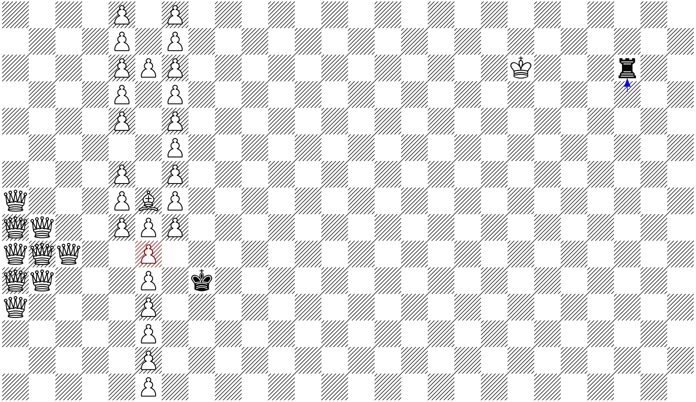
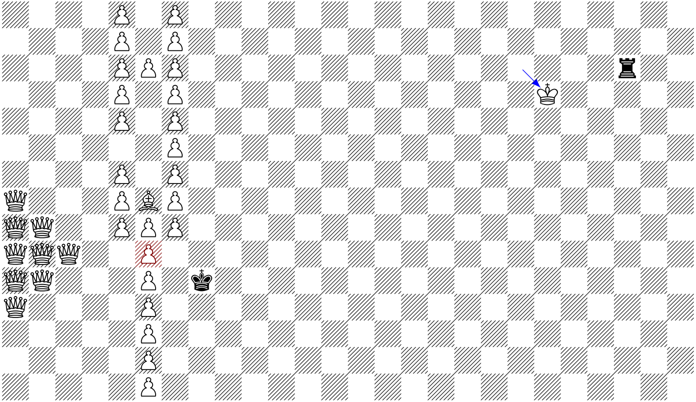


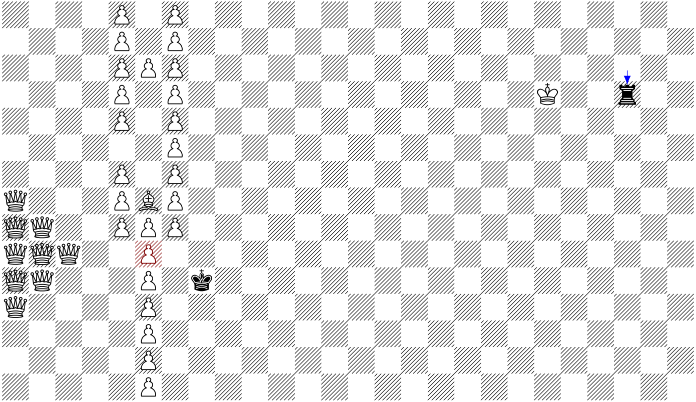
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

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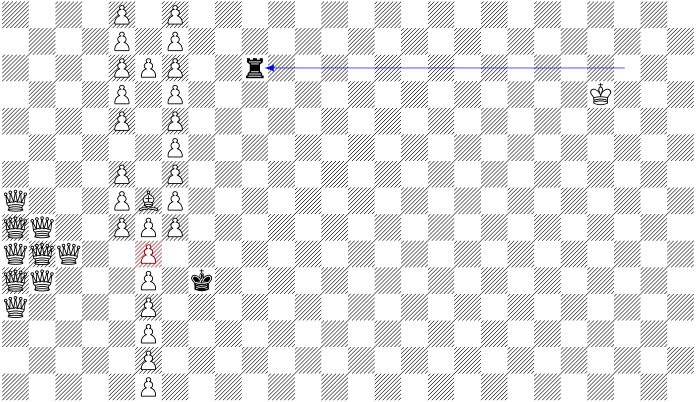
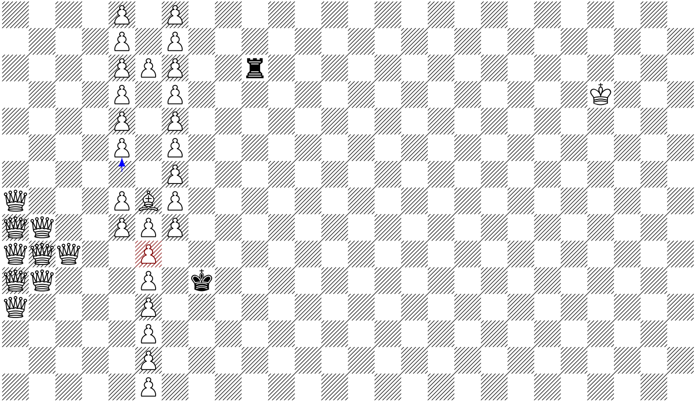
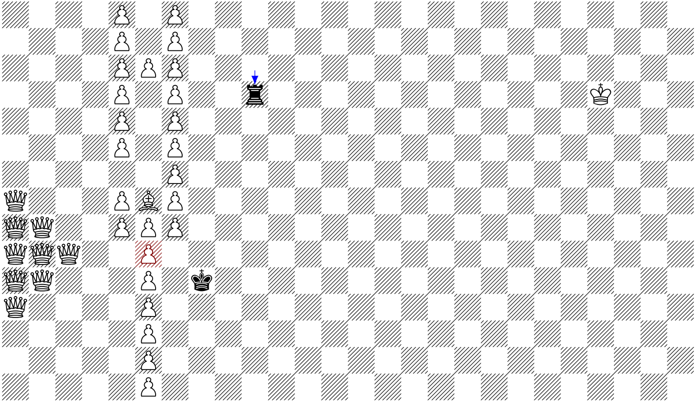


(Black should actually move arbitrary distance to the right.)

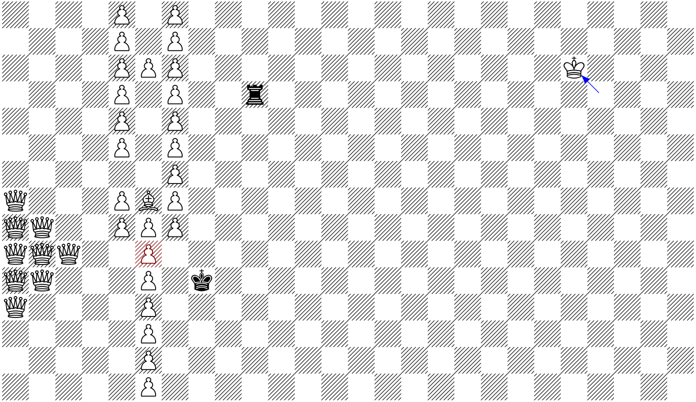
Releasing the Hordes, with value ω^2



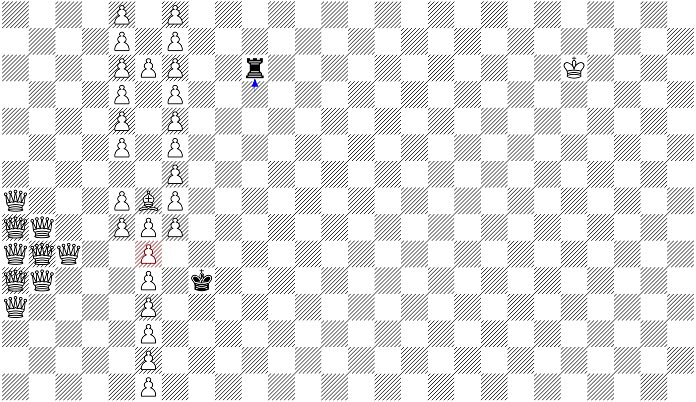
Releasing the Hordes, with value ω^2



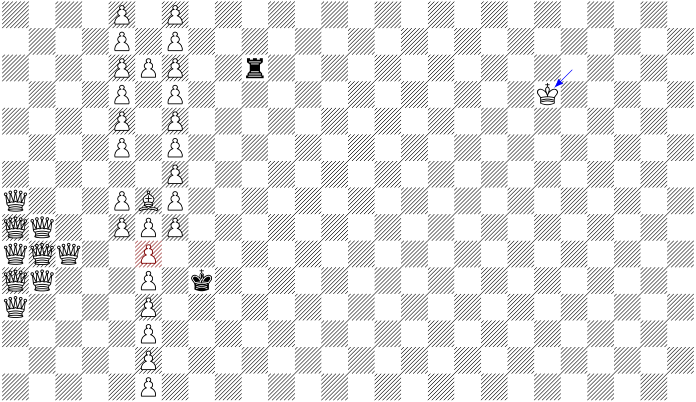
Releasing the Hordes, with value ω^2



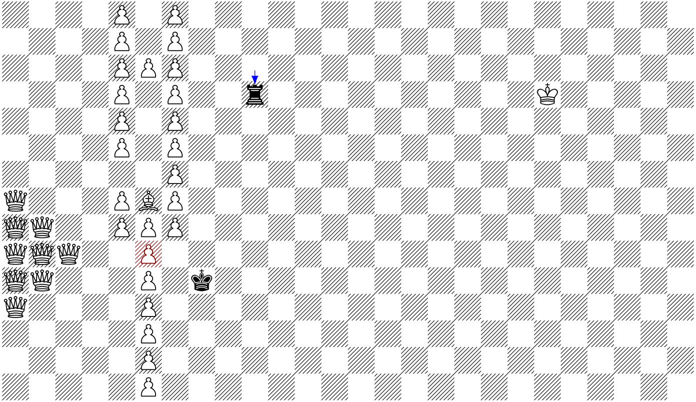
Releasing the Hordes, with value ω^2



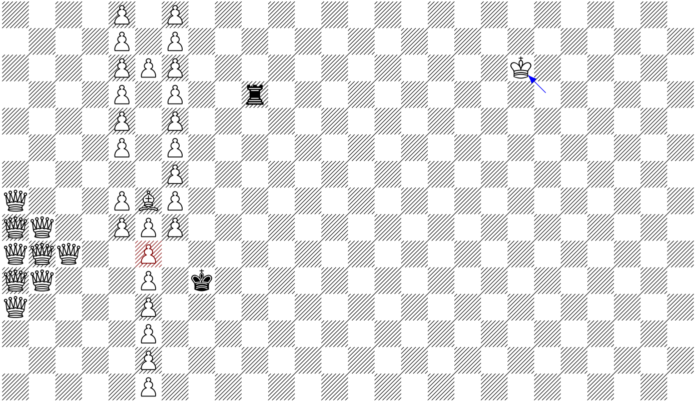
Releasing the Hordes, with value ω^2



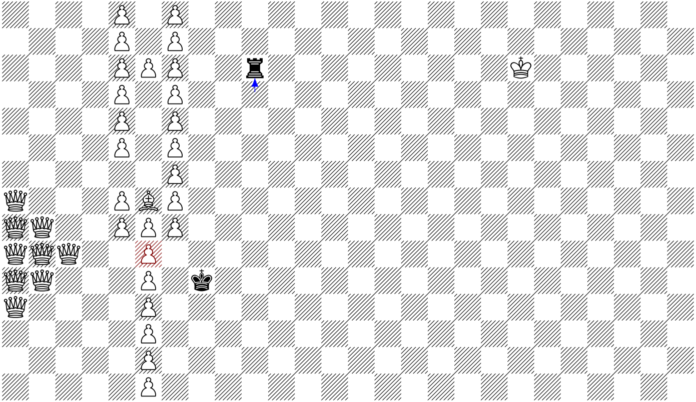
Releasing the Hordes, with value ω^2



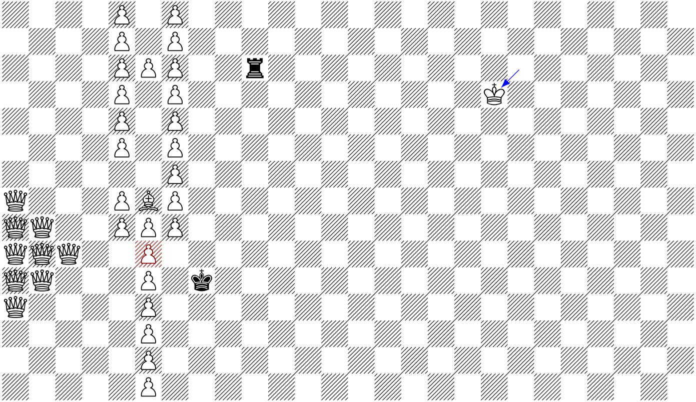
Releasing the Hordes, with value ω^2



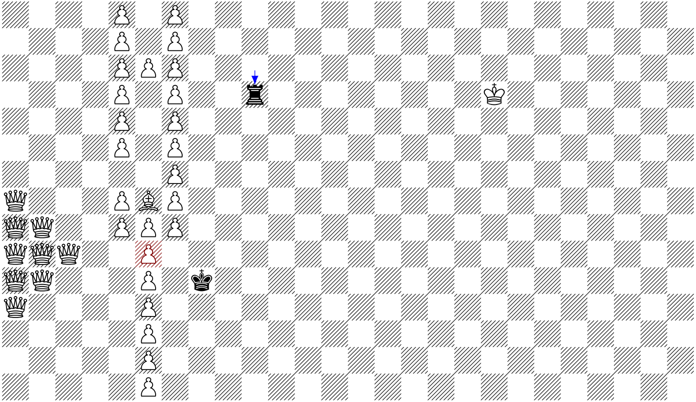
Releasing the Hordes, with value ω^2



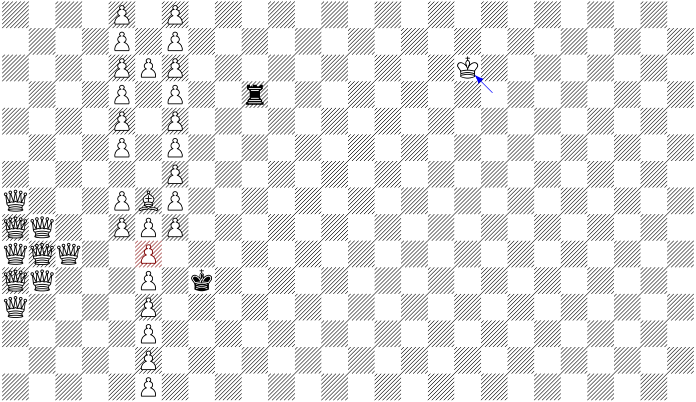
Releasing the Hordes, with value ω^2



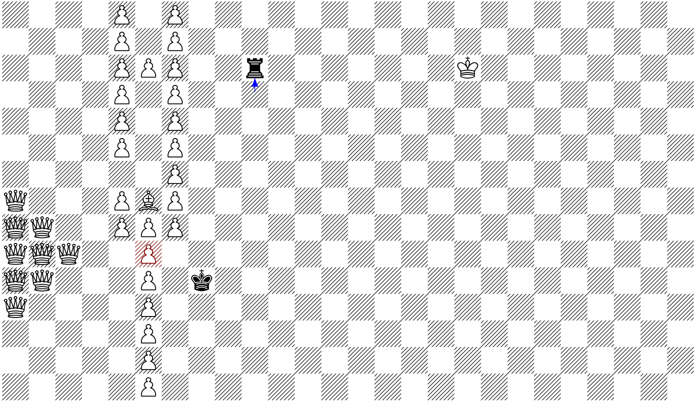
Releasing the Hordes, with value ω^2



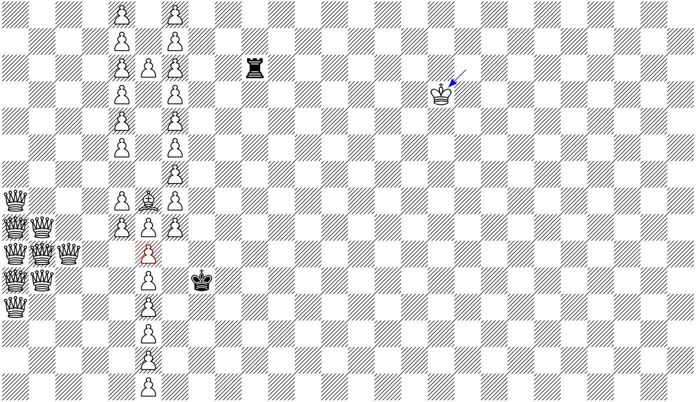
Releasing the Hordes, with value ω^2



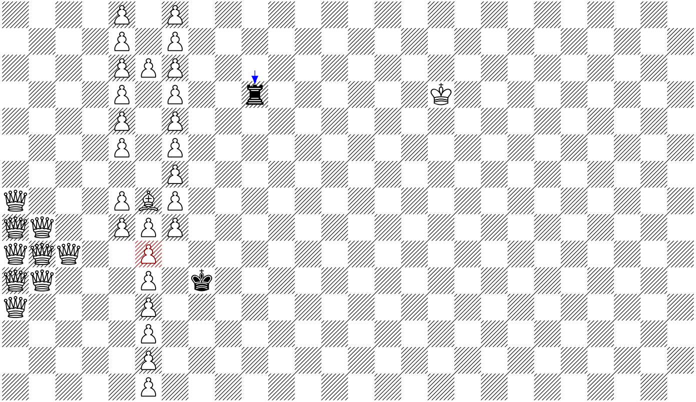
Releasing the Hordes, with value ω^2



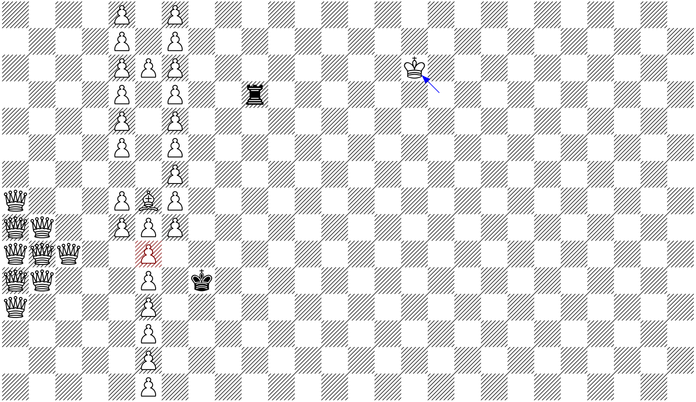
Releasing the Hordes, with value ω^2



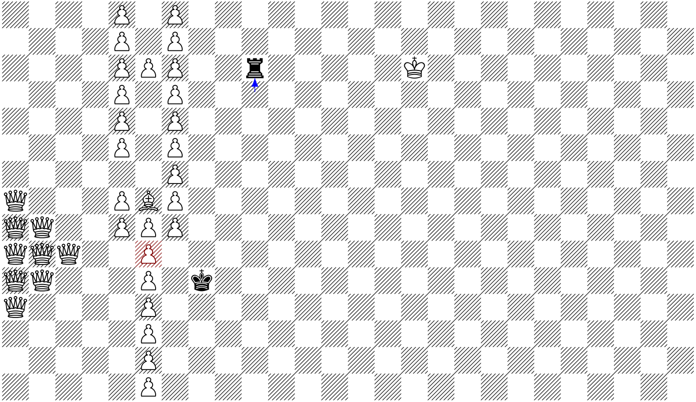
Releasing the Hordes, with value ω^2



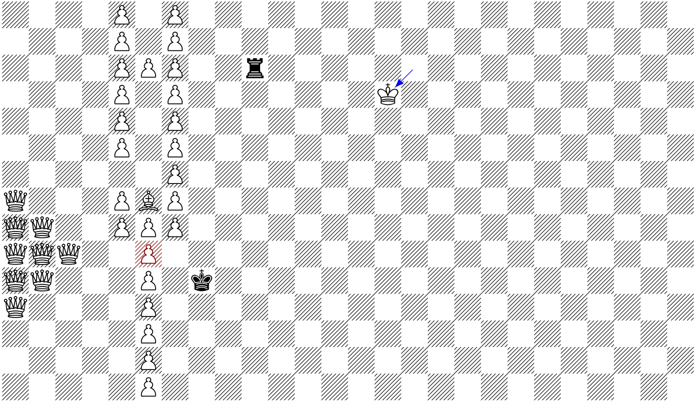
Releasing the Hordes, with value ω^2



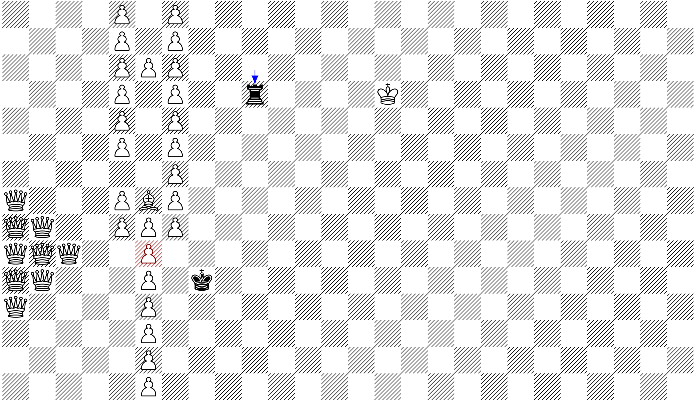
Releasing the Hordes, with value ω^2



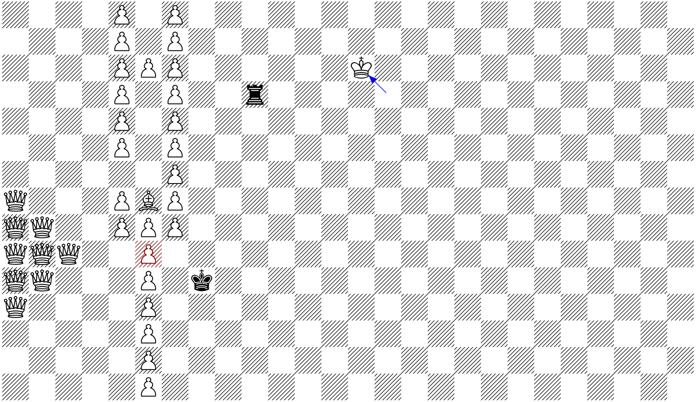
Releasing the Hordes, with value ω^2



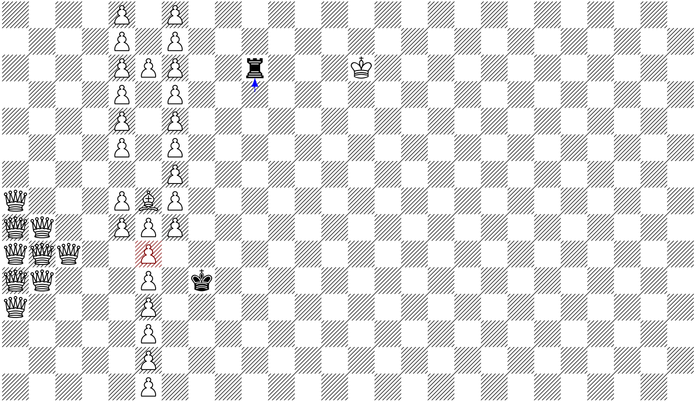
Releasing the Hordes, with value ω^2



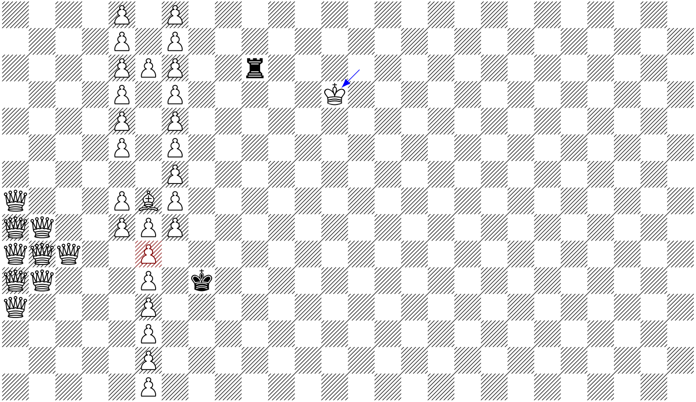
Releasing the Hordes, with value ω^2



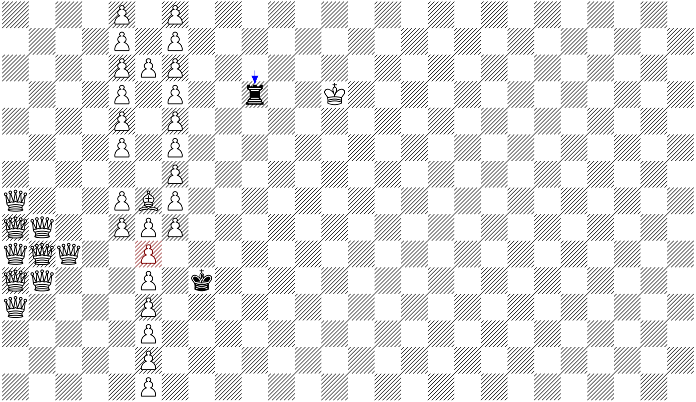
Releasing the Hordes, with value ω^2



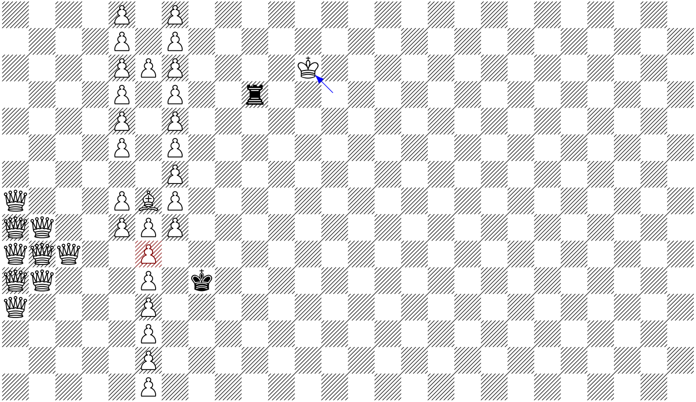
Releasing the Hordes, with value ω^2



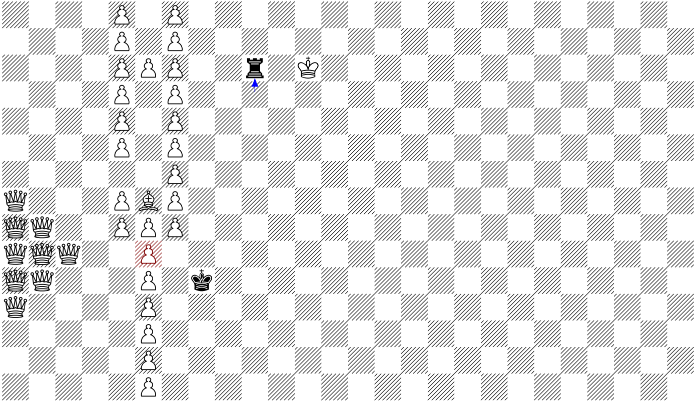
Releasing the Hordes, with value ω^2



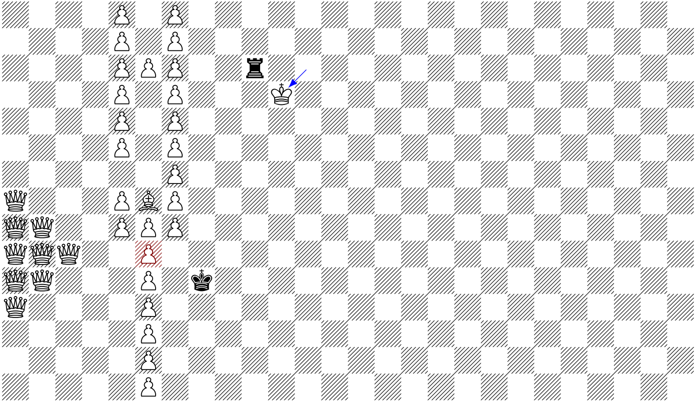
Releasing the Hordes, with value ω^2



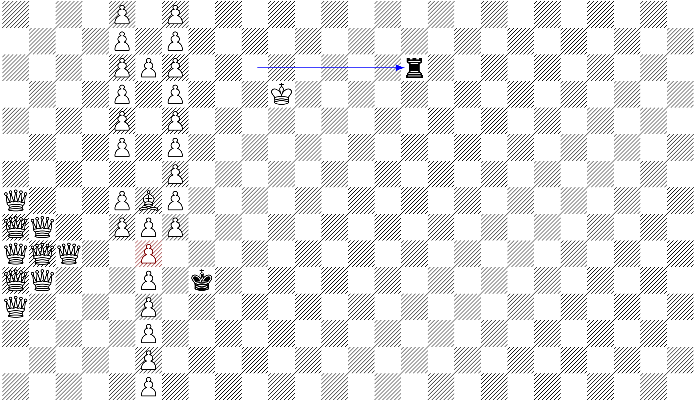
Releasing the Hordes, with value ω^2



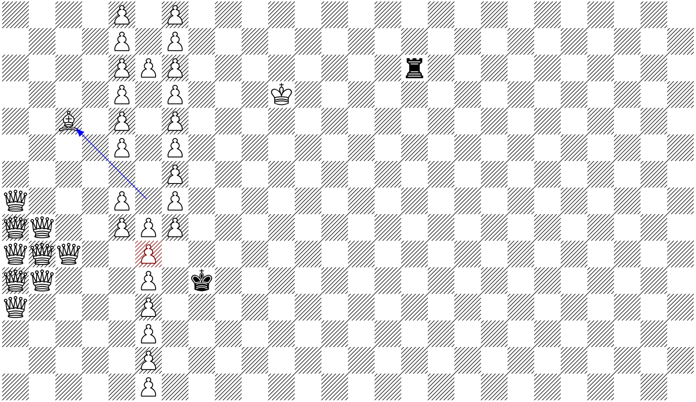
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

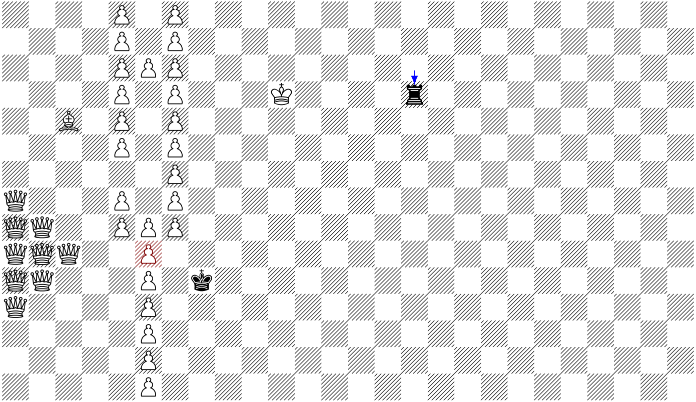


Releasing the Hordes, with value ω^2

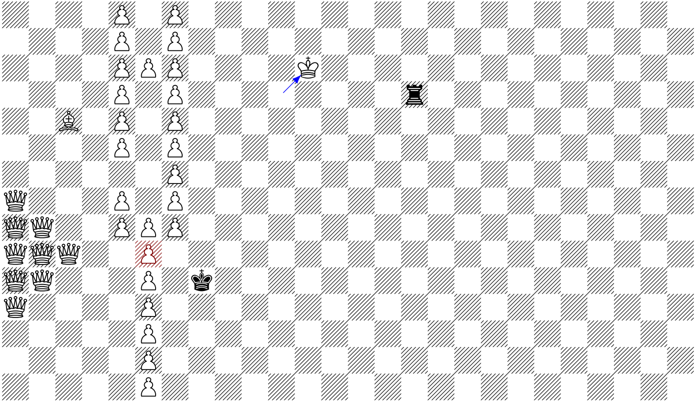


The bishop unlocks the door.

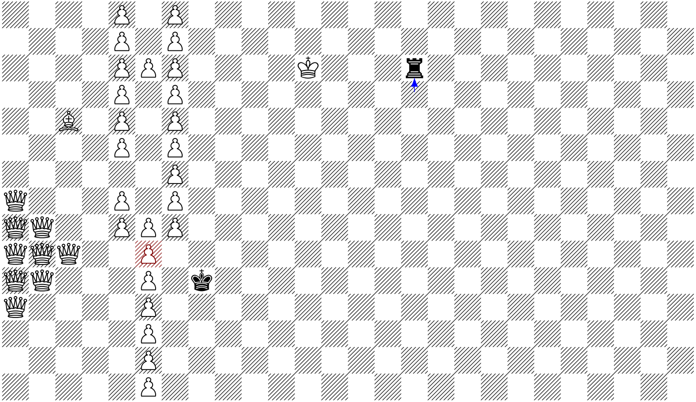
Releasing the Hordes, with value ω^2



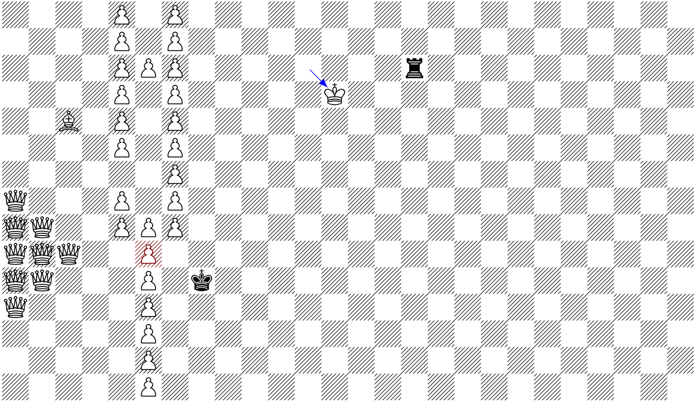
Releasing the Hordes, with value ω^2



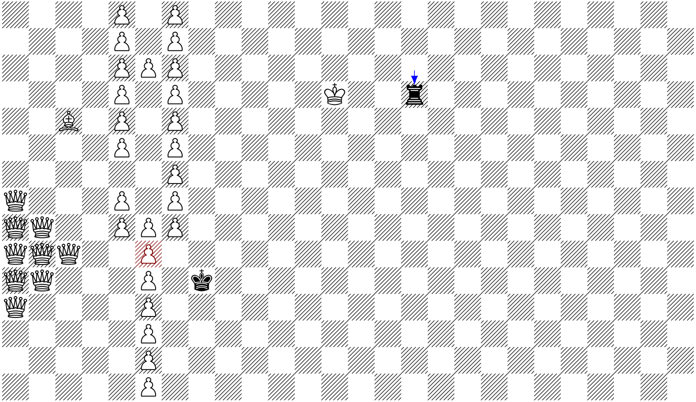
Releasing the Hordes, with value ω^2



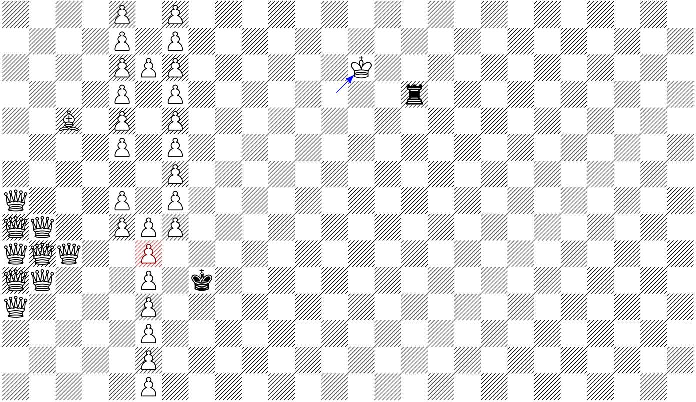
Releasing the Hordes, with value ω^2



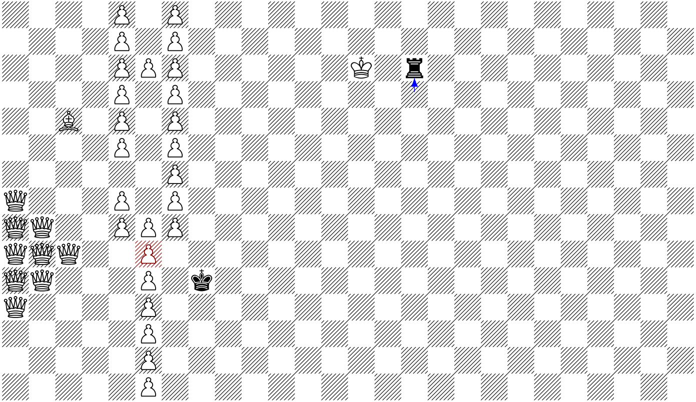
Releasing the Hordes, with value ω^2



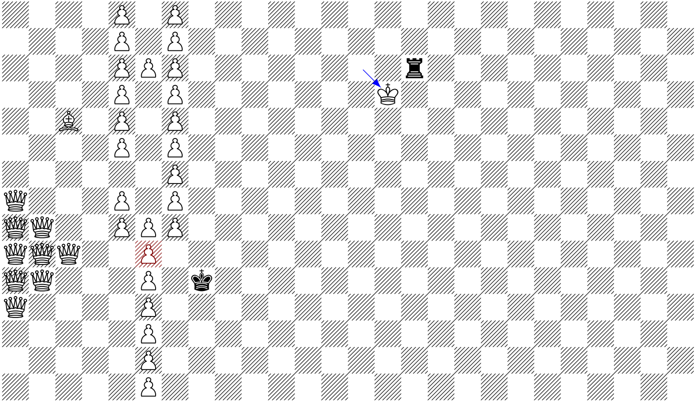
Releasing the Hordes, with value ω^2



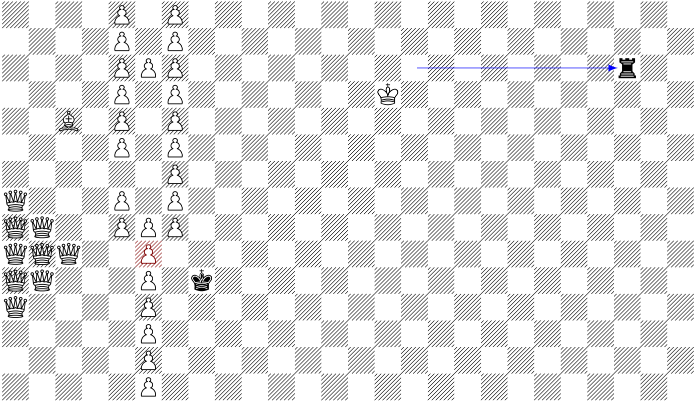
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

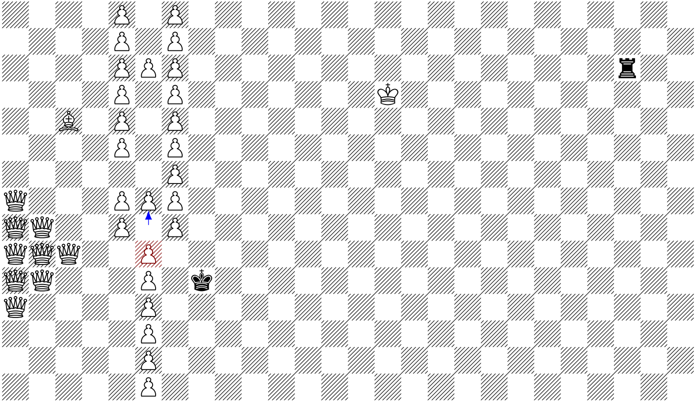


Releasing the Hordes, with value ω^2

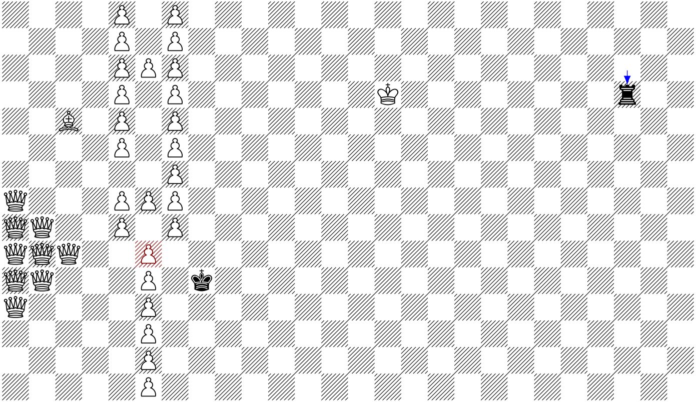


Black can move rook arbitrary distance.

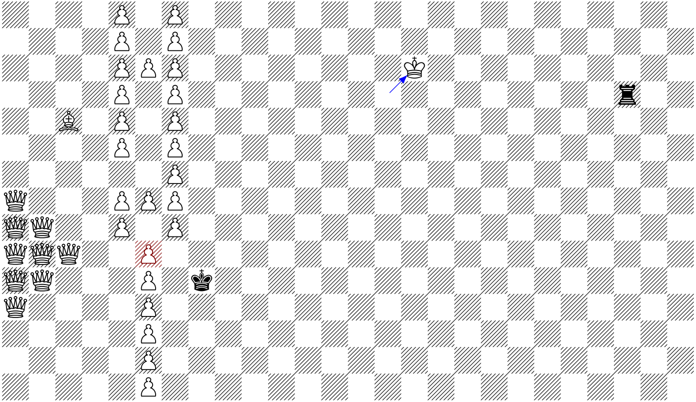
Releasing the Hordes, with value ω^2



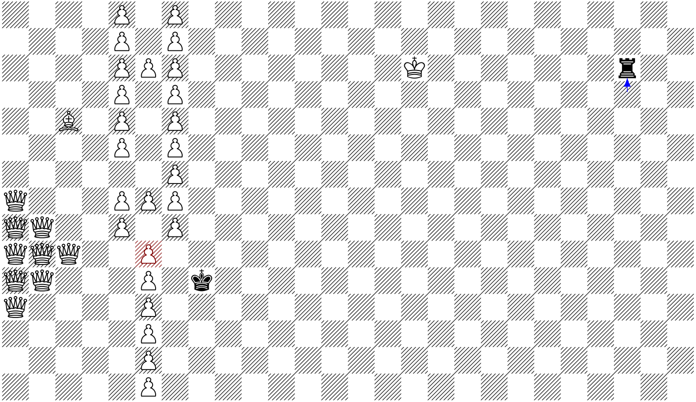
Releasing the Hordes, with value ω^2



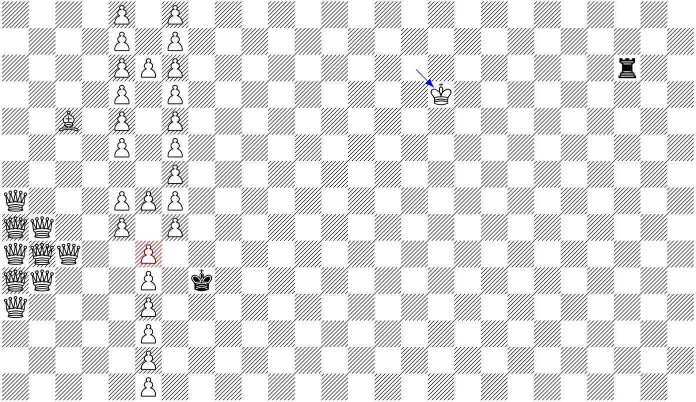
Releasing the Hordes, with value ω^2



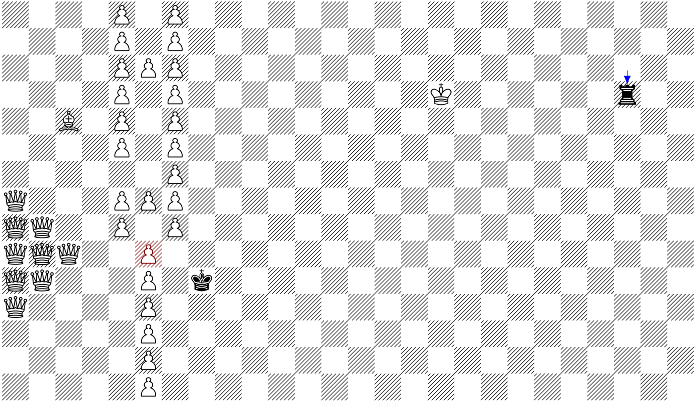
Releasing the Hordes, with value ω^2



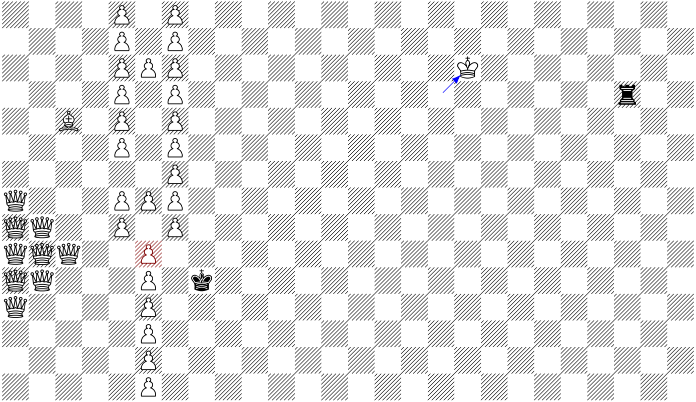
Releasing the Hordes, with value ω^2



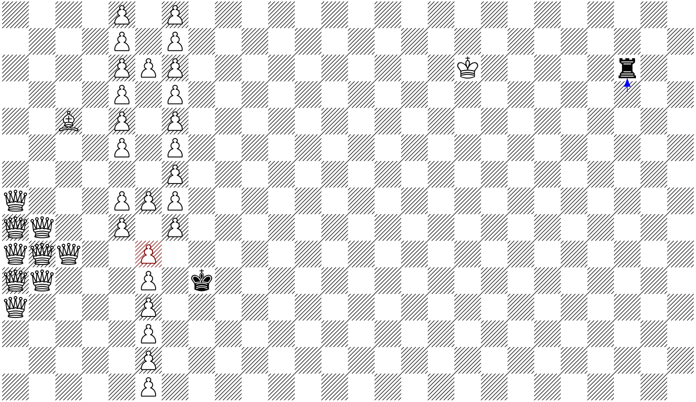
Releasing the Hordes, with value ω^2



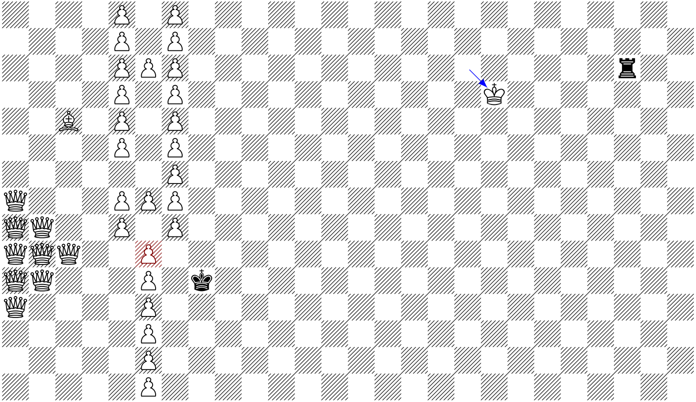
Releasing the Hordes, with value ω^2



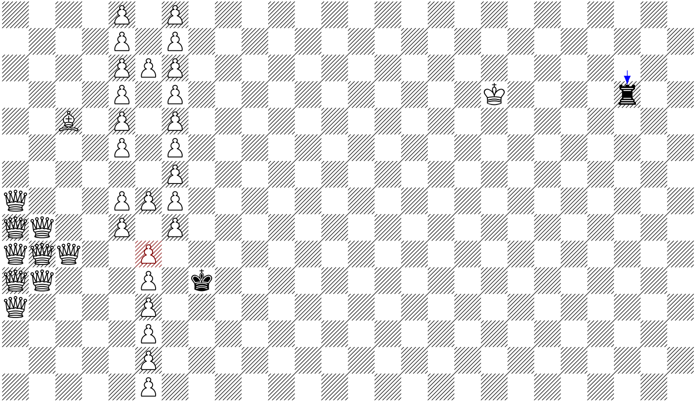
Releasing the Hordes, with value ω^2



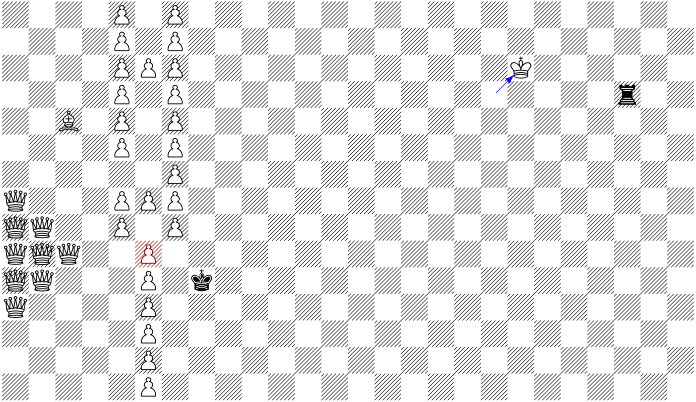
Releasing the Hordes, with value ω^2



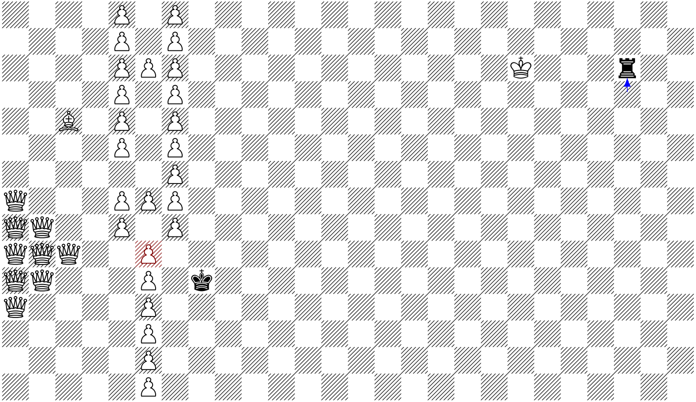
Releasing the Hordes, with value ω^2



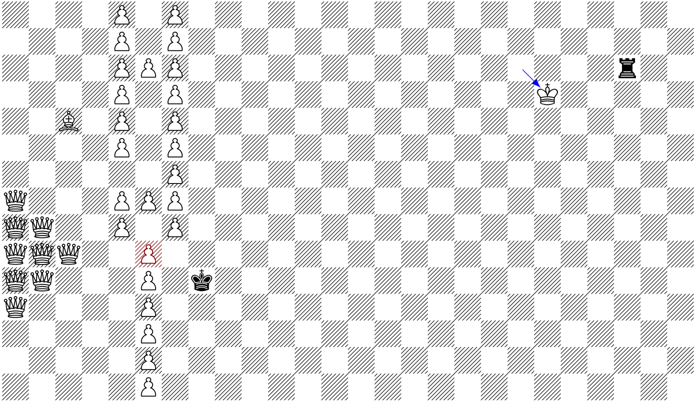
Releasing the Hordes, with value ω^2



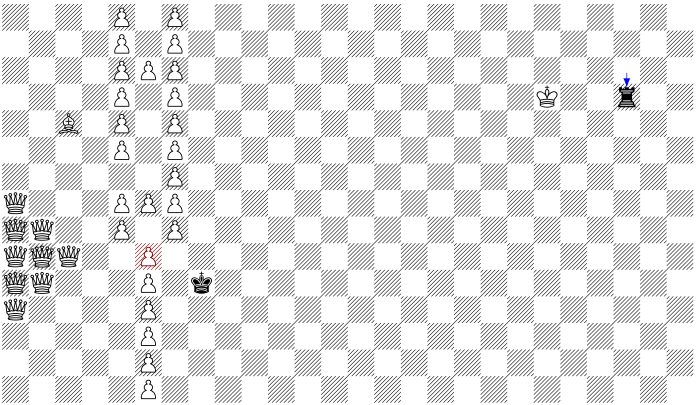
Releasing the Hordes, with value ω^2



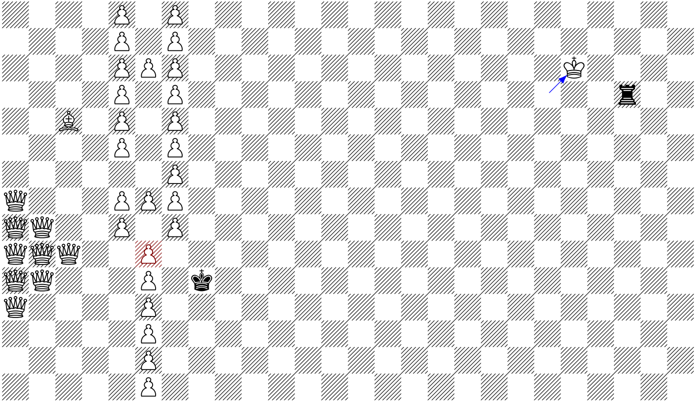
Releasing the Hordes, with value ω^2



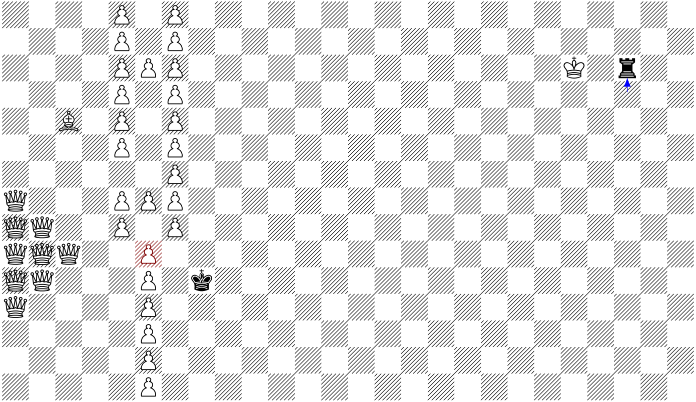
Releasing the Hordes, with value ω^2



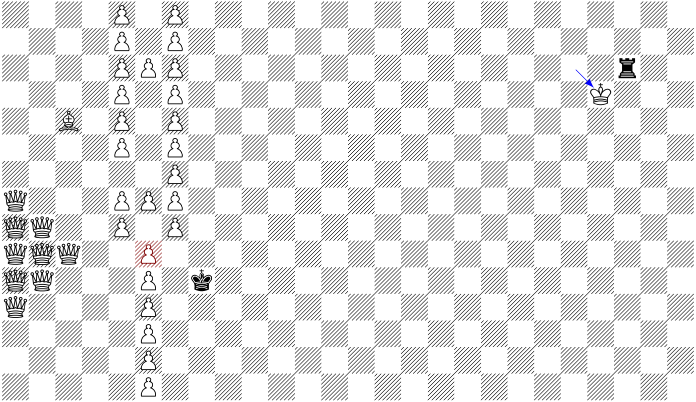
Releasing the Hordes, with value ω^2



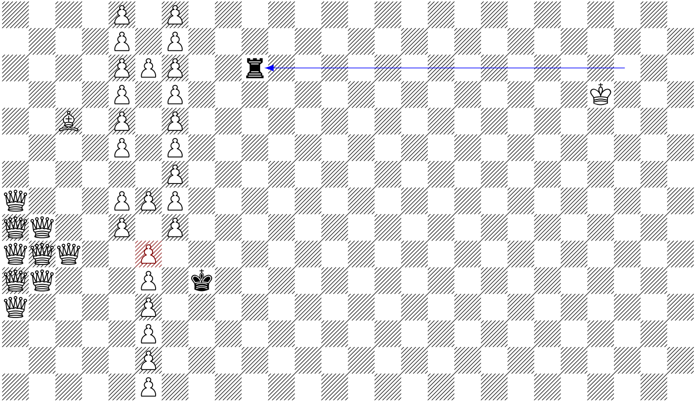
Releasing the Hordes, with value ω^2



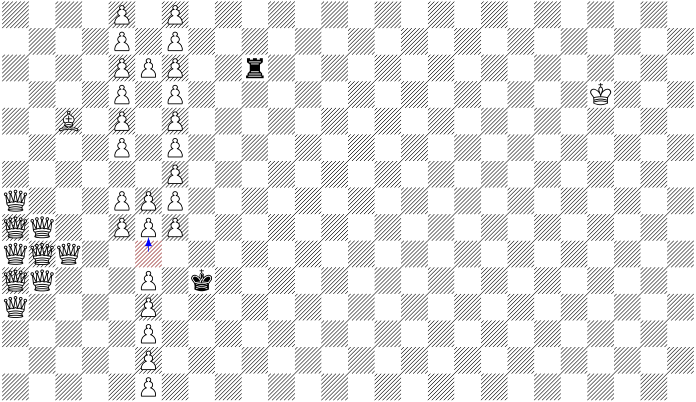
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2

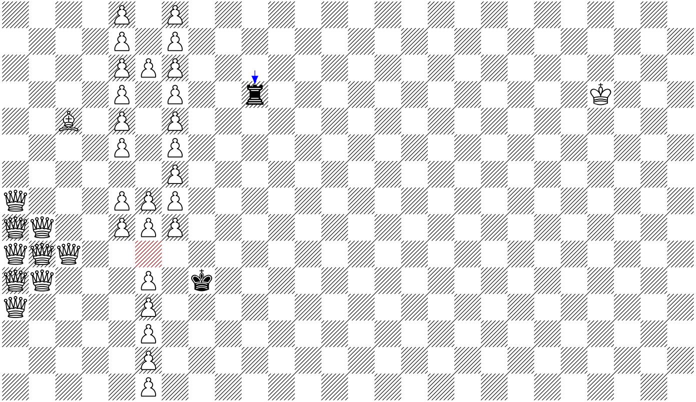


Releasing the Hordes, with value ω^2

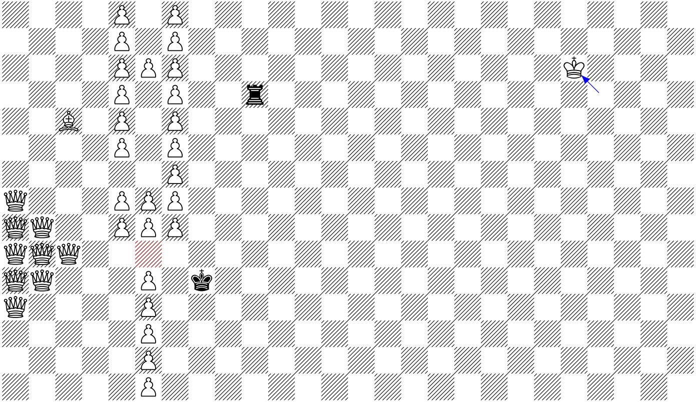


The portcullis opens...

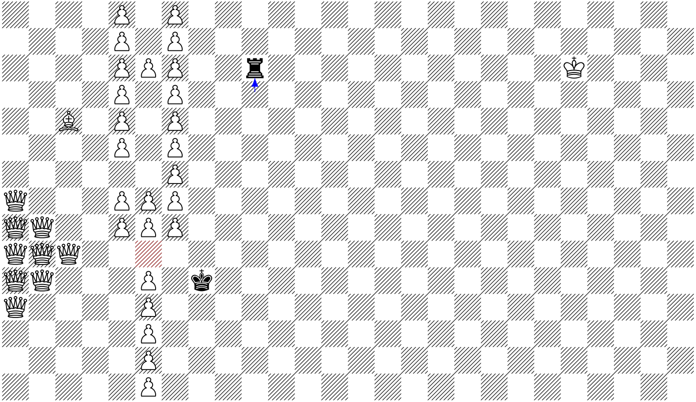
Releasing the Hordes, with value ω^2



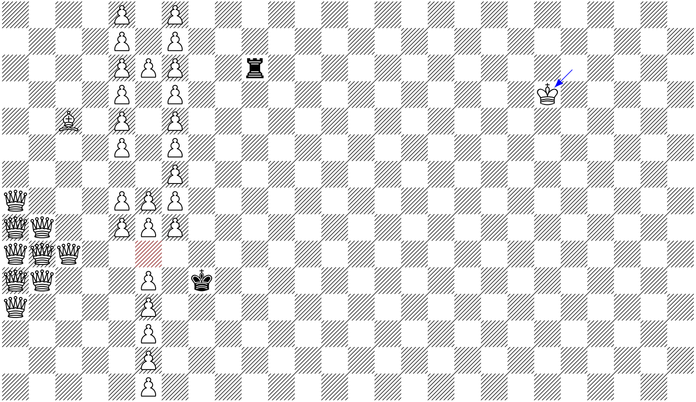
Releasing the Hordes, with value ω^2



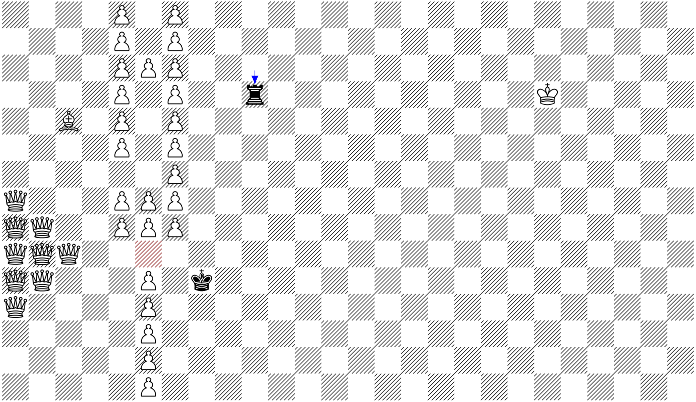
Releasing the Hordes, with value ω^2



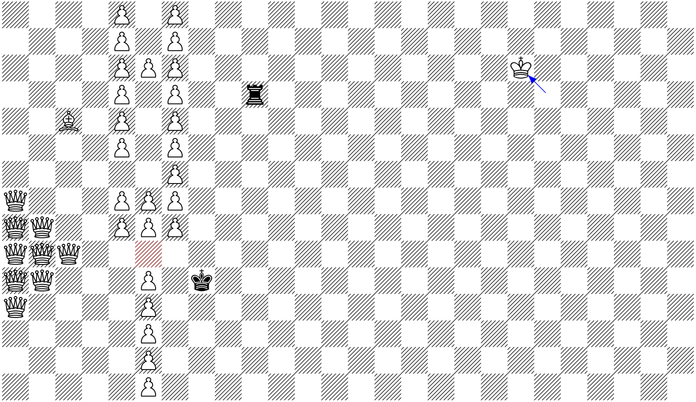
Releasing the Hordes, with value ω^2



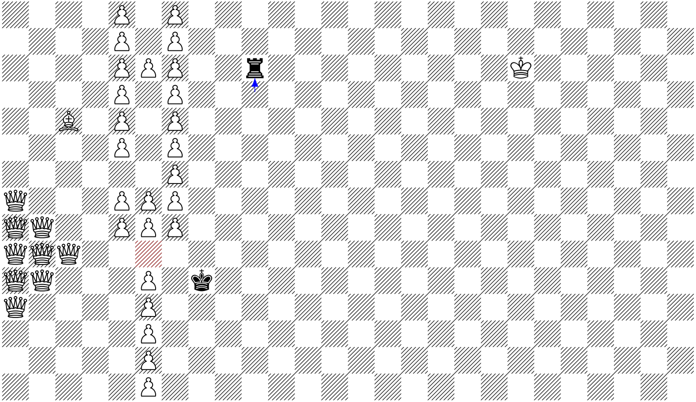
Releasing the Hordes, with value ω^2



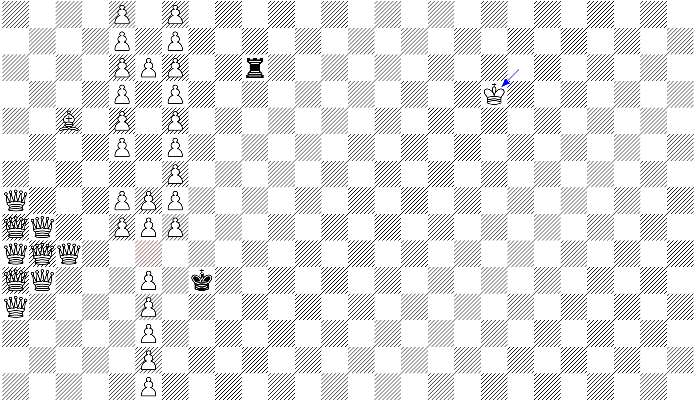
Releasing the Hordes, with value ω^2



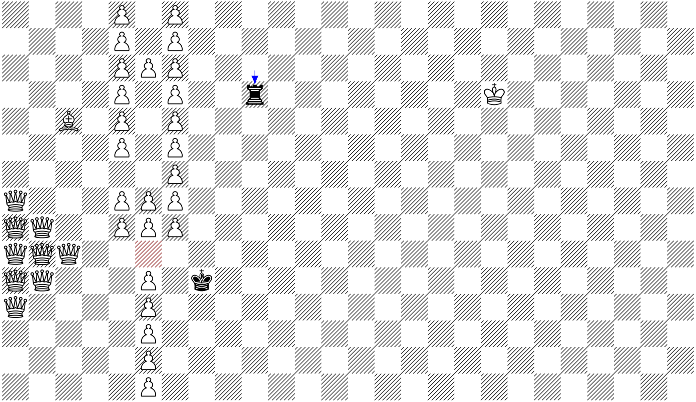
Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



Releasing the Hordes, with value ω^2



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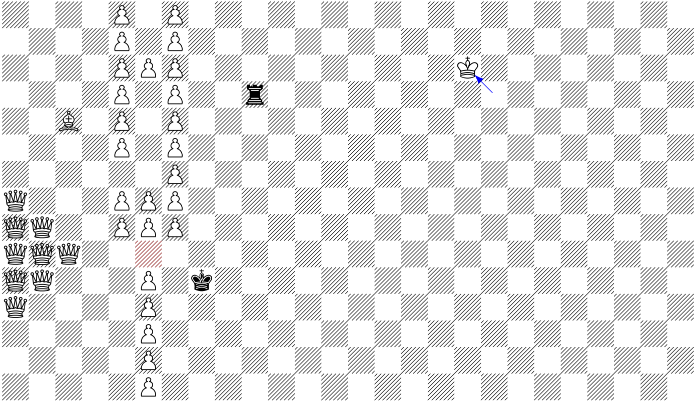

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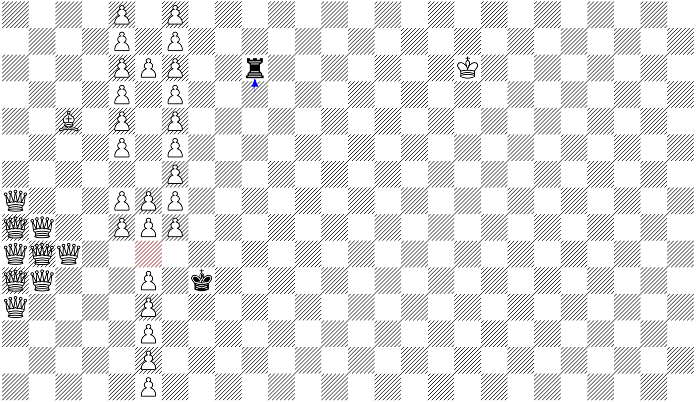

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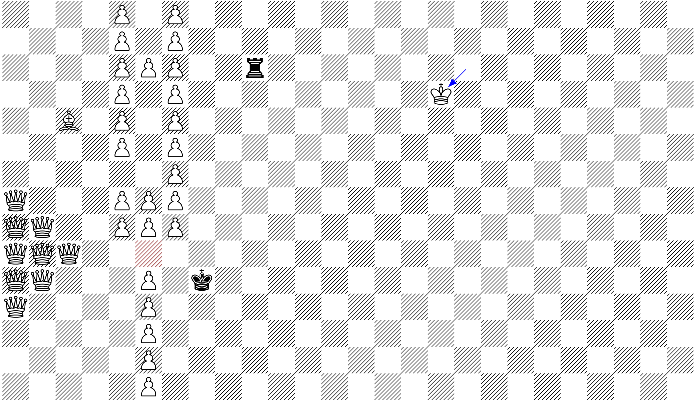

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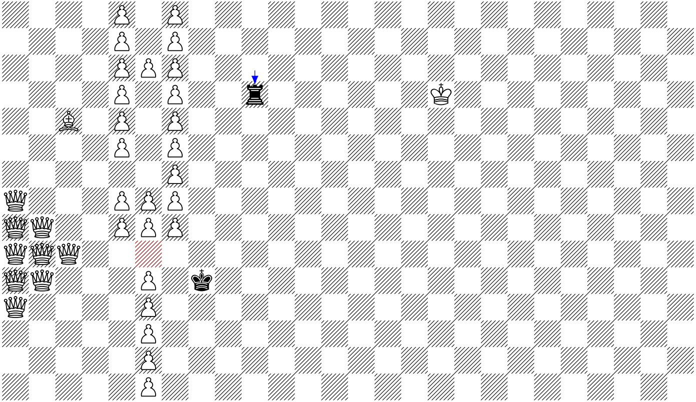

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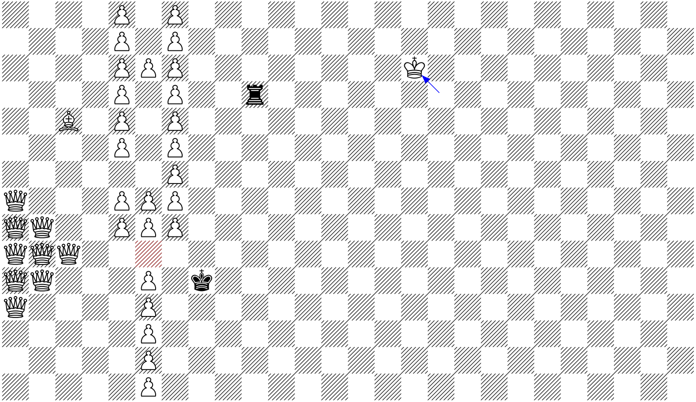

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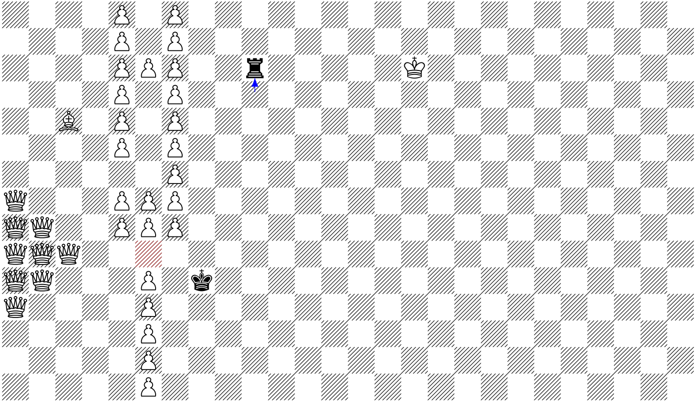

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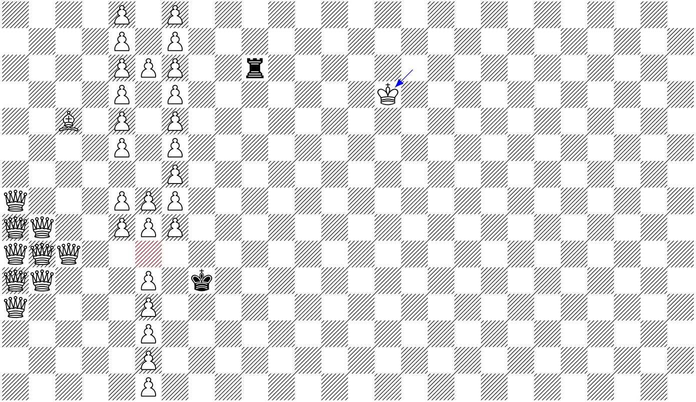

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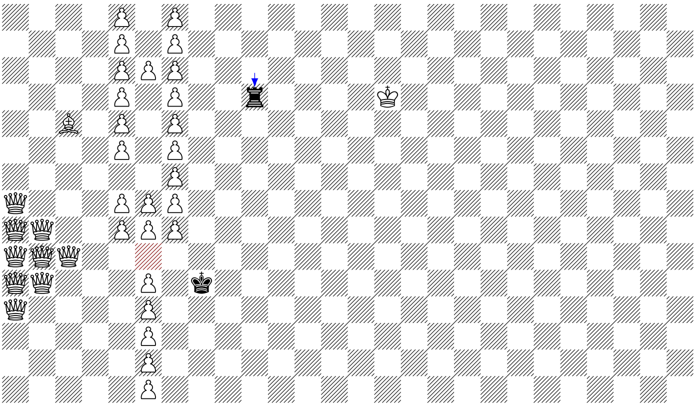

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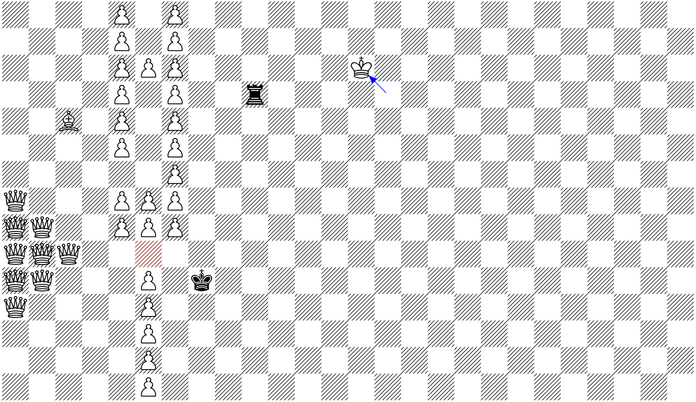

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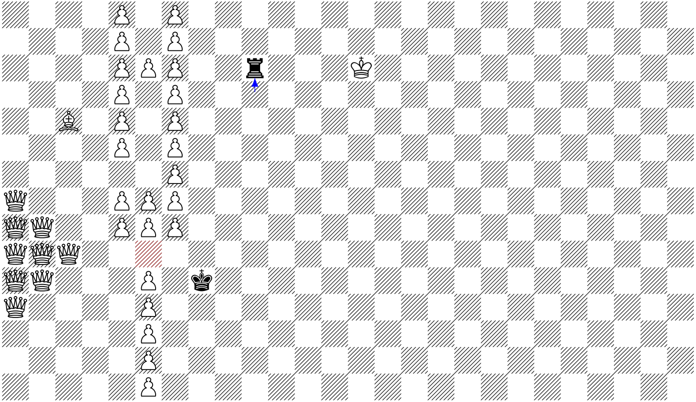

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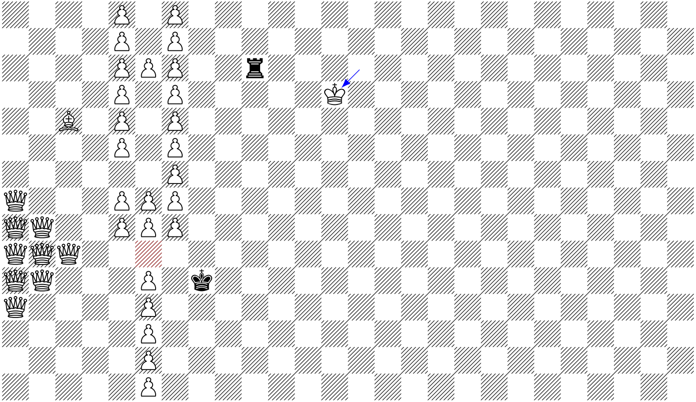

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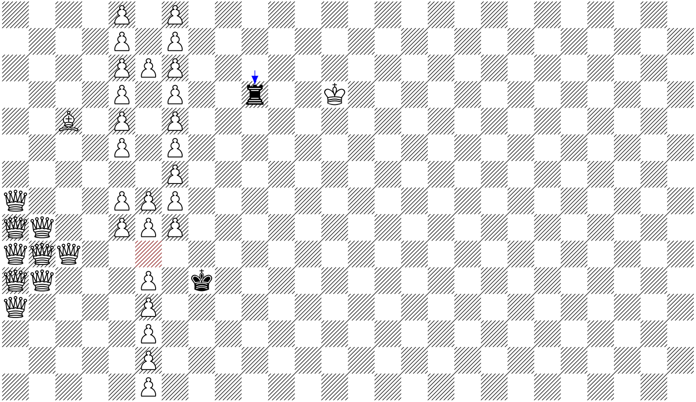

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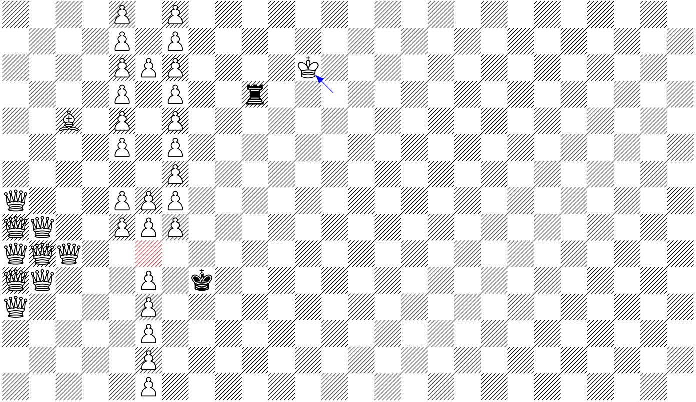

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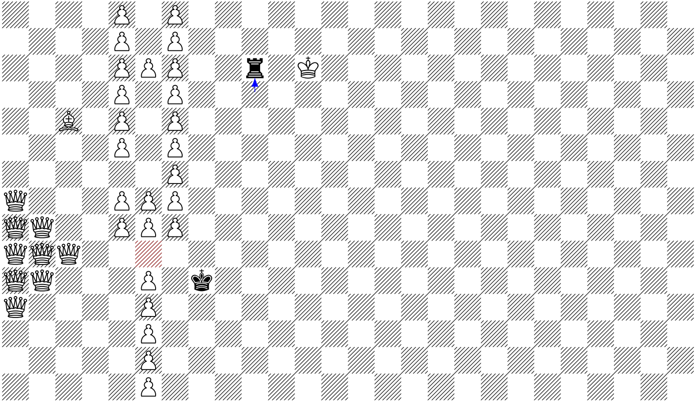

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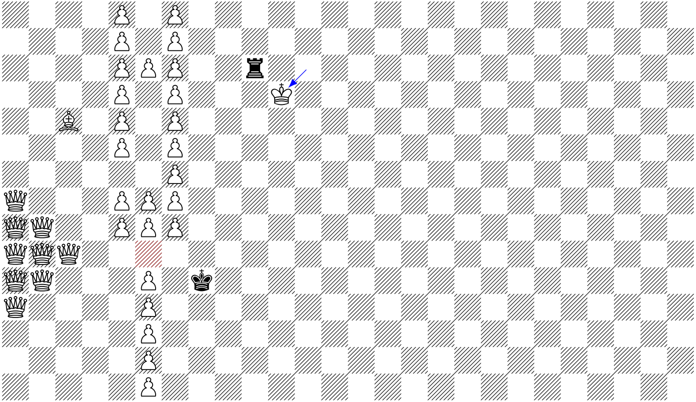

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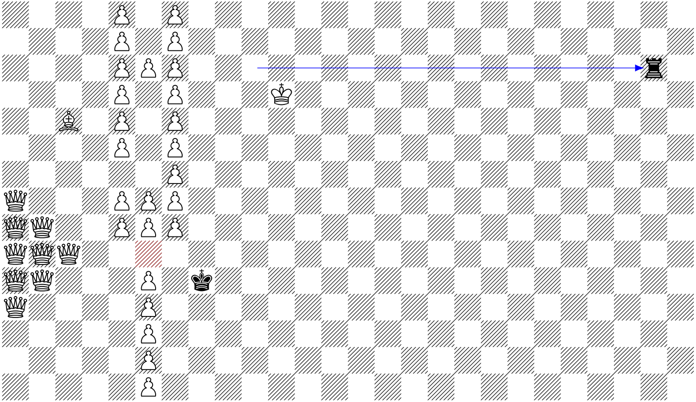

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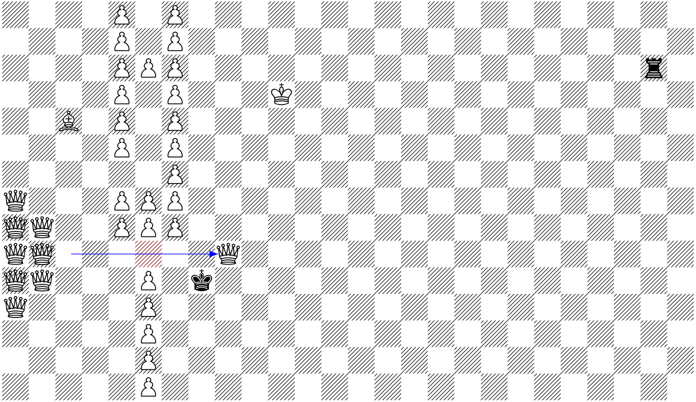
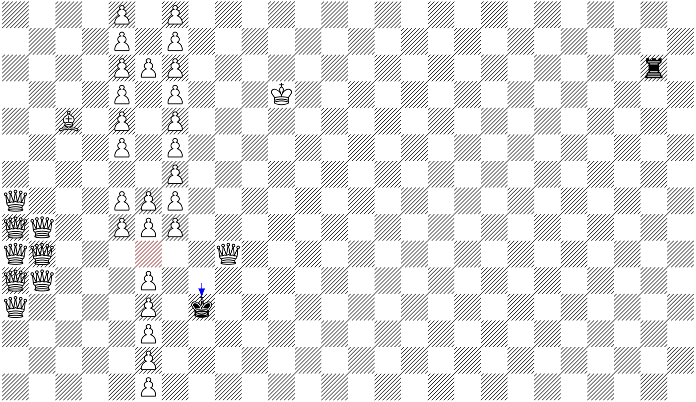


Queens enter the mating chamber.

Releasing the Hordes, with value ω^2



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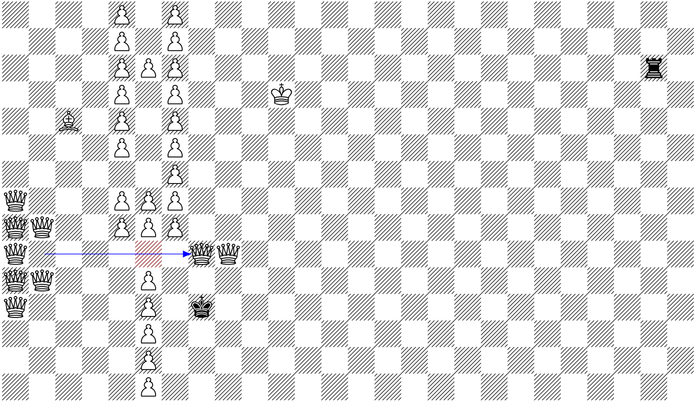
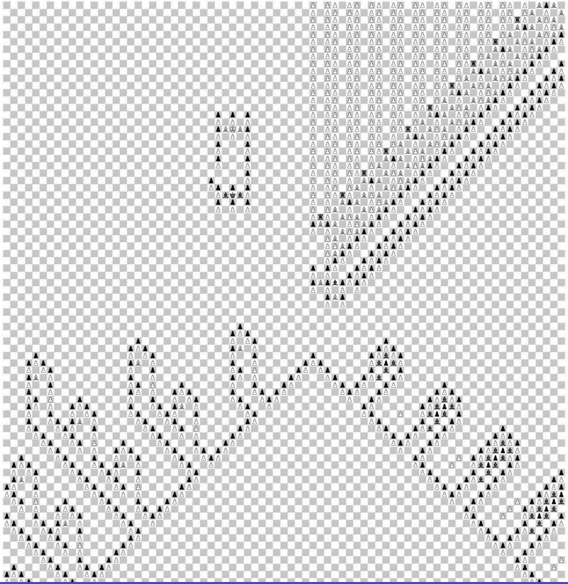
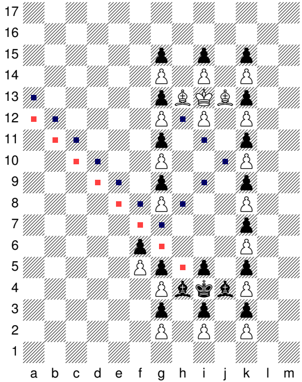


Checkmate

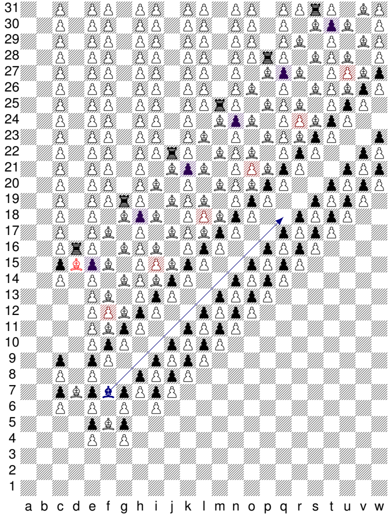
State of the art: a position with value ω^4



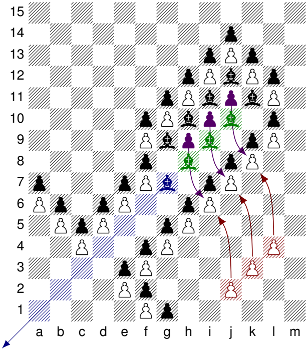
The throne room



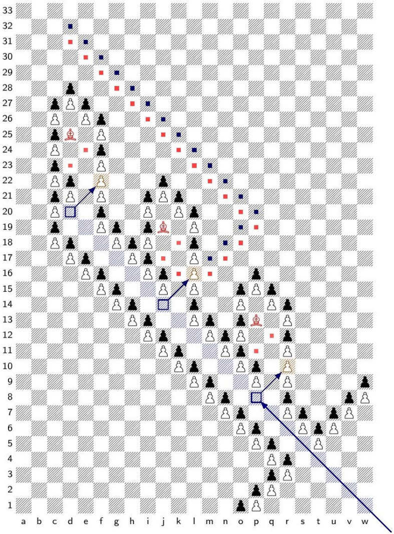
The rook towers



Bishop cannon



Bishop gateway terminal



3D chess

Let us turn briefly to three-dimensional chess.

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The history of 3D chess spans more centuries than you expect:

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- 23rd century: Spock and Kirk play on several Star Trek episodes.

Infinite 3D chess

We consider *infinite* three-dimensional chess, 3D chess on infinite boards with no boundary.

One must specify the piece movement; there is room for reasonable disagreement.

Ordinal values in infinite 3D chess

Cory Evans and I proved that every countable ordinal arises as a game value in infinite 3D chess.

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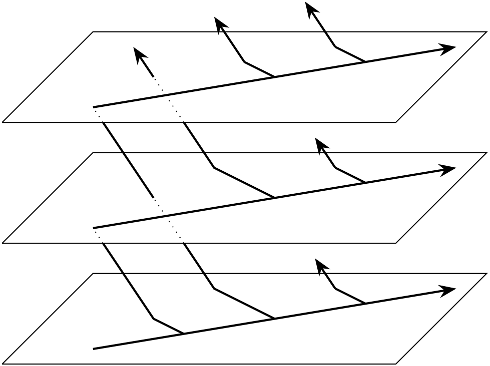
Let me discuss a few ideas from the proof.

Embedding a tree into 3 space

Main idea is to embed complex trees into infinite 3D chess.



We embed the tree, with the infinite-branching nodes of T simulated individually on separate layers

Ascending stairs in 3D chess

Our positions involve embedded trees, with connections ascending to higher levels along various stairway passages.

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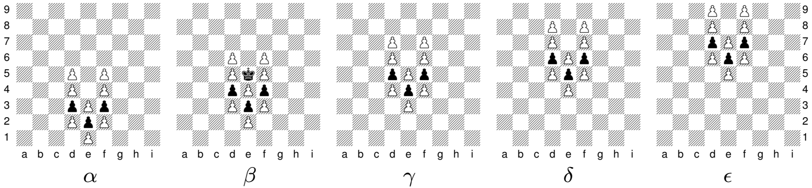
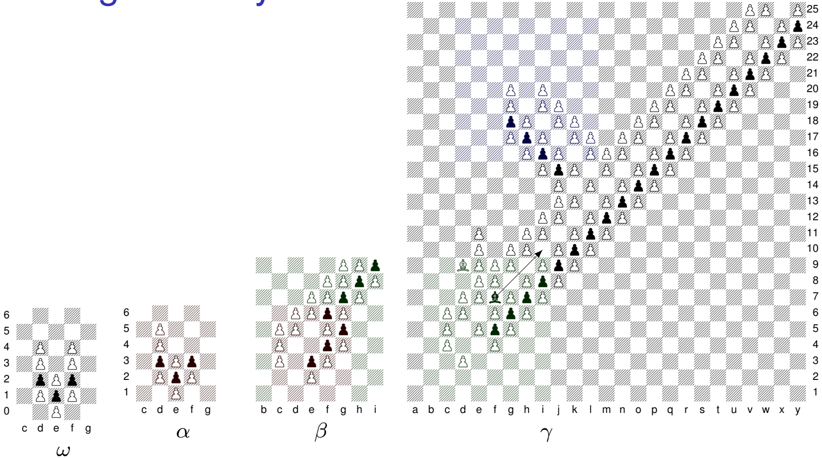


The black king is forced to ascend the stairs via

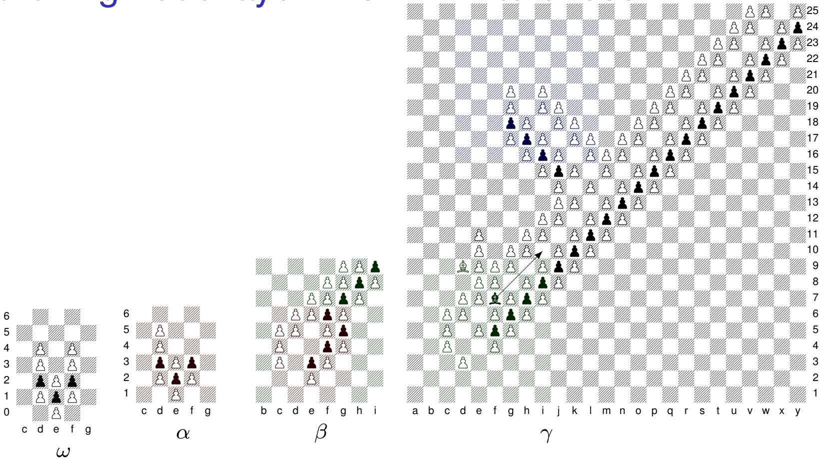
1.αe4+ Kγe6 2.βe5+ Kδe7 3.γe6+ Kϵe8 4.δe7+.

Branching node layer in 3D infinite chess



Black king is forced to ascend stairway into this realm, where a choice will be made about which branching node to follow.

Branching node layer in 3D infinite chess



Black king is forced to ascend stairway into this realm, where a choice will be made about which branching node to follow.

Main point: embedding trees $\rightarrow$ high game values

Thank you.

Slides and articles available on http://jdh.hamkins.org.

Joel David Hamkins
University of Notre Dame

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