The Set-theoretical Multiverse

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Set Theory

Set theory is the study of sets, particularly the transfinite, with a focus on well-founded transfinite recursion.

The subject began in earnest with Cantor in late 19th century and became mature in the mid 20th century, with the rise of forcing and other sophisticated techniques.

Set theory today is vast. We have independence proofs, large cardinals, forcing, infinite combinatorics, set theory of the continuum, descriptive set theory,...

Set theory serves as an ontological foundation for the rest of mathematics.

Set Theory as a Foundation

Mathematical objects can be viewed fundamentally as sets.

On this view, everything in mathematics is a set. Being precise in mathematics amounts to specifying an object in set theory.

For example, a *topological space* is an underlying set together with its open sets. A *group* is a set with a binary operation having certain properties. The real numbers are Dedekind cuts in the rationals (or equivalence classes of Cauchy sequences). Even the rational numbers and the natural numbers can be viewed as special kinds of sets.

Set theory consequently speaks to or with other mathematical subjects, particularly on foundational matters.

Axioms of set theory

The fundamental axioms of set theory are the Zermelo-Fraenkel ZFC axioms, which concern set existence.

The ZFC axioms were chosen to include all of the most general set construction principles used in the foundation of mathematics in set theory.

Thus, one can view all these mathematical arguments as taking place within set theory, using the ZFC axioms.

Models of set theory

A model of set theory is a mathematical structure in which the ZFC axioms are true.

Each such model is a complete mathematical world, in which any mathematician could be at home. Every model of ZFC has its own versions of all the mathematical objects that the typical mathematician deals with: the natural numbers, the real numbers, Hilbert spaces and so on.

All of classical mathematics can be carried out inside any one of these models of set theory.

From a classical perspective, these models of set theory appear indistinguishable.....but they are not.

Alternative worlds

An analogy with Non-Euclidean geometry:

Just as we have many alternative geometrical worlds—some Euclidean, some non-Euclidean—we also now have many alternative set-theoretical worlds.

Surely this situation was profound for geometry. It is more profound for set theory, I argue, because set theory is a foundation for all mathematics.

Thus, we have discovered the multiverse of alternative mathematical worlds...

And although at first these worlds seem indistinguishable, set theory reveals their enormous differences.

Independence

For example, Gödel provided a model of ZFC in which the Continuum Hypothesis holds, and Cohen provided one in which it fails.

This means that the Continuum Hypothesis cannot be settled on the basis of the fundamental axioms—it is *independent* of ZFC.

Set theorists have powerful methods to construct such models. e.g. forcing (Cohen 1963). We now have thousands.

As set theory has matured, the fundamental object of study has become: the model of set theory.

Second order set theory

As a result, set theory now exhibits a category-theoretic nature.

What we have is a vast cosmos of models of set theory, each its own mathematical universe, connected by forcing extensions and large cardinal embeddings.

The thesis of this talk is that, as a result, set theory now exhibits an essential second-order nature.

Two emerging developments

Two emerging developments are focused on second-order features of the set theoretic universe.

- Modal Logic of forcing. Upward-oriented, looking from a model of set theory to its forcing extensions.
- Set-theoretic geology. Downward-oriented, looking from a model of set theory down to its ground models.

This analysis engages pleasantly with various philosophical views on the nature of mathematical existence.

In particular, the two perspectives are unified by and find motivation in a multiverse view of set theory, the philosophical view that there are many set-theoretic worlds.

Philosophy of mathematical existence

Mathematical Platonism. Many set theorists hold that there is just one universe of set theory, and our task is to understand it.

Paradoxically, however, the most powerful tools in set theory are actually methods of constructing alternative universes. We build new models of set theory from existing models, via forcing and ultrapowers. These other models offer us glimpses of alternative universes and alternative truths.

The Multiverse View. This philosophical position accepts these alternative universes as fully existing mathematically.

This is realism, not formalism, but rejects the uniqueness of the mathematical universe. This philosophical view has guided the research on which I speak.

Forcing

Forcing (Cohen 1963) is a principal method of building models of set theory. It was used initially to prove the independence of Axiom of Choice and the Continuum Hypothesis.

Subsequent explosion of applications: enormous variety of models of set theory.

$$V \subseteq V[G]$$

The forcing extension $V[G]$ is built from the ground model V by adding a new ideal object G.

The extension $V[G]$ is closely related to the ground model V, but exhibits new truths in a way that can be carefully controlled.

Relations between Models

Set-theorists now often focus on the relations *between* models of set theory. How are set-theoretical properties affected by forcing?

Theorem (Yuzuru Kakuda 1981, indep. M. Magidor)

Suppose that $V \subseteq V[G]$ is a κ-c.c. forcing extension. An ideal on κ is precipitous in V if and only if the ideal it generates in $V[G]$ is precipitous in $V[G]$.

This was an early example illustrating the focus on connections between V and forcing extensions $V[G]$.

Affinity of Forcing & Modal Logic

Since a ground model has access, via names and the forcing relation, to the objects and truths of the forcing extension, there is a natural Kripke model lurking here.

- The *possible worlds* are the models of set theory.
- The *accessibility relation* relates a model M to its forcing extensions $M[G]$.

Many set theorists habitually operate within this Kripke model, even if they wouldn't describe it that way.

Modal operators

- A sentence φ is *possible* or *forceable*, written $\Diamond\varphi$, when it holds in a forcing extension.
- A sentence φ is *necessary*, written $\Box\varphi$, when it holds in all forcing extensions.

The modal assertions are expressible in set theory:

$$\Diamond\varphi \quad \leftrightarrow \quad \exists\mathbb{P} \ \Vdash_{\mathbb{P}} \varphi$$

$$\Box\varphi \quad \leftrightarrow \quad \forall\mathbb{P} \ \Vdash_{\mathbb{P}} \varphi$$

While $\Diamond$ and $\Box$ are eliminable, we nevertheless retain them here, because we are interested in what principles these operators must obey.

Easy forcing validities

- K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- Dual $\Box\neg\varphi \leftrightarrow \neg\Diamond\varphi$
- S $\Box\varphi \to \varphi$
- 4 $\Box\varphi \to \Box\Box\varphi$
- .2 $\Diamond\Box\varphi \to \Box\Diamond\varphi$

Theorem

Any S4.2 modal assertion is a valid principle of forcing.

$\varphi(p_0, \ldots, p_n)$ is a *valid principle of forcing* if $\varphi(\psi_0, \ldots, \psi_n)$ holds for any set theoretical ψ_i.

Question

What are the valid principles of forcing?

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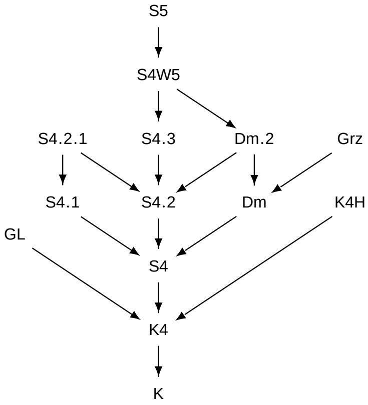
Beyond S4.2

5	$\Diamond\Box\varphi \to \varphi$
M	$\Box\Diamond\varphi \to \Diamond\Box\varphi$
W5	$\Diamond\Box\varphi \to (\varphi \to \Box\varphi)$
.3	$\Diamond\varphi \wedge \Diamond\psi \to (\Diamond(\varphi \wedge \Diamond\psi) \vee \Diamond(\varphi \wedge \psi) \vee \Diamond(\psi \wedge \Diamond\varphi))$
Dm	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Diamond\Box\varphi \to \varphi)$
Grz	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$
Löb	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$
H	$\varphi \to \Box(\Diamond\varphi \to \varphi)$

It is a fun forcing exercise to show that these are invalid in some or all models of ZFC. Counterexamples built from such assertions as $V \neq L$, $\omega_1^L < \omega_1$, CH, or Boolean combinations of these.

Some common modal theories

S5	=	S4 + 5
S4W5	=	S4 + W5
S4.3	=	S4 + .3
S4.2.1	=	S4 + .2 + M
S4.2	=	S4 + .2
S4.1	=	S4 + M
S4	=	K4 + S
Dm.2	=	S4.2 + Dm
Dm	=	S4 + Dm
Grz	=	K + Grz
GL	=	K4 + Löb
K4H	=	K4 + H
K4	=	K + 4
K	=	K + Dual



Valid principles of forcing

Theorem (Hamkins,Löwe)

If ZFC is consistent, then the ZFC-provably valid principles of forcing are exactly S4.2.

We know S4.2 is valid. The difficult part is to show that nothing else is valid.

Given S4.2 $\nvdash \varphi$, we must provide ψ_i such that $\varphi(\psi_0, \ldots, \psi_n)$ fails in some model of set theory.

Buttons and Switches

- A *switch* is a statement φ such that both φ and $\neg\varphi$ are necessarily possible.
- A *button* is a statement φ such that φ is (necessarily) possibly necessary.

Fact. Every statement in set theory is either a switch, a button or the negation of a button.

Theorem

If $V = L$, then there is an infinite independent family of buttons and switches.

Buttons: $b_n \quad = \quad$ " $\aleph_n^L$ is collapsed "

Switches: $s_m \quad = \quad$ " GCH holds at $\aleph_{\omega+m}$ "

Kripke models and frames

Kripke models provide a general modal semantics. A *Kripke model* is a collection of propositional *worlds*, with an underlying accessibility relation called the *frame*.

Partial pre-order implies S4. Directed pre-order implies S4.2

Fact

If S4.2 $\nvdash \varphi$, then φ fails in a Kripke model whose frame is a finite directed partial pre-order.

Improved Fact

If S4.2 $\nvdash \varphi$, then φ fails in a Kripke model whose frame is a finite pre-lattice.

Simulating Kripke models

Lemma

If $W \models \mathrm{ZFC}$ has buttons and switches, then for any Kripke model M on a finite pre-lattice frame, any $w \in M$, there is $p_i \mapsto \psi_i$ so that for any φ:

$$(M, w) \models \varphi(p_1, \ldots, p_n) \quad \leftrightarrow \quad W \models \varphi(\psi_1, \ldots, \psi_n).$$

Consequently, if S4.2 $\nvdash \varphi$, then there is a substitution instance such that $W \models \neg\varphi(\psi_1, \ldots, \psi_n)$.

Main Theorem

If ZFC is consistent, then the ZFC-provable forcing validities are exactly S4.2.

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Validities in a model

If $W \models \mathrm{ZFC}$, then Force^W denotes the set of valid principles of forcing over W.

Theorem

S4.2 $\subseteq \mathrm{Force}^W \subseteq$ S5.

Both endpoints occur for various W.

Question

Can we have S4.2 $\subsetneq \mathrm{Force}^W \subsetneq$ S5?

Question

If φ is valid for forcing over W, does it remain valid for forcing over all extensions of W? Equivalently, is Force^W normal?

Surprising entry of large cardinals

Theorem. The following are equiconsistent:

1. S5($\mathbb{R}$) is valid.
2. S4W5($\mathbb{R}$) is valid for forcing.
3. Dm($\mathbb{R}$) is valid for forcing.
4. There is a stationary proper class of inaccessible cardinals.

Theorem

1. *(Welch,Woodin) If S5($\mathbb{R}$) is valid in all forcing extensions (using $\mathbb{R}$ of extension), then $\mathrm{AD}^{L(\mathbb{R})}$.*
2. *(Woodin) If $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular, then it is consistent that S5($\mathbb{R}$) is valid in all extensions.*

A new perspective

Forcing is naturally viewed as a method of building *outer* as opposed to *inner* models of set theory.

Nevertheless, a simple switch in perspective allows us to view forcing as a method of producing inner models as well.

Namely, we look for how the universe V might itself have arisen via forcing. Given V, we look for classes $W \subseteq V$ of which the universe V is a forcing extension by some W-generic filter $G \subseteq \mathbb{P} \in W$

$$W \subseteq W[G] = V$$

This change in viewpoint results in the subject we call *set-theoretic geology*. Many open questions remain. Here, I give a few of the most attractive initial results.

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Digging for Grounds

A transitive class W is a *ground* of the universe V if it is a model of ZFC over which V is obtained by set forcing, that is, if there is some forcing notion $\mathbb{P} \in W$ and a W-generic filter $G \subseteq \mathbb{P}$ such that $V = W[G]$.

Theorem (Laver, independently Woodin)

Every ground W is a definable class in its forcing extensions $W[G]$, using parameters in W.

Laver's proof used my methods on approximation and covering.

Definition (Hamkins,Reitz)

The *Ground Axiom* is the assertion that the universe V has no nontrivial grounds.

The Ground Axiom is first order

The Ground Axiom GA asserts that the universe was not obtained by set forcing over an inner model.

At first, this appears to be a second order assertion, because it quantifies over grounds. But:

Theorem (Reitz)

The Ground Axiom is first order expressible in set theory.

The Ground Axiom holds in many canonical models of set theory: L, $L[0^\sharp]$, $L[\mu]$, many instances of K.

Question: To what extent are the highly regular features of these models consequences of GA?

Answer: Not at all.

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Consequences of the Ground Axiom

Theorem (Reitz)

Every model of ZFC *has an extension, preserving any desired* V_α*, which is a model of* GA.

Thus, the Ground Axiom does not imply any of the usual combinatorial set-theoretic regularity features $\Diamond$, GCH, etc.

What about $V = \mathrm{HOD}$? Reitz's method obtains GA by forcing very strong versions of $V = \mathrm{HOD}$.

Theorem (Hamkins,Reitz,Woodin)

Every model of set theory has an extension which is a model of GA *plus* $V \neq \mathrm{HOD}$.

Preparatory forcing, followed by Silver iteration. Very robust.

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Bedrocks

W is a *bedrock* of V if it is a ground of V and minimal with respect to the forcing extension relation.

Equivalently, W is a bedrock of V if it is a ground of V and satisfies GA.

Open Question

Is the bedrock unique when it exists?

Theorem (Reitz)

It is relatively consistent with ZFC *that the universe* V *has no bedrock model.*

Such models are *bottomless*.

The Mantle

We now carry the investigation deeper underground.

The principal new concept is the *Mantle*.

Definition

The *Mantle* $\mathbb{M}$ is the intersection of all grounds.

The analysis engages with an interesting philosophical view:

Ancient Paradise. This is the philosophical view that there is a highly regular core underlying the universe of set theory, an inner model obscured over the eons by the accumulating layers of debris heaped up by innumerable forcing constructions since the beginning of time. If we could sweep the accumulated material away, we should find an ancient paradise.

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Every model is a mantle

Although the *Ancient Paradise* philosophical view is highly appealing, our main theorem tends to refute it.

Main Theorem (Fuchs, Hamkins, Reitz)

Every model of ZFC is the mantle of another model of ZFC.

By sweeping away the accumulated sands of forcing, what we find is not a highly regular ancient core, but rather: an arbitrary model of set theory.

The grounds form a parameterized family

Theorem

There is a parameterized family $\{ W_r \mid r \in V \}$ of classes such that

1. *Every W_r is a ground of V and $r \in W_r$.*
2. *Every ground of V is W_r for some r.*
3. *The relation "$x \in W_r$" is first order.*

Reducing Second to First order

The parameterized family $\{\, W_r \mid r \in V \,\}$ of grounds reduces 2nd order properties about grounds to 1st order properties about parameters.

- The Ground Axiom holds if and only if $\forall r\, W_r = V$.
- W_r is a bedrock if and only if $\forall s\, (W_s \subseteq W_r \to W_s = W_r)$.
- The Mantle is defined by $\mathrm{M} = \{\, x \mid \forall r\, (x \in W_r)\,\}$.

Downward directedness

Definition

1. The grounds are *downward directed* if for every r and s there is t such that $W_t \subseteq W_r \cap W_s$.
2. The grounds are *downward set-directed* if for every A there is t with $W_t \subseteq \bigcap_{r\in A} W_r$.

Question

Is the bedrock unique when it exists? Are the grounds downward directed? Downward set directed?

In every model for which we can determine the answer, the answer is yes.

The Mantle under directedness

If the grounds are downward directed, the Mantle is well behaved.

Theorem

1. *If the grounds are downward directed, then the Mantle is constant across the grounds, and* $\mathrm{M} \models \mathrm{ZF}$.
2. *If they are downward set-directed, then* $\mathrm{M} \models \mathrm{ZFC}$.

The Generic Mantle

We defined the Mantle to be the intersection of all grounds of V.

Let the *Generic Mantle*, denoted gM, be the intersection of all grounds of all forcing extensions of V.

Any ground of V is a ground of any forcing extension of V, so the generic Mantle is the intersection of more models.

Thus, $\mathrm{gM} \subseteq \mathrm{M}$.

The generic multiverse

The *Generic Multiverse* is the family of universes obtained by closing under forcing extensions and grounds.

There are various philosophical motivations to study the generic multiverse.

Woodin introduced the generic multiverse essentially to reject it, to defeat a certain multiverse view of truth.

Our view is that the generic multiverse is a natural context for set-theoretic investigation, and it should be a principal focus of study.

A *multiverse path* is $\langle U_0, \ldots, U_n \rangle$, where each U_{i+1} is either a ground or forcing extension of U_i.

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The Generic Mantle

Theorem

The generic Mantle gM *is a parameter-free uniformly definable class, invariant by forcing, containing all ordinals and* $\mathrm{gM} \models \mathrm{ZF}$*.*

Since the generic Mantle is invariant by forcing, it follows that:

Corollary

The generic Mantle gM is constant across the multiverse. In fact, gM is the intersection of the generic multiverse.

On this view, the generic Mantle is a canonical, fundamental feature of the generic multiverse, deserving of intense study.

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Grounds and generic Grounds

Theorem

If the generic grounds are downward directed, then the grounds are dense below the generic multiverse, and so $\mathrm{M} = \mathrm{gM}$.

In this case, the generic multiverse is exhausted by the ground extensions $W_r[G]$. (So multiverse paths of length 2 suffice.)

Observation

It is relatively consistent that the generic grounds do not exhaust the generic multiverse.

Proof.

If $V[c]$ and $V[d]$ are not amalgamable, then $V[d]$ is not a generic ground of $V[c]$, but they have the same multiverse. □

The Generic HOD

HOD is the class of hereditarily ordinal definable sets.

$$\mathrm{HOD} \models \mathrm{ZFC}$$

The *generic* HOD, introduced by Fuchs, is the intersection of all HODs of all forcing extensions.

$$\mathrm{gHOD} = \bigcap_G \mathrm{HOD}^{V[G]}$$

The original motivation was to identify a very large canonical forcing invariant class.

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The Generic HOD

Facts

1. gHOD is constant across the generic multiverse.
2. The HODs of all forcing extensions are downward set-directed.
3. Consequently, $\mathrm{gHOD} \models \mathrm{ZFC}$.
4. The following inclusions hold.

$$\begin{array}{ccccc} \mathrm{HOD} & & & & \\ \cup & & & & \\ \mathrm{gHOD} & \subseteq & \mathrm{gM} & \subseteq & M \end{array}$$

Separating the notions

$$\begin{array}{ccccc} \mathrm{HOD} & & & & \\ \cup & & & & \\ \mathrm{gHOD} & \subseteq & \mathrm{gM} & \subseteq & M \end{array}$$

To what extent can we control and separate these classes?

We answer with our main theorems.

First Main Theorem

First, we can control the classes to keep them all low.

Main Theorem

If $V \models \mathrm{ZFC}$, then there is a class extension $V[G]$ in which

$$V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{HOD}^{V[G]}$$

In particular, as mentioned earlier, every model of ZFC is the mantle and generic mantle of another model of ZFC.

It follows that we cannot expect to prove ANY regularity features about the mantle or the generic mantle.

Proof ideas

The initial idea goes back to McAloon (1971), to make sets definable by forcing.

For an easy case, consider an arbitrary real $x \subseteq \omega$. It may not happen to be definable in V.

With an infinite product, we can force the GCH to hold at $\aleph_n$ exactly when $x(n) = 1$.

In the resulting forcing extension $V[G]$, the original real x is definable, without parameters.

Proof sketch

For the main theorem, start in $V \models \mathrm{ZFC}$. Want $V[G]$ with $V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{HOD}^{V[G]}$.

Let $\mathbb{Q}_\alpha$ generically decide whether to force GCH or $\neg\mathrm{GCH}$ at $\aleph_\alpha$ (*). Let $\mathbb{P} = \Pi_\alpha \mathbb{Q}_\alpha$, with set support. Consider $V[G]$ for generic $G \subseteq \mathbb{P}$.

Every set in V becomes coded unboundedly into the continuum function of $V[G]$. Hence, definable in $V[G]$ and all extensions.

So $V \subseteq \mathrm{gHOD}$. Consequently $V \subseteq \mathrm{gHOD} \subseteq \mathrm{gM} \subseteq \mathrm{M}$ and $V \subseteq \mathrm{HOD}$.

Every tail segment $V[G^\alpha]$ is a ground of $V[G]$. Also, $\cap_\alpha V[G^\alpha] = V$. Thus, $\mathrm{M} \subseteq V$. Consequently, $V = \mathrm{gHOD} = \mathrm{gM} = \mathrm{M}$.

$\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^{V[G^\alpha]}$, since $\mathbb{P} \upharpoonright \alpha$ is densely almost homogeneous.

So $\mathrm{HOD}^{V[G]} \subseteq V$.

In summary, $V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{HOD}^{V[G]}$, as desired.

Second Main Theorem: Mantles low, HOD high

Main Theorem 2

If $V \models \mathrm{ZFC}$, then there is a class extension $V[G]$ in which

$$V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} \quad \text{but} \quad \mathrm{HOD}^{V[G]} = V[G]$$

Proof ideas

Want an extension $V[G]$ with

$$V = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]} = \mathrm{gHOD}^{V[G]} \quad \text{but} \quad \mathrm{HOD}^{V[G]} = V[G]$$

- Balance the forces on M, gM, gHOD and HOD.
- Force to $V[G]$ where every set in V is coded unboundedly in the GCH pattern.
- Also ensure that G is definable, but not robustly.
- The proof uses *self-encoding forcing*:

  Add a subset $A \subseteq \kappa$. Then code this set A into the GCH pattern above κ. Then code THOSE sets into the GCH pattern, etc. Get extension $V[G_{(\kappa)}]$ in which $G_{(\kappa)}$ is definable.

Keeping HODs low, Mantles high

Next, we keep the HODs low and the Mantles high, seeking $V[G]$ with $V = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]}$ but $\mathrm{M}^{V[G]} = V[G]$.

Such a model $V[G]$ will of course be a model of the Ground Axiom plus $V \neq \mathrm{HOD}$. Recall

Theorem (Hamkins,Reitz,Woodin)

Every $V \models \mathrm{ZFC}$ has a class forcing extension $V[G] \models \mathrm{GA} + V \neq \mathrm{HOD}$.

We modified the argument to obtain:

Theorem

If $V \models \mathrm{ZFC}$, then there is a class extension $V[G]$ in which

$$V = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]} \qquad \textit{but} \qquad \mathrm{M}^{V[G]} = V[G]$$

Mantles high, HODs high

Lastly,

Theorem

If $V \models \mathrm{ZFC}$, then there is $V[G]$ in which

$$V[G] = \mathrm{HOD}^{V[G]} = \mathrm{gHOD}^{V[G]} = \mathrm{M}^{V[G]} = \mathrm{gM}^{V[G]}$$

This is possible by forcing the Continuum Coding Axiom CCA.

The Inner Mantles

When the Mantle M is a model of ZFC, we may consider the Mantle of the Mantle, iterating to reveal the *inner Mantles*:

$$\mathrm{M}^1 = \mathrm{M} \qquad \mathrm{M}^{\alpha+1} = \mathrm{M}^{\mathrm{M}^\alpha} \qquad \mathrm{M}^\lambda = \bigcap_{\alpha<\lambda} \mathrm{M}^\alpha$$

Continue as long as the model satisfies ZFC.

The *Outer Core* is reached if M^α has no grounds, $M^\alpha \models \mathrm{ZFC} + \mathrm{GA}$.

Conjecture. Every model of ZFC is the α^{th} inner Mantle of another model, for arbitrary $\alpha \leq \mathrm{ORD}$.

Philosophical view: ancient paradise?

Thank you.

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